ARithmetische Geometrie OberSeminar

Bonn, Wintersemester 2019/20

Prismatic Dieudonné theory

In the study of abelian varieties A over number fields K, a key role is played by the study of their p^n-torsion subgroups $A[p^n] \subset A$. For example, these can be used to define their p-adic Tate module $T_pA = \varprojlim_n A[p^n](\overline{K})$, giving interesting representations of the absolute Galois group of K. On the other hand, interesting new phenomena appear when A is defined over the ring of integers and one takes the fibres in characteristic p. Indeed, one has to consider the torsion subgroup $A[p^n] \subset A$ as a finite flat group scheme, which may be highly nonreduced. For varying n, these can be assembled into the p-divisible group $A[p^\infty] = \varinjlim_n A[p^n]$, giving an analogue of the Tate module. An important result is that p-divisible groups over perfect fields k can be classified in terms of semi-linear algebra over the ring of p-typical Witt vectors $W(k)$ of k:

Theorem. The category of p-divisible groups over a perfect field k is equivalent to the category of finite free $W(k)$-modules M equipped with a σ-linear operator $F : M \to M$ and a σ^{-1}-linear operator $V : M \to M$ satisfying $FV = VF = p$.

Various generalizations of such a classification have been studied in the literature. First, one studied p-divisible groups G over rings R of characteristic p, and has obtained a classification in terms of F-crystals: Roughly, locally free sheaves on the crystalline site of R equipped with a Frobenius. The construction of the functor is due to Berthelot-Breen-Messing, [3], and various results on fully faithfulness and essential surjectivity have been obtained by de Jong and others. If R is perfect, a direct generalization of the theorem holds true; this is a theorem of Gabber relying on previous work of Berthelot, and reproved by Lau.

Obtaining results in mixed characteristic is more difficult. Partial results had been obtained by Fontaine and then generalized by Breuil and Kisin. However, it remained unclear what the general target category should be. Using windows in place of crystals, Zink has obtained some results that were further generalized by Lau.

Recently, Anschütz and Le Bras have obtained a generalization of most known classification results for p-divisible groups, in terms of the prismatic site.

Theorem. Let R be a quasisyntomic ring that is flat over $\mathbb{Z}/p^n$ for some n, or over $\mathbb{Z}_p$. Then the category of p-divisible groups over R is equivalent to a certain category of locally free sheaves on the prismatic site of R equipped with a filtration and a Frobenius.

The goal of this ARGOS will be to study the proof of their result.

Talks

1. Talk: Dieudonné theory over perfect fields

Recall the definition of finite flat group schemes and p-divisible groups, as well as Cartier duality and the relation to abelian schemes, following [13, 1.1, 1.2, 1.3, 2.1]. When the base is a perfect field k, recall the classification in terms of Dieudonné modules, cf. [6, p. 71]. To do this, recall the decomposition of finite flat group schemes over k into étale, multiplicative, and unipotent parts, [6, p. 39]. Recall the explicit description of the Dieudonné modules of $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞}.

2. Talk: Construction of the crystalline Dieudonné module

In this talk, the piece-meal definition of the Dieudonné module is replaced by a conceptual definition that works over any base. First, recall without proof the relation of the Dieudonné module of $A[p^\infty]$ with the first crystalline cohomology $H^1_{\mathrm{crys}}(A/W(k))$ of an abelian variety A/k. Then give the general definition of the crystal of a finite locally free group scheme G following Berthelot-Breen-Messing, [3, Section 3.1]. Explain Breen's method [3, Section 2.1] (cf. also [1, Section 4.4]) for computing the relevant Ext-groups, and show that this functor is well-behaved, cf. [3, Corollaire 3.1.3, Proposition 3.1.6, Theorem 3.3.3], by reduction to the case of abelian schemes (and prove the identification with crystalline cohomology of abelian schemes for this definition of the Dieudonné module, [3, Theorem 2.5.6]).

3. Talk: Dieudonné theory over perfect rings

Describe the category of p-divisible groups over perfect rings in terms of Dieudonné modules, by following the work of Berthelot, [4, Section 3], for perfect valuation rings, and using the v-descent results of [5] to handle the general case (a similar argument with slightly different emphasis is carried out in [11, Appendix to Lecture XVII]). (An alternative proof is due to Lau, [9].)

4. Talk: Dieudonné theory over $\mathcal{O}_C$

For an algebraically closed nonarchimedean field $C/\mathbb{Q}_p$, describe the category of p-divisible groups over $\mathcal{O}_C$ in terms of pairs (T, W) of a finite free $\mathbb{Z}_p$-module T and a filtration $W \subset T \otimes_{\mathbb{Z}_p} C$, cf. [7], [12, Section 5.2] (you may restrict to the case that C is spherically complete with value group $\mathbb{R}_{>0}$), and after recalling Fontaine's definition of A_{inf}, in terms of Breuil-Kisin-Fargues modules, [11, Theorem 14.4.1] (this is due to Fargues, [8]).

5. Talk: Prisms and perfectoid rings

Introduce the categories of prisms and perfectoid rings [1, Section 2.1], and prismatic cohomology, [1, Section 3.1]. Moreover, introduce the classes of quasisyntomic and quasiregular semiperfectoid rings, [1, Sections 3.3, 3.4], and state (without proof) [1, Proposition 3.4.2, Lemma 3.4.3, Theorem 3.4.6]. (Do not discuss the part of [1, Section 3.4] on TC.)

6. Talk: Filtered prismatic Dieudonné crystals

For a quasisyntomic ring R, define the category of filtered prismatic Dieudonné crystals, [1, Section 4.1]. Cover all the material up to (including) [1, Proposition 4.1.22].

7. Talk: Definition of the functor

Define the functor from p-divisible groups G over R to filtered prismatic Dieudonné crystals and compare it to known constructions in characteristic p and over $\mathcal{O}_C$, following [1, Sections 4.2, 4.3] up to Corollary 4.3.8.

8. Talk: Finiteness results

Prove that the functor defined in the previous talk indeed defines locally free modules, and is exact, by first analyzing the case of abelian schemes [1, Section 4.5] and then reducing the general case to this [1, Section 4.6].

9. Talk: The q-logarithm

Following [1, Section 4.7], describe explicitly the prismatic Dieudonné modules of $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞}, and the functor in terms of the q-logarithm. Using the input from [2], prove the fully faithfulness for maps $\mathbb{Q}_p/\mathbb{Z}_p \to \mu_{p^\infty}$, [1, Theorem 4.7.6]. Depending on time, give some indications on the required background from algebraic K-theory and topological cyclic homology.

10. Talk: Fully faithfulness

Following [1, Section 4.8] along with [1, Section 3.6], prove fully faithfulness of the functor in general, by reduction to the previous talk.

11. Talk: Essential surjectivity

Prove essential surjectivity following [1, Section 4.9] by lifting to perfectoid rings; also discuss the relevant parts on windows from [1, Section 4.1].

12. Talk: Complements

Discuss [1, Section 5.1] on the classification of finite locally free group schemes over perfectoid rings (for $p > 2$, this has also been proved by Lau, [10]), and [1, Section 5.2] about the classification of p-divisible groups over regular rings of mixed characteristic.

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