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Submit the solutions in groups of two at the lecture on Tuesday, 2018-06-12

Definition. The *BMO (for "bounded mean oscillation") norm* of a (measurable) function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_{\mathrm{BMO}} := \sup_I \inf_{c \in \mathbb{R}} |I|^{-1} \int_I |f - c|,$$

where the supremum is taken over all subintervals of $\mathbb{R}$. The *dyadic BMO norm* is defined similarly with a supremum over dyadic intervals I (intervals of the form $2^j((0,1] + k)$ for $j, k \in \mathbb{Z}$).
The space of functions with finite BMO (resp. dyadic BMO) norm is denoted by BMO (resp. BMO_d)

Exercise 1. (a) Show that $\|f\|_{\mathrm{BMO}} \le \|f\|_\infty$

(b) Show that the function $\log|x|$ is in BMO.

(c) Show that the function $1_{x>0} \log|x|$ is in BMO_d, but not in BMO.

(d) Show that

$$\|f\|_{\mathrm{BMO}} \le \sup_I |I|^{-1} \int_I |f - f_I| \le 2\|f\|_{\mathrm{BMO}}, \quad f_I = |I|^{-1} \int_I f.$$

Exercise 2. Let $K_j(x) = x_j |x|^{-d-1}$ be the j-th *Riesz kernel* on $\mathbb{R}^d$, where $j \in \{1, \dots, d\}$. Define the corresponding principal value tempered distribution by

$$\mathrm{p.v.} K_j f := \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B(x,\epsilon)} f(y) K_j(y) dy.$$

We define the jth *Riesz transform* as the convolution

$$R_j f(x) = \mathrm{p.v.} K_j * f.$$

(a) Verify that $\mathrm{p.v.} K_j$ indeed defines a tempered distribution.

(b) Verify that K_j is a Calderón-Zygmund kernel (satisfies size and regularity conditions).

(c) Show that there is a constant such that $\widehat{\mathrm{p.v.} K_j}(\xi) = C\xi_j / |\xi|$. Conclude that R_j extends to a bounded operator on L^2.

(d) Let $u, f : \mathbb{R}^d \to \mathbb{C}$ be Schwartz functions and suppose that $\Delta u = f$. Show that $\|\partial_j \partial_l u\|_p \lesssim_p \|f\|_p$ for all $1 < p < \infty$ and $1 \le j, l \le d$.

This was one of the original motivations for the theory of singular integrals.