# Evaluation of an integral

Dr. E. Assing

The goal of this note is to prove the integral identity contained in the following lemma.

**Lemma 1.** *For $n \in \mathbb{N} \cup \{0\}$, $\Re(\nu) > -\frac{1}{2}$ and $a > 0$ we have*

$$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^{(\nu)}\left(\frac{x}{a}-1\right) e^{-bx} dx = (-1)^n 2\pi \frac{\Gamma(2\nu+n)}{n!\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab} I_{\nu+n}(ab).$$

Here $C_n^{(\nu)}(z)$ is the Gegenbauer Polynomial (see [3, 8.930]) and $I_\lambda(z)$ is the usual $I$-Bessel function (see [3, 8.406]).

**Remark 1.** *Note that for $n = 0$ we have $C_0^{(\nu)}(z) = 1$ and*

$$2\pi \frac{\Gamma(2\nu)}{\Gamma(\nu)} = \sqrt{\pi} 2^{2\nu} \Gamma(\nu + \frac{1}{2}).$$

*The latter is a well known duplication formula for the $\Gamma$-function. Thus for $n = 0$ we recover*

$$\begin{aligned} \int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} e^{-bx} dx &= 2\pi \frac{\Gamma(2\nu)}{\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab} I_\nu(ab) \qquad (1) \\ &= \sqrt{\pi}\Gamma(\nu+\frac{1}{2}) \left(\frac{2a}{b}\right)^{\nu} e^{-ab} I_\nu(ab). \end{aligned}$$

*This agrees with [3, 3.388.1] (with $\nu \rightsquigarrow \nu + \frac{1}{2}$, $u = a$ and $\mu = b$). However, the evaluation provided in Lemma 1 differs from [3, 7.322] by a factor of 2. This is exactly the factor by which [3, 7.322] differs from [3, 3.388.1] when specialised to $n = 0$.*

*This mistake seems to go back to [1, 4.11.(9) (p.171)]. The same issue manifests itself in [1, 5.15.(6)] after taking the Inverse Laplace Transform. Indeed, when specialising [1, 5.15.(6)] to $n = 0$ it differs from [1, 5.15.(5)] by the same factor of 2.*

**Corollary 1.** *For $n \in \mathbb{N} \cup \{0\}$, $\Re(\nu) > -\frac{1}{2}$ and $a > 0$ we have*

$$\begin{aligned} &\frac{1}{2\pi i} \int_{(c)} z^{-\nu} I_{\nu+n}(az) e^{xz} dz \\ &\qquad = \mathbb{1}_{(-a,a)}(x) \cdot (-1)^n \frac{n!\Gamma(\nu)}{2\pi\Gamma(2\nu+n)} \cdot \left(\frac{2}{a}\right)^{\nu} \cdot (a^2 - x^2)^{\nu-\frac{1}{2}} \cdot C_n^{(\nu)}\left(\frac{x}{a}\right). \end{aligned}$$

*Our formula is consistent with [1, 5.15(5) (p.277)] when specialised to $n = 0$ but differs from [1, 5.15(6) (p.277)] by a factor of $\frac{1}{2}$.*

*Proof.* This follows directly from Lemma 1 after taking the Inverse Laplace Transform. □

## Preliminaries

We will derive Lemma 1 from (1) using a recursive relation for the Gegenbauer Polynomials. It seems fair to take (1) for granted as it is a classical integral representation for $I_\nu(z)$ and is given in [3, 3.388.1]. (One can also verify (1) using *Mathematica* for example.)

For the record we state the following version of (1):

$$\int_{-1}^{1}[1-x^2]^{\nu-\frac{1}{2}}e^{-bx}dx = 2\pi\frac{\Gamma(2\nu)}{\Gamma(\nu)}(2b)^{\nu}I_\nu(b). \tag{2}$$

Indeed this follows from (1) when setting $a=1$ and changing variables $x \mapsto x+1$.

Finally we need the recursion

$$(1-x^2)^{\nu-\frac{1}{2}}C_{n+1}^{(\nu)}(x) = -\frac{2\nu}{(n+1)(n+2\nu+1)}\cdot\frac{d}{dx}\left[(1-x^2)^{\nu+\frac{1}{2}}C_n^{(\nu+1)}(x)\right]. \tag{3}$$

The latter follows directly from [2, 18.9(iii)] (see `http://dlmf.nist.gov/18.9.E20`).

Note that we have already used the well known (see [3, 8.335.1]) duplication formula

$$2\sqrt{\pi}\frac{\Gamma(2s)}{\Gamma(x)} = 2^{2x}\Gamma(x+\frac{1}{2}).$$

We will also make use of the simple formula

$$\frac{x}{\Gamma(x+1)} = \frac{1}{\Gamma(x)}$$

given by [3, 8.331.1].

## Proof of Lemma 1

We first compute the following simpler integral.

**Lemma 2.** *For $n \in \mathbb{N}\cup\{0\}$, $\Re(\nu) > -\frac{1}{2}$ and $a>0$ we have*

$$\int_{-1}^{1}[1-x^2]^{\nu-\frac{1}{2}}C_n^{(\nu)}(x)e^{-bx}dx = (-1)^n 2\pi\frac{\Gamma(2\nu+n)}{n!\Gamma(\nu)}(2b)^{-\nu}I_{\nu+n}(b).$$

*Proof.* Note that, if $n=0$, the statement is given in (2). Thus we can proceed by induction and assume that the integral has been computed for some $n\in\mathbb{N}\cup\{0\}$ and arbitrary $\nu$ with $\Re(\nu) > -\frac{1}{2}$.

For notational sake we write

$$f_n^\nu(x) = (1-x^2)^{\nu-\frac{1}{2}}C_n^\nu(x) \text{ and}$$
$$a_{n+1,\nu} = -\frac{2\nu}{(n+1)(n+2\nu+1)}.$$

With this notation (3) reads

$$f_{n+1}^\nu = a_{n+1,\nu}\cdot\frac{d}{dx}f_n^{\nu+1}(x)$$

and we rewrite our integral as

$$\int_{-1}^{1} [1-x^2]^{\nu-\frac{1}{2}} C_{n+1}^{(\nu)}(x) e^{-bx} dx = a_{n+1,\nu} \cdot \int_{-1}^{1} e^{-bx} \frac{d}{dx} f_n^{\nu+1}(x) dx.$$

Notice that $f_n^{\nu+1}(\pm 1) = 0$, so that integration by parts yields

$$\begin{aligned}
\int_{-1}^{1} [1-x^2]^{\nu-\frac{1}{2}} C_{n+1}^{(\nu)}(x) e^{-bx} dx &= u \cdot a_{n+1,\nu} \cdot \int_{-1}^{1} e^{-bx} f_n^{\nu+1}(x) dx \\
&= b \cdot a_{n+1,\nu} \cdot \int_{-1}^{1} [1-x^2]^{\nu+\frac{1}{2}} C_n^{(\nu+1)}(x) e^{-bx} dx \\
&= b \cdot a_{n+1,\nu} \cdot (-1)^n 2\pi \frac{\Gamma(2\nu+n+2)}{n!\Gamma(\nu+1)} (2b)^{-\nu-1} I_{\nu+n+1}(b) \\
&= (-1)^{n+1} 2\pi \frac{\Gamma(2\nu+n+1)}{(n+1)!\Gamma(\nu)} (2b)^{-\nu} I_{\nu+n+1}(b).
\end{aligned}$$

This concludes the induction step and we are done. □

*Proof of Lemma 1:* The remaining task is to reduce the statement of Lemma 1 to Lemma 2 by suitable changes of variables. Indeed we compute

$$\begin{aligned}
\int_{0}^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^{(\nu)} \left(\frac{x}{a}-1\right) e^{-bx} dx &= a^{2\nu} \int_{0}^{2} [x(2-x)]^{\nu-\frac{1}{2}} C_n^{(\nu)} (x-1) e^{-abx} dx \\
&= a^{2\nu} e^{-ab} \int_{-1}^{1} [1-x^2]^{\nu-\frac{1}{2}} C_n^{(\nu)} (x) e^{-abx} dx.
\end{aligned}$$

Now we apply Lemma 2 and get

$$\begin{aligned}
\int_{0}^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^{(\nu)} \left(\frac{x}{a}-1\right) e^{-bx} dx &= a^{2\nu} e^{-ab} \cdot (-1)^n 2\pi \frac{\Gamma(2\nu+n)}{n!\Gamma(\nu)} (2ab)^{-\nu} I_{\nu+n}(ab) \\
&= (-1)^n 2\pi \frac{\Gamma(2\nu+n)}{n!\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab} I_{\nu+n}(ab).
\end{aligned}$$

This is the desired formula and we are done. □

# References


[1] Harry Bateman, *Tables of integral transforms [volumes i & ii]*, vol. 1, McGraw-Hill Book Company, 1954.

[2] *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/, Release 1.1.1 of 2021-03-15, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, seventh ed., Elsevier/Academic Press, Amsterdam, 2007, Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX). MR 2360010