Unstable extremal surfaces of the "Shiffman functional" spanning rectifiable boundary curves *

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Bonn, Januar 2005

*This research was supported by the Deutsche Forschungsgemeinschaft. The author would also like to thank Prof. Dr. S. Hildebrandt and Prof. Dr. H. von der Mosel.

Contents

1 Introduction and main result

In this paper we generalize the result of [**?**], a sufficient condition for the existence of unstable extremal surfaces of a parametric functional with a dominant area term via the "*mountain pass* principle", from polygonal to arbitrary closed rectifiable boundary curves $\Gamma \subset \mathbb{R}^3$ that merely have to satisfy a chord-arc condition (**??**). Hence, we give a precise proof of a "*mountain pass* theorem" claimed by Shiffman in [**?**] who only outlined a very sketchy and incomplete proof in the author's opinion.

Shiffman considered Plateau's problem for the 2-dimensional parametric functional

$$\mathcal{J}(X) := \int_B F(X_u \wedge X_v) + k \mid X_u \wedge X_v \mid dudv =: \mathcal{F}(X) + k\,\mathcal{A}(X),$$

on surfaces $X \in H^{1,2}(B, \mathbb{R}^3)$ of the type of the disc $B := \mathring{\mathbb{D}}^2 \subset \mathbb{R}^2$. The Lagrangian F is assumed to satisfy the following requirements:

$$F \in C^0(\mathbb{R}^3) \cap C^1(\mathbb{R}^3 \setminus \{0\}), \tag{1.1}$$

$$F \text{ is convex}, \tag{1.2}$$

$$F(tz) = t\,F(z) \qquad \forall\, t \geq 0, \quad \forall\, z \in \mathbb{R}^3, \tag{1.3}$$

$$m_1 \mid z \mid \leq F(z) \leq m_2 \mid z \mid \qquad \forall\, z \in \mathbb{R}^3, \quad 0 < m_1 \leq m_2. \tag{1.4}$$

Moreover we assume that

$$k \overset{!}{>} \max_{\mathbb{S}^2} F = m_2. \tag{1.5}$$

Thus $\mathcal{J}$ is a controlled perturbation of the area functional $\mathcal{A}$, where F depends only on the normal $X_u \wedge X_v$, but not on the position vector X itself. Moreover with respect to some closed rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ we consider the Plateau class $\mathcal{C}^*(\Gamma)$ of surfaces $X \in H^{1,2}(B, \mathbb{R}^n)$ whose L^2-traces $X \mid_{\partial B}$ are continuous, monotonic mappings of $\mathbb{S}^1$ onto Γ satisfying a three-point-condition:

$$X \mid_{\partial B} (e^{i\psi_k}) \overset{!}{=} P_k, \quad \psi_k := \frac{2\pi k}{3}, \; k = 0, 1, 2, \tag{1.6}$$

where P_0, P_1, P_2 are three fixed points on Γ. Furthermore we topologize $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ by the $C^0(\bar{B}, \mathbb{R}^3)$-norm. We are going to prove (see Def. **??** and **??** in Subsection **??** and Def. 3.5 in [**?**])

Theorem 1.1 (Main result) *Let Γ be an arbitrary closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying a chord-arc condition (**??**). If there exist two different conformally parametrized surfaces $X_1 \neq X_2$ in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \| \cdot \|_{C^0(\bar{B})})$ that are in a mountain pass situation with respect to $\mathcal{J}$, then there exists an unstable $\mathcal{J}$-extremal surface X^* in $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$.*

Following Shiffman we replace $\mathcal{J}$ by its *dominance* functional

$$\mathcal{I}(X) := \int_B F(X_u \wedge X_v) + \frac{k}{2} \mid DX \mid^2 dudv = \mathcal{F}(X) + k\, \mathcal{D}(X).$$

Now the crucial tools which allow a generalization of the *mountain-pass* result in [**?**] to the above theorem are the compactness result Theorem **??** for minimizers of $\mathcal{I}$ within boundary value classes $H^{1,2}_{\varphi}(B, \mathbb{R}^3)$, termed $\mathcal{I}$-surfaces, which is derived from a fundamental "quasi maximum principle" for $\mathcal{I}$-surfaces, Theorem **??**, the closedness of the set of $\mathcal{I}$-surfaces with respect to $C^0(\bar{B}, \mathbb{R}^3)$-convergence, Theorem **??**, and a "continuity theorem" for $\mathcal{I}$ applied to conformally parametrized $\mathcal{I}$-surfaces, Corollary **??**, which is achieved by the "continuity theorem" for $\mathcal{A}$ applied to harmonic surfaces on ring domains due to Morse and Tompkins [**?**]. Shiffman realized the importance of these tools in [**?**] but he only outlined incomplete proofs. Possessing these results one is able to follow the lines of Heinz' paper [**?**] in which Heinz tackled the analogous problem for the H-surface functional instead of $\mathcal{J}$ resp. $\mathcal{I}$ successfully by approximating Γ by a sequence of closed polygons and applying his results of [**?**] and the "finite dimensional" mountain pass lemma.

2 A quasi maximum principle and a compactness result for $\mathcal{I}$-surfaces

In this chapter we prove a "*quasi maximum principle*", Theorem **??**, for the unique minimizers of $\mathcal{I}$ within boundary value classes $H^{1,2}_{\varphi}(B,\mathbb{R}^3)$, which we term $\mathcal{I}$-surfaces (see Def. 2.1 and Theorem 4.3 in [**?**]), and derive a fundamental compactness result, Theorem **??**, for sequences of those surfaces. Shiffman claimed these results in Sections 6 and 7 of [**?**] but his proof of Theorem **??** is incomplete. In footnote 7 on p. 552 in [**?**] Shiffman gave an incorrect proof of the following fundamental lemma which turned out to be a rather involved topological question.

Lemma 2.1 *The restriction $g\,|_{\mathbb{S}^2}$ of an <u>even</u> function $g \in C^1(B^3_{1+\delta}(0)\setminus B^3_{1-\delta}(0))$, $\delta \in (0,\frac{1}{8})$, possesses at least three linearly independent critical points, i.e. there are at least three linear independent unit vectors $a_1,a_2,a_3 \in \mathbb{S}^2$ at which $\nabla g(a_j) = r_j\, a_j^{\top}$, for some $r_j \in \mathbb{R}$, $j=1,2,3$.*

In order to combine this result with the method of "levelling" real valued functions on $\bar{B}$ used by Shiffman in Section 6 of [**?**] and by McShane in [**?**] we need

Definition 2.1 *Let $f \in C^0(\bar{B})$ and $G \subseteq B$ be an open subset of B. We set*

$$\mathrm{m}_G(f) := \max\{\max_{\bar{G}} f - \max_{\partial G} f, \min_{\partial G} f - \min_{\bar{G}} f\} \tag{2.1}$$

and call $\mathrm{md}(f) := \sup_{G\subseteq B}\mathrm{m}_G(f)$ the monotonic diefficiency of f, where the supremum is taken over all open subsets $G \subseteq B$.

Now let F be a fixed integrand and $g(x) := F(x) + F(-x)$. By Lemma **??** the function g gives rise to a matrix $A := (a_1,a_2,a_3)^{\top} \in GL_3(\mathbb{R})$, having chosen three linearly independent critical points a_1,a_2,a_3 of $g\,|_{\mathbb{S}^2}$ arbitrarily. Now we can state the two results of this chapter (see Lemma 2.2 and Theorems 4.3 and 5.2 in [**?**]).

Theorem 2.1 *Let $\varphi \in C^0(\partial B,\mathbb{R}^3)\cap H^{\frac{1}{2},2}(\partial B,\mathbb{R}^3)$ be prescribed boundary values. Then the corresponding $\mathcal{I}$-surface $X^* \in H^{1,2}_{\varphi}(B,\mathbb{R}^3)\cap C^0(\bar{B},\mathbb{R}^3)$, i.e. the unique minimizer of $\mathcal{I}$ in $H^{1,2}_{\varphi}(B,\mathbb{R}^3)$, satisfies $\mathrm{md}((A\,X^*)_i) = 0$ for $i=1,2,3$.*

Theorem 2.2 *Let $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces with $\mathcal{D}(X^n) \le const.$, $\forall\, n \in \mathbb{N}$, and with equicontinuous and uniformly bounded boundary values. Then there exists a subsequence $\{X^{n_j}\}$ such that*

$$X^{n_j} \longrightarrow \bar{X} \quad \text{in } C^0(\bar{B},\mathbb{R}^3) \quad \text{and} \quad X^{n_j} \rightharpoonup \bar{X} \quad \text{in } H^{1,2}(B,\mathbb{R}^3), \tag{2.2}$$

for a surface $\bar{X} \in H^{1,2}(B,\mathbb{R}^3)\cap C^0(\bar{B},\mathbb{R}^3)$ with $\mathrm{md}((A\,\bar{X})_i) = 0$, $i=1,2,3$.

2.1 Proof of Lemma ??

Firstly we need

Definition 2.2 *Let Z be a C^∞-vector field on a smooth manifold and* Sing(Z) *its set of singularities. A compact subset P of* Sing(Z) *which can be seperated from* $\mathrm{Sing(Z)} \setminus P$ *by some open neighborhood U, i.e. $P = \bar{U} \cap \mathrm{Sing(Z)}$, will be termed a part of* Sing(Z).

Definition 2.3 *Let Z be a C^∞-vector field on a 2-dimensional manifold and P a part of $\mathrm{Sing}(Z)$ which is contained in a chart (V, h), i.e. $h : V \overset{\cong}{\to} B_r^2(0)$, $r > 0$, and which possesses an open seperating neighborhood $U \subset\subset V$ with a smooth boundary and such that $\bar{U} \cong \mathbb{D}^2$. We set $\tilde{U} := h(\bar{U})$, $\tilde{P} := h(P)$. Then we define the index of Z around P by*

$$\mathrm{Ind}(Z, P) := \mathrm{Ind}(h_*(Z), \tilde{P}) := \deg\Big(\frac{h_*(Z) \mid_{\partial\tilde{U}}}{\mid h_*(Z) \mid_{\partial\tilde{U}}\mid}\Big),$$

thus $2\pi\ Ind(Z, P) = \int_{\partial\tilde{U}} \big(\frac{h_*(Z)}{|h_*(Z)|} \mid_{\partial\tilde{U}}\big)^*(\omega_{\mathbb{S}^1})$, where $\omega_{\mathbb{S}^1}$ denotes the volume form $y_1\, dy_2 - y_2\, dy_1$ of $\mathbb{S}^1$ and $h_*(Z)(y) := Dh_{h^{-1}(y)}(Z(h^{-1}(y)))$.

Remark 2.1 *For our further argumentation we have to ensure that the above notion of degree, the "de Rham-degree", coincides with its counterpart in singular homology with real and integral coefficients. This can easily be carried out by using the naturality of the de Rham isomorphism*

$$R^* : H^*_{dR}(M) \overset{\cong}{\longrightarrow} H^*(M, \mathbb{R}) \cong \mathrm{Hom}(H_*(M, \mathbb{R}), \mathbb{R})$$

for a smooth, closed, orientable and connected manifold M, whose definition implies in particular

$$\langle R^n([\omega]), [M]\rangle = \int_M \omega \qquad \forall\, \omega \in \Omega^n(M),$$

where $[M]$ denotes the fundamental class of M and $\langle\cdot\,,\cdot\rangle$ the Kronecker product, and the naturality of the isomorphism

$$H_*(M, \mathbb{Z}) \otimes \mathbb{R} \overset{\cong}{\longrightarrow} H_*(M, \mathbb{R}), \qquad \text{given by} \quad [\alpha] \otimes r \longmapsto [\alpha \otimes r],$$

*derived from the universal coefficient theorem (see [**?**], pp. 263, 264). Applying this to the map $\tilde{Z} \mid_{\partial\tilde{U}} := \frac{h_*(Z)}{|h_*(Z)|} \mid_{\partial\tilde{U}} : \partial\tilde{U} \longrightarrow \mathbb{S}^1$ and using $R^1 : [\omega_{\mathbb{S}^1}] \mapsto 2\pi\, [\mathbb{S}^1]^*$ we obtain:*

$$2\pi \deg_{dR}(\tilde{Z} \mid_{\partial\tilde{U}}) = \int_{\partial\tilde{U}} (\tilde{Z} \mid_{\partial\tilde{U}})^*(\omega_{\mathbb{S}^1}) = \langle R^1([(\tilde{Z} \mid_{\partial\tilde{U}})^*(\omega_{\mathbb{S}^1})]), [\partial\tilde{U}]\rangle$$
$$= \langle (\tilde{Z} \mid_{\partial\tilde{U}})^*(R^1([\omega_{\mathbb{S}^1}])), [\partial\tilde{U}]\rangle = \langle 2\pi\, (\tilde{Z} \mid_{\partial\tilde{U}})^*([\mathbb{S}^1]^*), [\partial\tilde{U}]\rangle = 2\pi\, \langle [\mathbb{S}^1]^*, (\tilde{Z} \mid_{\partial\tilde{U}})_*([\partial\tilde{U}])\rangle$$
$$= 2\pi\, \langle [\mathbb{S}^1]^*, \deg_{Sing_{\mathbb{R}}}(\tilde{Z} \mid_{\partial\tilde{U}})\, [\mathbb{S}^1]\rangle = 2\pi \deg_{Sing_{\mathbb{R}}}(\tilde{Z} \mid_{\partial\tilde{U}}) = 2\pi \deg_{Sing_{\mathbb{Z}}}(\tilde{Z} \mid_{\partial\tilde{U}}).$$

Now we have to verify the independence of Def. **??** from the choice of the chart (V,h) and of U. This can be done by the use of the properties of the fixed point index $I(\phi(\,\cdot\,,t)\,|_U)$ (as defined in [**?**], p. 205) of the flow

$$\phi(x,t) := \pi(x - t\,Z(x)) \tag{2.3}$$

on $\mathbb{S}^2$, where we restrict $t \in [0,t_0]$ for some sufficiently small t_0 such that the orthogonal projection π from $B^3_2(0)\backslash B^3_\delta(0)$, $\delta \in (0,\frac{1}{8})$, onto $\mathbb{S}^2$ can be applied to the points $x - t\,Z(x)$. Our assertion follows immediately from

Proposition 2.1 *Let $P \subset U \subset\subset V$ and h be as in Def.* **??**. *Then we have*

$$\mathrm{Ind}(Z,P) = I(\phi(\,\cdot\,,t)\,|_U) \tag{2.4}$$

for any choice of $t \in (0,t_0]$, U and the chart (V,h).

Proof: We choose some $t_0 > 0$ sufficiently small such that $\phi(\,\cdot\,,t) : \bar{U} \longrightarrow V\ \forall\, t \in [0,t_0]$, abbreviate $\phi := \phi(\,\cdot\,,t)$ for some fixed $t \in (0,t_0]$ and set $\tilde{\phi}\,|_{\tilde{U}} := h\circ\phi\circ h^{-1}\,|_{\tilde{U}} \colon \tilde{U} \longrightarrow B^2_r(0)$ for $\tilde{U} := h(\bar{U})$. From the commutativity of the fixed point index (see [**?**], p. 206) we infer immediately:

$$I(\tilde{\phi}\,|_{\overset{\circ}{\tilde{U}}}) = I(\phi\circ h^{-1}\circ h\,|_U) = I(\phi\,|_U). \tag{2.5}$$

Furthermore we have the following commutative diagram, where we use singular homology with integral coefficients:

$$\begin{array}{ccccccc}
H_2(\tilde{U},\tilde{U}\setminus\tilde{P}) & \xleftarrow{i_*} & H_2(\tilde{U},\partial\tilde{U}) & \xrightarrow{\partial_*} & H_1(\partial\tilde{U}) & \xrightarrow{=} & H_1(\partial\tilde{U}) \\
((id-\tilde{\phi})\,|_{\tilde{U}})_*\downarrow & & ((id-\tilde{\phi})\,|_{\tilde{U}})_*\downarrow & & \downarrow((id-\tilde{\phi})\,|_{\partial\tilde{U}})_* & & \downarrow\left(\frac{(id-\tilde{\phi})|_{\partial\tilde{U}}}{|(id-\tilde{\phi})|_{\partial\tilde{U}}|}\right)_* \\
H_2(\mathbb{R}^2,\mathbb{R}^2\setminus\{0\}) & \xrightarrow{=} & H_2(\mathbb{R}^2,\mathbb{R}^2\setminus\{0\}) & \xrightarrow{\partial_*} & H_1(\mathbb{R}^2\setminus\{0\}) & \xrightarrow{\cong} & H_1(\mathbb{S}^1).
\end{array}$$

Due to $\tilde{U} \cong \mathbb{D}^2$ the exact homology sequence of the pair $(\tilde{U},\partial\tilde{U})$ yields

$$\partial_* : H_2(\tilde{U},\partial\tilde{U}) \xrightarrow{\cong} H_1(\partial\tilde{U}) \cong \mathbb{Z},$$

thus the isomorphism ∂_* takes a generator o of $H_2(\tilde{U},\partial\tilde{U})$ into a fundamental class $[\partial\tilde{U}]$ of $\partial\tilde{U}$. Now let $o_{\tilde{P}} \in H_2(\tilde{U},\tilde{U}\setminus\tilde{P})$ denote a fundamental class around $\tilde{P}$ and o_0 a generator of $H_2(\mathbb{R}^2,\mathbb{R}^2\setminus\{0\})$. From [**?**], pp. 269–270, we infer that i_* maps o into $o_{\tilde{P}}$. Hence, chasing the diagram along the lower way we obtain by the excision-isomorphism $H_2(\tilde{U},\tilde{U}\setminus\tilde{P}) \cong H_2(\overset{\circ}{\tilde{U}},\overset{\circ}{\tilde{U}}\setminus\tilde{P})$ and by the definition of the fixed point index (see [**?**], S. 203):

$$o \mapsto o_{\tilde{P}} \mapsto I(\tilde{\phi}\,|_{\overset{\circ}{\tilde{U}}})\,o_0 \mapsto I(\tilde{\phi}\,|_{\overset{\circ}{\tilde{U}}})\,[\mathbb{S}^1].$$

On the other hand following the upper way we obtain by the definition of the degree in singular homology:

$$o \mapsto [\partial\tilde{U}] \mapsto deg\Big(\frac{id_{\partial\tilde{U}} - \tilde{\phi}\,|_{\partial\tilde{U}}}{|\,id_{\partial\tilde{U}} - \tilde{\phi}\,|_{\partial\tilde{U}}|}\Big)[\mathbb{S}^1].$$

Hence, by the commutativity of the diagram we proved

$$I(\tilde{\phi}\mid_{\mathring{\tilde{U}}}) = deg\Big(\frac{id_{\partial\tilde{U}} - \tilde{\phi}\mid_{\partial\tilde{U}}}{\mid id_{\partial\tilde{U}} - \tilde{\phi}\mid_{\partial\tilde{U}}\mid}\Big). \tag{2.6}$$

Moreover we calculate:

$$\frac{d}{dt}\tilde{\phi}(y,t) = \frac{d}{dt}(h\circ\phi(h^{-1}(y),t)) = Dh_{\phi(h^{-1}(y),t)}\Big(\frac{d}{dt}\phi(h^{-1}(y),t)\Big).$$

Together with $\frac{d}{dt}\phi(x,t) = D\pi_{(x-t\,Z(x))}(-Z(x))$ the evaluation in $t=0$ yields:

$$\begin{aligned}\frac{d}{dt}\tilde{\phi}(y,t)\mid_{t=0} = Dh_{h^{-1}(y)}(D\pi_{h^{-1}(y)}(-Z(h^{-1}(y)))) &= Dh_{h^{-1}(y)}(-Z(h^{-1}(y))) \\ &= -h_*(Z)(y),\end{aligned}$$

$\forall\, y\in\tilde{U}$. We insert this into the taylor expansion of $\tilde{\phi}(y,t)$ w. r. to t:

$$\tilde{\phi}(y,t) = y - t\,h_*(Z)(y) + t^2 r(y,t),$$

where the remainder $r(y,t)$ depends smoothly on $y\in\tilde{U}$ and $t\in(0,t_0]$ (see [?], p. 135). Hence we obtain for $y\in\partial\tilde{U}$ and $t\in(0,t_0]$:

$$\frac{y-\tilde{\phi}(y,t)}{\mid y-\tilde{\phi}(y,t)\mid} = \frac{h_*(Z)(y) - t\,r(y,t)}{\mid h_*(Z)(y) - t\,r(y,t)\mid}.$$

Now combining this with (??), (??), the homotopy invariance of the degree and Def. ?? we conclude:

$$\begin{aligned}I(\phi\mid_U) = I(\tilde{\phi}\mid_{\mathring{\tilde{U}}}) = deg\Big(\frac{id_{\partial\tilde{U}} - \tilde{\phi}\mid_{\partial\tilde{U}}}{\mid id_{\partial\tilde{U}} - \tilde{\phi}\mid_{\partial\tilde{U}}\mid}\Big) &= deg\Big(\frac{(h_*(Z)(\cdot) - t\,r(\cdot,t))\mid_{\partial\tilde{U}}}{\mid (h_*(Z)(\cdot) - t\,r(\cdot,t))\mid_{\partial\tilde{U}}\mid}\Big) \\ &= deg\Big(\frac{h_*(Z)\mid_{\partial\tilde{U}}}{\mid h_*(Z)\mid_{\partial\tilde{U}}\mid}\Big) = Ind(Z,P).\end{aligned}$$

The homotopy invariance of the fixed point index and its independence of the choice of the seperating neighborhood U of P (see [?], p. 206) finally proves the assertion.

◇

Now we consider the "right", "left","upper" and "lower" closed hemispheres

$$\mathbb{S}^2_{r(l)} := \{x\in\mathbb{S}^2 \mid x_1 \geq (\leq)\, 0\} \quad \text{and} \quad \mathbb{S}^2_{+(-)} := \{x\in\mathbb{S}^2 \mid x_3 \geq (\leq)\, 0\}.$$

Moreover we construct charts $h^{r,l} : H^{r,l} \xrightarrow{\cong} B^2_{1+\rho}(0)$, $h^- : H^- \xrightarrow{\cong} B^2_{1+\rho}(0)$, for some $\rho > 0$, using the stereographic projection, projecting from the points $(-1,0,0)$, $(1,0,0)$ and $(0,0,1)$ respectively, where we have set

$$H^r := \{x\in\mathbb{S}^2 \mid x_1 > -\delta\}, \quad H^l := \{x\in\mathbb{S}^2 \mid x_1 < \delta\}, \quad H^- := \{x\in\mathbb{S}^2 \mid x_3 < \delta\}, \tag{2.7}$$

for a $\delta > 0$ depending on ρ. Explicitly we set

$$h^r(x_1, x_2, x_3) := \Big(\frac{-x_2}{1+x_1}, \frac{x_3}{1+x_1}\Big), \qquad h^l(x_1, x_2, x_3) := \Big(\frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}\Big)$$
$$h^-(x_1, x_2, x_3) := \Big(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\Big). \tag{2.8}$$

Moreover one easily verifies that, for example,

$$(h^l)^{-1}(y) = \frac{1}{\mid y \mid^2 +1}(\mid y \mid^2 -1, 2y_1, 2y_2) \qquad \text{for } y = (y_1, y_2) \in B^2_{1+\rho}(0),$$

which yields for the transition function $\psi^l := h^- \circ (h^l)^{-1} : h^l(H^l \cap H^-) \xrightarrow{\cong} h^-(H^l \cap H^-)$ by an easy calculation:

$$\psi^l(y) = \frac{1}{1+ \mid y \mid^2 -2y_2}(\mid y \mid^2 -1, 2y_1) \qquad \text{and} \qquad \det D\psi^l(y) = \frac{4}{(1+ \mid y \mid^2 -2y_2)^2} > 0$$

$\forall\, y \in h^l(H^l \cap H^-)$. Hence, ψ^l yields an orientation preserving change of coordinates and analogously $\psi^r := h^- \circ (h^r)^{-1}$. We note that the fraction $\frac{1}{1+|y|^2-2y_2}$ has its only singularity in the point $(0,1)$ which is mapped by $(h^l)^{-1}$ to the point of projection $(0,0,1)$ of h^-, where h^- is not defined. Now we consider a smooth Jordan curve

$$\gamma : [0,\pi] \longrightarrow \mathbb{S}^2_r \cap \mathbb{S}^2_+ \cap H^- \qquad \text{satisfying} \tag{2.9}$$
$$\gamma(0) = (0,-1,0) \quad \text{and} \quad \gamma(\pi) = (0,1,0),$$

such that the closure of this curve by the reflection R^{x_1} at the (x_2, x_3)-plane, i.e. $\gamma(t) := R^{x_1}(\gamma(2\pi - t))$ for $t \in [\pi, 2\pi]$, is smooth on $[0, 2\pi]$. Moreover we consider its reflection by R^{x_3} at the (x_1, x_2)-plane, i.e. $\bar{\gamma} := R^{x_3}(\gamma)$ on $[0, 2\pi]$, and set

$$\tilde{\gamma}^r := h^r \circ \gamma \mid_{[0,\pi]}, \quad \tilde{\gamma}^l := h^l \circ \gamma \mid_{[\pi,2\pi]}, \quad \tilde{\bar{\gamma}}^r := h^r \circ \bar{\gamma} \mid_{[0,\pi]}, \quad \tilde{\bar{\gamma}}^l := h^l \circ \bar{\gamma} \mid_{[\pi,2\pi]}. \tag{2.10}$$

Furthermore we choose a smooth parametrization

$$\beta : [0,1] \xrightarrow{\cong} \mathbb{S}^2_- \cap \mathbb{S}^2_r \cap \mathbb{S}^2_l \quad \text{with} \quad \beta(0) = (0,1,0) \quad \text{and} \quad \beta(1) = (0,-1,0) \tag{2.11}$$

and term $u := h^r(\beta) = h^l(-\beta)$. Now we are able to state

Proposition 2.2 *For fixed curves γ and β, as described above, and any odd C^∞-vector field Z on $\mathbb{S}^2$, i.e. $Z(x) = -Z(-x) \in \mathbb{R}^3$, with $\mathrm{Sing}(Z) \cap \mathrm{trace}(\gamma \oplus (-\beta)) = \emptyset$ we prove:*

$$\int_{\tilde{\gamma}^r \oplus u} (\tilde{Z}^r \mid_{\tilde{\gamma}^r \oplus u})^*(\omega_{\mathbb{S}^1}) = -\int_{\tilde{\bar{\gamma}}^l \oplus u} (\tilde{Z}^l \mid_{\tilde{\bar{\gamma}}^l \oplus u})^*(\omega_{\mathbb{S}^1}), \tag{2.12}$$
$$\int_{\tilde{\bar{\gamma}}^r \oplus u} (\tilde{Z}^r \mid_{\tilde{\bar{\gamma}}^r \oplus u})^*(\omega_{\mathbb{S}^1}) = -\int_{\tilde{\gamma}^l \oplus u} (\tilde{Z}^l \mid_{\tilde{\gamma}^l \oplus u})^*(\omega_{\mathbb{S}^1}),$$

where we have set $\tilde{Z}^r := \frac{h^r_*(Z)}{|h^r_*(Z)|} : B^2_{1+\rho}(0) \setminus (\mathrm{Sing}(h^r_*(Z))) \longrightarrow \mathbb{S}^1$ and $\tilde{Z}^l$ analogously.

Proof: We have:

$$(\tilde{Z}^r)^*(\omega_{\mathbb{S}^1}) = (\tilde{Z}^r)^*(y_1\, dy_2 - y_2\, dy_1) = \tilde{Z}^r_1\, d\tilde{Z}^r_2 - \tilde{Z}^r_2\, d\tilde{Z}^r_1$$
$$= \begin{pmatrix} \tilde{Z}^r_1 \frac{\partial \tilde{Z}^r_2}{\partial y_1} - \tilde{Z}^r_2 \frac{\partial \tilde{Z}^r_1}{\partial y_1} \\ \tilde{Z}^r_1 \frac{\partial \tilde{Z}^r_2}{\partial y_2} - \tilde{Z}^r_2 \frac{\partial \tilde{Z}^r_1}{\partial y_2} \end{pmatrix} \cdot \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} =: \begin{pmatrix} W^r_1 \\ W^r_2 \end{pmatrix} \cdot \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix}.$$

Furthermore, by our choice of the charts h^r and h^l we see $(h^r)^{-1}(y) = -(h^l)^{-1}(\bar{y})$ $\forall\, y \in B^2_{1+\rho}(0)$. Thus, since Z is odd we obtain:

$$h^r_*(Z)(y) = Dh^r_{(h^r)^{-1}(y)}(Z((h^r)^{-1}(y))) = Dh^r_{(h^r)^{-1}(y)}(-Z((h^l)^{-1}(\bar{y})))$$
$$= \overline{Dh^l_{(h^l)^{-1}(\bar{y})}(Z((h^l)^{-1}(\bar{y})))} = \overline{h^l_*(Z)(\bar{y})},$$

$\forall\, y \in B^2_{1+\rho}(0)$, hence $\tilde{Z}^r(y) = \overline{\tilde{Z}^l(\bar{y})}$ $\forall\, y \in B^2_{1+\rho}(0) \setminus (Sing(h^r_*(Z)))$ and

$$D(\tilde{Z}^r)(y) = \begin{pmatrix} \nabla \tilde{Z}^l_1(\bar{y}) \\ -\nabla \tilde{Z}^l_2(\bar{y}) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{Z}^l_1}{\partial y_1} & -\frac{\partial \tilde{Z}^l_1}{\partial y_2} \\ -\frac{\partial \tilde{Z}^l_2}{\partial y_1} & \frac{\partial \tilde{Z}^l_2}{\partial y_2} \end{pmatrix} (\bar{y}).$$

Thus we arrive at:

$$W^r(y) = \begin{pmatrix} -\tilde{Z}^l_1 \frac{\partial \tilde{Z}^l_2}{\partial y_1} + \tilde{Z}^l_2 \frac{\partial \tilde{Z}^l_1}{\partial y_1} \\ \tilde{Z}^l_1 \frac{\partial \tilde{Z}^l_2}{\partial y_2} - \tilde{Z}^l_2 \frac{\partial \tilde{Z}^l_1}{\partial y_2} \end{pmatrix} (\bar{y}) =: \begin{pmatrix} -W^l_1(\bar{y}) \\ W^l_2(\bar{y}) \end{pmatrix},$$

$\forall\, y \in B^2_{1+\rho}(0) \setminus (Sing(h^r_*(Z)))$. Finally we note $\tilde{\bar{\gamma}}^l(t) = \overline{\tilde{\gamma}^r(t-\pi)}$, $\forall\, t \in [\pi, 2\pi]$, yielding

$$(\tilde{\bar{\gamma}}^l)'(t) = \left(\overline{\tilde{\gamma}^r}\right)'(t-\pi) = \overline{(\tilde{\gamma}^r)'(t-\pi)}, \qquad \forall\, t \in [\pi, 2\pi].$$

Hence, altogether we can calculate:

$$\int_{\tilde{\gamma}^r} (\tilde{Z}^r\,|_{\tilde{\gamma}^r})^*(\omega_{\mathbb{S}^1}) = \int_{\tilde{\gamma}^r} \langle W^r(y), dy\rangle = \int_0^\pi \langle W^r(\tilde{\gamma}^r), (\tilde{\gamma}^r)'\rangle\, dt$$
$$= \int_0^\pi -W^l_1(\overline{\tilde{\gamma}^r})\,(\tilde{\gamma}^r_1)' + W^l_2(\overline{\tilde{\gamma}^r})\,(\tilde{\gamma}^r_2)'\, dt = \int_\pi^{2\pi} -W^l_1(\tilde{\bar{\gamma}}^l)\,(\tilde{\bar{\gamma}}^l_1)' + W^l_2(\tilde{\bar{\gamma}}^l)\,(-\tilde{\bar{\gamma}}^l_2)'\, dt$$
$$= -\int_\pi^{2\pi} \langle W^l(\tilde{\bar{\gamma}}^l), (\tilde{\bar{\gamma}}^l)'\rangle\, dt = -\int_{\tilde{\bar{\gamma}}^l} \langle W^l(y), dy\rangle = -\int_{\tilde{\bar{\gamma}}^l} (\tilde{Z}^l\,|_{\tilde{\bar{\gamma}}^l})^*(\omega_{\mathbb{S}^1}).$$

Moreover we note that

$$(\tilde{Z}^r_1, \tilde{Z}^r_2) \equiv (-\tilde{Z}^l_1, \tilde{Z}^l_2) \qquad \text{on} \quad u,$$

implying $W^r \equiv -W^l$ on u, i.e.

$$\int_u (\tilde{Z}^r\,|_u)^*(\omega_{\mathbb{S}^1}) = -\int_u (\tilde{Z}^l\,|_u)^*(\omega_{\mathbb{S}^1}). \tag{2.13}$$

Hence, the first assertion in (**??**) is proved. The second equation in (**??**) follows analogously due to $\overline{\tilde{\gamma}^l(t)} = \tilde{\bar{\gamma}}^r(t-\pi)$, $\forall\, t \in [\pi, 2\pi]$, together with (**??**).

$\diamond$

Proof of Lemma **??**: We may suppose that g is not constant, otherwise we are done. Thus, as g is required to be even, a global maximizer and a global minimizer of g cannot coincide or be antipodal points, hence are linearly independent. Let G be the great circle which is determined by an antipodal pair of global maximizers and minimizers of g. We may assume $G = \mathbb{S}^1$. Now we assume that there does not exist any further critical point of $g\,|_{\mathbb{S}^2}$ on $\mathbb{S}^2 \setminus \mathbb{S}^1$. We mollify g by means of even Dirac kernels φ_ϵ:

$$g_\epsilon(\,\cdot\,) := \int_{B^3_{1+\delta}(0)\setminus B^3_{1-\delta}(0)} \varphi_\epsilon(\,\cdot\, - \bar{x})\, g(\bar{x})\, d\bar{x} \in C^\infty_c(\mathbb{R}^3 \setminus B^3_{1-2\delta}(0)),$$

for $\delta \in (0, \frac{1}{8})$ and $\epsilon \in (0, \delta)$. One verifies easily:

$$g_\epsilon \longrightarrow g \qquad in\ \ C^1(B^3_{1+\frac{\delta}{2}}(0) \setminus B^3_{1-\frac{\delta}{2}}(0)), \tag{2.14}$$

and that g_ϵ is even, just like g and φ_ϵ. Next we define the vector fields

$$a(x) := \nabla g(x) - \langle \nabla g(x), x\rangle\, x, \quad a_\epsilon(x) := \nabla g_\epsilon(x) - \langle \nabla g_\epsilon(x), x\rangle\, x,$$

and the flows

$$\phi(x,t) := \pi(x - t\, a(x)), \quad \phi_\epsilon(x,t) := \pi(x - t\, a_\epsilon(x))$$

on $\mathbb{S}^2$, for $\epsilon \in (0,\delta)$, where we restrict $t \in [0, t_0]$ for some sufficiently small t_0, as explained in (**??**). We note:

$$a_\epsilon \longrightarrow a \quad in \quad C^0(\mathbb{S}^2, \mathbb{R}^3), \tag{2.15}$$

$$\phi_\epsilon(\,\cdot\,, t) \longrightarrow \phi(\,\cdot\,, t) \quad in \quad C^0(\mathbb{S}^2, \mathbb{S}^2), \tag{2.16}$$

$$\textit{Hausdorff dist.}(\textit{Sing}(a_\epsilon), \textit{Sing}(a)) \longrightarrow 0, \tag{2.17}$$

$$\textit{Sing}(a_\epsilon) = \textit{Fix}(\phi_\epsilon(\,\cdot\,, t)) \quad \forall\, t \in (0, t_0],\ \forall\, \epsilon \in (0, \delta), \tag{2.18}$$

and that a_ϵ is an odd C^∞-vector field on $\mathbb{S}^2$. Since g is assumed to be not constant the mean value theorem (in integrated form) yields that $\textit{Sing}(a) \neq \mathbb{S}^1$. Thus, since $\textit{Sing}(a)$ ($\subset \mathbb{S}^1$ by hypothesis) is compact and symmetric, i.e. $\textit{Sing}(a) = -\textit{Sing}(a)$, there exists a point $x^* \in \mathbb{S}^1$ and a $\sigma > 0$ such that $\textit{Sing}(a) \subset \mathbb{S}^1 \setminus (B^3_\sigma(x^*) \cup B^3_\sigma(-x^*))$. We may assume that $x^* = (0,1,0)$. Hence, by property (**??**) of the family $\{a_\epsilon\}$ we can choose a smooth Jordan curve $\gamma : [0,\pi] \longrightarrow \mathbb{S}^2_r \cap \mathbb{S}^2_+ \cap H^-$, whose closure by the reflection R^{x_1} at the (x_2, x_3)-plane is smooth on $[0, 2\pi]$ (as below (**??**)) which we call γ again and which satisfies the following two requirements:

$$h^-(\textit{Sing}(a_\epsilon)) \subset\subset \overline{\textit{int}(h^-(\gamma))} =: K_1 \quad \forall\, \epsilon \in (0, \epsilon_0), \tag{2.19}$$

for ϵ_0 sufficiently small, where we infer $K_1 \cong \mathbb{D}^2$ from Schoenflies' theorem on account of trace$(\gamma) \cong \mathbb{S}^1$ (see [**?**], p. 68). Applying the reflection R^{x_3} at the (x_1, x_2)-plane to γ and setting $\bar{\gamma} := R^{x_3}(\gamma)$ we require secondly:

$$h^-(\textit{Sing}(a_\epsilon)) \cap \overline{\textit{int}(h^-(\bar{\gamma}))} = \emptyset \qquad \forall\, \epsilon \in (0, \epsilon_0), \tag{2.20}$$

for ϵ_0 sufficiently small. We term $K_2 := \overline{int(h^-(\bar\gamma))}$. Moreover we will use the curves β and u as defined in (??). Now we fix an $\epsilon \in (0,\epsilon_0)$, push a_ϵ forward by h^-, h^r and h^l and term $\tilde a^l_\epsilon := \frac{h^l_*(a_\epsilon)}{|h^l_*(a_\epsilon)|} : B^2_{1+\rho}(0) \setminus (Sing(h^l_*(a_\epsilon))) \longrightarrow \mathbb{S}^1$ and $\tilde a^r_\epsilon$, $\tilde a^-_\epsilon$ analogously (as in Prop. ??). Moreover we consider the orientation preserving transition maps $\psi^l := h^- \circ (h^l)^{-1} : h^l(H^l \cap H^-) \xrightarrow{\cong} h^-(H^l \cap H^-)$ and $\psi^r := h^- \circ (h^r)^{-1}$ as discussed in (??). Now we show that

$$\tilde a^-_\epsilon \circ \psi^l \simeq \tilde a^l_\epsilon \qquad \text{on } h^l((H^l \cap H^-) \setminus Sing(a_\epsilon)). \tag{2.21}$$

By (??) we know that $\det(D\psi^l(y))^{-1} > 0 \ \ \forall y \in h^l(H^l \cap H^-)$, thus

$$deg\Big(\frac{(D\psi^l(y))^{-1}}{|\,(D\psi^l(y))^{-1}\,|}\ |_{\mathbb{S}^1}\Big) = sign\ \det(D\psi^l(y))^{-1} = 1,$$

implying that $\frac{(D\psi^l(y))^{-1}}{|(D\psi^l(y))^{-1}|}\ |_{\mathbb{S}^1} \simeq\ id_{\mathbb{S}^1}\ \ \forall y \in h^l(H^l \cap H^-)$ by deforming the angle of $\frac{(D\psi^l(y))^{-1}}{|(D\psi^l(y))^{-1}|}\ |_{\mathbb{S}^1}$ "linearly" (see [?], p. 54). In order to state this homotopy explicitly we need the "argument function" $\arg(\,\cdot\,) := \exp(2\pi i\,\cdot)^{-1} : \mathbb{S}^1 \xrightarrow{\cong} [0,1]/(0 \sim 1)$. By [?], p. 53, any continuous $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ gives rise to a unique continuous function $\varphi : [0,1]/(0\sim 1) \longrightarrow \mathbb{R}$ such that $f(z) = f((1,0)) \cdot \exp(2\pi i\,\varphi(arg(z)))$ on $\mathbb{S}^1$, where "$\cdot$" denotes complex multiplication. Thus for every $y \in h^l(H^l \cap H^-)$ we obtain a unique function φ^l_y such that

$$\frac{(D\psi^l(y))^{-1}}{|\,(D\psi^l(y))^{-1}\,|}(z) = \frac{(D\psi^l(y))^{-1}}{|\,(D\psi^l(y))^{-1}\,|}((1,0))\ \cdot \exp(2\pi i\,\varphi^l_y(arg(z))), \tag{2.22}$$

$\forall z \in \mathbb{S}^1$. Now following [?], p. 54, we construct the homotopy

$$H^l_y(z,t) := A^l_y(t)\Big(\frac{(D\psi^l(y))^{-1}}{|\,(D\psi^l(y))^{-1}\,|}((1,0))\Big)\cdot \exp(2\pi i\,((1-t)\,\varphi^l_y(arg(z)) + t\,arg(z))), \tag{2.23}$$

from $\frac{(D\psi^l(y))^{-1}}{|(D\psi^l(y))^{-1}|}\ |_{\mathbb{S}^1}$ to $id_{\mathbb{S}^1}$, where $\{A^l_y(t)\}_{t\in[0,1]}$ is a family of rotations turning $\frac{(D\psi^l(y))^{-1}}{|(D\psi^l(y))^{-1}|}((1,0))$ into $(1,0)$ clockwise, given by

$$A^l_y(t) := \begin{pmatrix} \cos(-t\phi^l_y) & -\sin(-t\phi^l_y) \\ \sin(-t\phi^l_y) & \cos(-t\phi^l_y) \end{pmatrix},$$

where $\phi^l_y := 2\pi\ \arg\Big(\frac{(D\psi^l(y))^{-1}}{|(D\psi^l(y))^{-1}|}((1,0))\Big)$. Having noted that the family $\{H^l_y\}$ depends continuously on $y \in h^l(H^l \cap H^-)$ we insert $\tilde a^-_\epsilon \circ \psi^l$ into $H^l_y(\,\cdot\,,t)$ to obtain the homotopy $F^l : h^l((H^l \cap H^-) \setminus Sing(a_\epsilon)) \times [0,1] \longrightarrow \mathbb{S}^1$,

$$F^l(y,t) := H^l_y(\tilde a^-_\epsilon \circ \psi^l(y), t), \tag{2.24}$$

between $\tilde a^-_\epsilon \circ \psi^l$ and $\tilde a^l_\epsilon$, just as asserted in (??). Similarily one achieves a homotopy $F^r(\,\cdot\,,t) := H^r_{(\cdot)}(\tilde a^-_\epsilon \circ \psi^r(\,\cdot\,), t)$ between $\tilde a^-_\epsilon \circ \psi^r$ and $\tilde a^r_\epsilon$ on $h^r((H^r \cap H^-) \setminus Sing(a_\epsilon))$. Now on account of (**??**) and (**??**) we can choose a sequence of smooth closed curves $\{c_j\}$ in $h^l((H^l \cap H^-) \setminus \mathit{Sing}(a_\epsilon))$ that approximate $\tilde{\gamma}^l \oplus u$ at its two corners, as in Prop. 4 on p. 125 in [**?**], and gain by Prop. 9.26 and Corollary 10.14 in [**?**] that

$$\int_{c_j} (\tilde{a}_\epsilon^- \circ \psi^l \mid_{c_j})^*(\omega_{\mathbb{S}^1}) = \int_{c_j} (\tilde{a}_\epsilon^l \mid_{c_j})^*(\omega_{\mathbb{S}^1})$$

$\forall\, j \in \mathbb{N}$. Hence, letting $j \to \infty$ we gain by the proof of Prop. 4 on p. 125 in [**?**] in the limit:

$$\int_{\tilde{\gamma}^l \oplus u} (\tilde{a}_\epsilon^- \circ \psi^l \mid_{\tilde{\gamma}^l \oplus u})^*(\omega_{\mathbb{S}^1}) = \int_{\tilde{\gamma}^l \oplus u} (\tilde{a}_\epsilon^l \mid_{\tilde{\gamma}^l \oplus u})^*(\omega_{\mathbb{S}^1}) \tag{2.25}$$

and analogously

$$\int_{\tilde{\gamma}^r \oplus u} (\tilde{a}_\epsilon^- \circ \psi^r \mid_{\tilde{\gamma}^r \oplus u})^*(\omega_{\mathbb{S}^1}) = \int_{\tilde{\gamma}^r \oplus u} (\tilde{a}_\epsilon^r \mid_{\tilde{\gamma}^r \oplus u})^*(\omega_{\mathbb{S}^1}), \tag{2.26}$$

where we used the notation of (**??**). Now we set $B^r := \{y \in B^2_{1+\rho}(0) \mid y_1 \geq 0\}$ and $B^l := \{y \in B^2_{1+\rho}(0) \mid y_1 \leq 0\}$. Since $\psi^r \mid_{\tilde{\gamma}^r \oplus u} \colon \tilde{\gamma}^r \oplus u \xrightarrow{\cong} \partial(K_1 \cap B^r)$ and $\psi^l \mid_{\tilde{\gamma}^l \oplus u} \colon \tilde{\gamma}^l \oplus u \xrightarrow{\cong} \partial(K_1 \cap B^l)$ are orientation preserving diffeomorphisms we infer from the transformation theorem on p. 168 in [**?**] together with the use of two sequences of smooth closed curves approximating $\tilde{\gamma}^r \oplus u$ resp. $\tilde{\gamma}^l \oplus u$ at their two corners as above:

$$\int_{\partial(K_1 \cap B^r)} (\tilde{a}_\epsilon^- \mid_{\partial(K_1 \cap B^r)})^*(\omega_{\mathbb{S}^1}) + \int_{\partial(K_1 \cap B^l)} (\tilde{a}_\epsilon^- \mid_{\partial(K_1 \cap B^l)})^*(\omega_{\mathbb{S}^1})$$
$$= \int_{\tilde{\gamma}^r \oplus u} (\psi^r)^*(\tilde{a}_\epsilon^- \mid_{\partial(K_1 \cap B^r)})^*(\omega_{\mathbb{S}^1}) + \int_{\tilde{\gamma}^l \oplus u} (\psi^l)^*(\tilde{a}_\epsilon^- \mid_{\partial(K_1 \cap B^l)})^*(\omega_{\mathbb{S}^1}).$$

Hence, together with (**??**) and (**??**) we arrive at

$$\int_{\partial(K_1 \cap B^r)} (\tilde{a}_\epsilon^- \mid_{\partial(K_1 \cap B^r)})^*(\omega_{\mathbb{S}^1}) + \int_{\partial(K_1 \cap B^l)} (\tilde{a}_\epsilon^- \mid_{\partial(K_1 \cap B^l)})^*(\omega_{\mathbb{S}^1})$$
$$= \int_{\tilde{\gamma}^r \oplus u} (\tilde{a}_\epsilon^r \mid_{\tilde{\gamma}^r \oplus u})^*(\omega_{\mathbb{S}^1}) + \int_{\tilde{\gamma}^l \oplus u} (\tilde{a}_\epsilon^l \mid_{\tilde{\gamma}^l \oplus u})^*(\omega_{\mathbb{S}^1}). \tag{2.27}$$

Since $\psi^r \mid_{\tilde{\tilde{\gamma}}^r \oplus u} \colon \tilde{\tilde{\gamma}}^r \oplus u \xrightarrow{\cong} \partial(K_2 \cap B^r)$ and $\psi^l \mid_{\tilde{\tilde{\gamma}}^l \oplus u} \colon \tilde{\tilde{\gamma}}^l \oplus u \xrightarrow{\cong} \partial(K_2 \cap B^l)$ (see again (**??**)) are also orientation preserving diffeomorphisms we obtain analogously due to (**??**):

$$\int_{\partial(K_2 \cap B^r)} (\tilde{a}_\epsilon^- \mid_{\partial(K_2 \cap B^r)})^*(\omega_{\mathbb{S}^1}) + \int_{\partial(K_2 \cap B^l)} (\tilde{a}_\epsilon^- \mid_{\partial(K_2 \cap B^l)})^*(\omega_{\mathbb{S}^1})$$
$$= \int_{\tilde{\tilde{\gamma}}^r \oplus u} (\tilde{a}_\epsilon^r \mid_{\tilde{\tilde{\gamma}}^r \oplus u})^*(\omega_{\mathbb{S}^1}) + \int_{\tilde{\tilde{\gamma}}^l \oplus u} (\tilde{a}_\epsilon^l \mid_{\tilde{\tilde{\gamma}}^l \oplus u})^*(\omega_{\mathbb{S}^1}). \tag{2.28}$$

Now we split up $\partial K_1 = \partial(K_1 \cap B^r) \oplus \partial(K_1 \cap B^l)$, combine (**??**) with (**??**) via Prop. **??** and apply Stokes' theorem (p. 183 in [**?**]):

$$
\begin{aligned}
2\pi\, deg(\tilde{a}_\epsilon^- \mid_{\partial K_1}) &= \int_{\partial K_1} (\tilde{a}_\epsilon^- \mid_{\partial K_1})^*(\omega_{\mathbb{S}^1}) \\
= \int_{\partial(K_1\cap B^r)} (\tilde{a}_\epsilon^- \mid_{\partial(K_1\cap B^r)})^*(\omega_{\mathbb{S}^1}) &+ \int_{\partial(K_1\cap B^l)} (\tilde{a}_\epsilon^- \mid_{\partial(K_1\cap B^l)})^*(\omega_{\mathbb{S}^1}) \\
= \int_{\tilde{\gamma}^r\oplus u} (\tilde{a}_\epsilon^r \mid_{\tilde{\gamma}^r\oplus u})^*(\omega_{\mathbb{S}^1}) &+ \int_{\tilde{\gamma}^l\oplus u} (\tilde{a}_\epsilon^l \mid_{\tilde{\gamma}^l\oplus u})^*(\omega_{\mathbb{S}^1}) \\
= -\int_{\tilde{\tilde{\gamma}}^l\oplus u} (\tilde{a}_\epsilon^l \mid_{\tilde{\tilde{\gamma}}^l\oplus u})^*(\omega_{\mathbb{S}^1}) &- \int_{\tilde{\tilde{\gamma}}^r\oplus u} (\tilde{a}_\epsilon^r \mid_{\tilde{\tilde{\gamma}}^r\oplus u})^*(\omega_{\mathbb{S}^1}) \\
= -\int_{\partial(K_2\cap B^l)} (\tilde{a}_\epsilon^- \mid_{\partial(K_2\cap B^l)})^*(\omega_{\mathbb{S}^1}) &- \int_{\partial(K_2\cap B^r)} (\tilde{a}_\epsilon^- \mid_{\partial(K_2\cap B^r)})^*(\omega_{\mathbb{S}^1}) \\
= -\int_{K_2\cap B^l} (\tilde{a}_\epsilon^- \mid_{K_2\cap B^l})^*(d\omega_{\mathbb{S}^1}) &- \int_{K_2\cap B^r} (\tilde{a}_\epsilon^- \mid_{K_2\cap B^r})^*(d\omega_{\mathbb{S}^1}) = 0, \qquad (2.29)
\end{aligned}
$$

since $\tilde{a}_\epsilon^- \mid_{K_2\cap B^l}$ and $\tilde{a}_\epsilon^- \mid_{K_2\cap B^r}$ are well defined, i.e. smooth, due to (**??**) and since $K_2 \cap B^l$ resp. $K_2 \cap B^r$ are compactly contained in $h^-((H^l \cap H^-) \setminus Sing(a_\epsilon))$ resp. $h^-((H^r \cap H^-) \setminus Sing(a_\epsilon))$ which are the images under ψ^l resp. ψ^r of those sets, on which the homotopies F^l resp. F^r are defined, and due to $d\omega_{\mathbb{S}^1} \in \Omega^2(\mathbb{S}^1) = \{0\}$ (see also p. 189 in [**?**]). Finally it should be mentioned that in (**??**) one has to work again with two sequences of smooth closed curves approximating $\partial(K_2 \cap B^l)$ resp. $\partial(K_2 \cap B^r)$ at their two corners in order to apply Stokes' theorem correctly.
Furthermore we note that $H(x, s) := \pi(x - s\, ta_\epsilon(x))$, for $s \in [0, 1]$, yields a homotopy $\phi_\epsilon(\,\cdot\,, t) \simeq id_{\mathbb{S}^2}$, for any $t \in (0, t_0]$. Hence, the Lefschetz number Λ of $(\phi_\epsilon(\,\cdot\,, t))_*$ reduces to the Euler characteristic χ of $\mathbb{S}^2$, which amounts to 2. Now using that $h^-(Sing(a_\epsilon)) \subset\subset K_1$ by (**??**), $K_1 \cong \mathbb{D}^2$ and that K_1 has a smooth boundary we finally infer from (**??**), Def. **??**, Prop. **??**, the excision property of the fixed point index (see [**?**], p. 206) and Dold's fixed point theorem (see [**?**], p. 209, resp. p. 212):

$$
\begin{aligned}
0 = deg(\tilde{a}_\epsilon^- \mid_{\partial K_1}) = Ind(a_\epsilon, Sing(a_\epsilon)) = I(\phi_\epsilon(\,\cdot\,, t) \mid_{(h^-)^{-1}(\mathring{K}_1)}) &= I(\phi_\epsilon(\,\cdot\,, t)) \\
&= \Lambda((\phi_\epsilon(\,\cdot\,, t))_*) = \chi(\mathbb{S}^2) = 2,
\end{aligned}
$$

which is a contradiction, thus Lemma **??** is proved.

◇

2.2 Further preparing propositions

At first we shall make use of Lemma **??**. To this end let F be a fixed integrand (as in the introduction), $g(x) := F(x) + F(-x)$, a_1, a_2, a_3 three linearly independent critical points of $g \mid_{\mathbb{S}^2}$ and $A := (a_1, a_2, a_3)^\top \in GL_3(\mathbb{R})$. We choose two vectors b_1, c_1, such that $O_1 := (a_1, b_1, c_1)^\top \in SO(3)$ and set $F' := F \circ O_1^{-1}$, $g' := g \circ O_1^{-1}$. We prove

Lemma 2.2 *There are real constants c_y and c_z such that*

$$F'((x,y,z)) - F'((x,0,0)) \geq c_y\, y + c_z\, z \tag{2.30}$$

$\forall\, x, y, z \in \mathbb{R}$.

Proof: Since $O_1^{-1} \cdot (1,0,0)^\top = O_1^\top \cdot (1,0,0)^\top = a_1$ and since a_1 is a critical point of $g \mid_{\mathbb{S}^2}$ we calculate:

$$\nabla g'((1,0,0)^\top) = \nabla g(a_1) \cdot O_1^{-1} = r_1\, a_1^\top \cdot O_1^\top = r_1\, (O_1 \cdot a_1)^\top = r_1\, (1,0,0),$$

for some $r_1 \in \mathbb{R}$. Hence, $(1,0,0)^\top$ is a critical point of $g' \mid_{\mathbb{S}^2}$, implying in particular the equations:

$$\begin{aligned} 0 = g'_y((1,0,0)) &= F'_y((1,0,0)) - F'_y((-1,0,0)), \\ 0 = g'_z((1,0,0)) &= F'_z((1,0,0)) - F'_z((-1,0,0)), \end{aligned}$$

where we dropped the "$\top$". Now using that $\nabla F'$ is positively homogenous of degree 0 on $\mathbb{R}^3 \setminus \{0\}$ by (**??**) we obain:

$$F'_y \equiv \text{const.} =: c_y, \qquad F'_z \equiv \text{const.} =: c_z$$

on the x-axis except $\{0\}$. Furthermore we infer from the convexity of $F' \in C^1(\mathbb{R}^3 \setminus \{0\})$ (by (**??**) and (**??**)) for $x \neq 0$:

$$\begin{aligned} F'((x,y,z)) - F'((x,0,0)) &\geq \langle \nabla F'((x,0,0)), (x,y,z) - (x,0,0) \rangle \\ &= F'_y((x,0,0))\, y + F'_z((x,0,0))\, z = c_y\, y + c_z\, z, \end{aligned} \tag{2.31}$$

$\forall\, y, z \in \mathbb{R}$. Now letting $x \longrightarrow 0$ in (**??**) and using $F' \in C^0(\mathbb{R}^3)$ we achieve the assertion (**??**) also for $x = 0$.

$\diamond$

If we choose vectors b_2, c_2, and b_3, c_3, such that $O_2 := (b_2, a_2, c_2)^\top, O_3 := (b_3, c_3, a_3)^\top \in SO(3)$ and set $F'^2 := F \circ O_2^{-1}$, $F'^3 := F \circ O_3^{-1}$, then we obtain analogously:

$$F'^2((x,y,z)) - F'^2((0,y,0)) \geq \text{const.}\, x + \text{const.}\, z \tag{2.32}$$

and

$$F'^3((x,y,z)) - F'^3((0,0,z)) \geq \text{const.}\, x + \text{const.}\, y \tag{2.33}$$

$\forall\, x, y, z \in \mathbb{R}$. Next we need

Definition 2.4 *Let $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{\frac{1}{2},2}(\partial B, \mathbb{R}^3)$ be prescribed boundary values. Then we define*

$$M(\varphi) := \{\{Y^n\} \subset C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3) \mid Y_n \mid_{\partial B} \longrightarrow \varphi \quad \text{in} \quad C^0(\partial B, \mathbb{R}^3)\}$$

and

$$m(\varphi) := \inf_{\{Y^n\} \in M(\varphi)} \liminf_{n \to \infty} \mathcal{I}(Y^n). \tag{2.34}$$

Clearly one has $m(\varphi) \le \inf_{H^{1,2}_\varphi(B)\cap C^0(\bar{B})} \mathcal{I}$ and

Proposition 2.3 *There exists a minimizing element $\{X^j\}$ for $\mathcal{I}$ in $M(\varphi)$, i.e. $\{X^j\} \in M(\varphi)$ satisfies*

$$\lim_{j\to\infty} \mathcal{I}(X^j) = m(\varphi).$$

Proof: By the definition of $m(\varphi)$ we can choose a minimizing sequence $\{\{Y^n\}^j\}_{j\in\mathbb{N}}$ of sequences for $\mathcal{I}$ in $M(\varphi)$, i.e. we have $\{\{Y^n\}^j\}_{j\in\mathbb{N}} \subset M(\varphi)$ such that

$$\lim_{j\to\infty} \liminf_{n\to\infty} \mathcal{I}(\{Y^n\}^j) = m(\varphi).$$

We set $m_j := \liminf_{n\to\infty} \mathcal{I}(\{Y^n\}^j)$. For each $j \in \mathbb{N}$ we can choose an integer $n(j)$ such that

$$\mid \mathcal{I}(\{Y^{n(j)}\}^j) - m_j \mid < \frac{1}{j} \quad \text{and} \quad \| \{Y^{n(j)}\}^j \mid_{\partial B} - \varphi \|_{C^0(\partial B)} < \frac{1}{j}.$$

Now we choose $X^j := \{Y^{n(j)}\}^j \ \ \forall j \in \mathbb{N}$ and see that $\{X^j\}$ is an element of $M(\varphi)$ which satisfies indeed

$$\mid \mathcal{I}(X^j) - m(\varphi) \mid \le \mid \mathcal{I}(X^j) - m_j \mid + \mid m_j - m(\varphi) \mid \longrightarrow 0.$$

◇

Proposition 2.4 *For any $X \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ there is a mollified family $\{X_\epsilon\} \subset C_c^\infty(B_{1+2\delta}(0), \mathbb{R}^3)$, for $\epsilon \in (0,\delta)$ and some $\delta > 0$, that satisfies:*

$$X_\epsilon \longrightarrow X \qquad \text{in } C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3). \tag{2.35}$$

Proof: Due to the continuation theorem for Sobolev functions there is a continuation $\hat{X} \in H^{1,2}(B_{1+\delta}(0), \mathbb{R}^3)$ of X, for some $\delta > 0$. An examination of this continuation, explicitly given in [?], p. 256, shows that we also have $\hat{X} \in C^0(\overline{B_{1+\frac{\delta}{2}}(0)}, \mathbb{R}^3)$ on account of $X \in C^0(\bar{B}, \mathbb{R}^3)$. Now we use a family $\{\varphi_\epsilon\}$ of even Dirac kernels, with $supp(\varphi_\epsilon) = \overline{B_\epsilon(0)}$, to mollify $\hat{X}$:

$$X_\epsilon(\,\cdot\,) := \int_{B_{1+\delta}(0)} \varphi_\epsilon(\,\cdot\, - w)\, \hat{X}(w)\, dw \in C_c^\infty(B_{1+2\delta}(0), \mathbb{R}^3)$$

for $\epsilon \in (0,\delta)$. Due to $\hat{X} \in H^{1,2}(B_{1+\delta}(0), \mathbb{R}^3)$ we firstly obtain by [?], p. 108, that

$$\| X_\epsilon - X \|_{H^{1,2}(B)} = \| X_\epsilon - \hat{X} \mid_B \|_{H^{1,2}(B)} \longrightarrow 0 \qquad \text{for } \epsilon \searrow 0.$$

Moreover, due to $supp(\varphi_\epsilon) = \overline{B_\epsilon(0)}$ and $\int_{B_{1+\delta}(0)} \varphi_\epsilon(y-w)\, dw = 1$, $\forall y \in \bar{B}$, $\forall \epsilon \in (0,\delta)$, we gain:

$$\| X_\epsilon - X \|_{C^0(\bar{B})} = \| X_\epsilon - \hat{X} \mid_{\bar{B}} \|_{C^0(\bar{B})} = \max_{y\in\bar{B}} \mid \int_{B_{1+\delta}(0)} \varphi_\epsilon(y-w)\, \hat{X}(w)\, dw - \hat{X}(y) \mid$$

$$= \max_{y\in\bar{B}} \mid \int_{B_{1+\delta}(0)} \varphi_\epsilon(y-w)(\hat{X}(w) - \hat{X}(y))\, dw \mid \le \max_{y\in\bar{B}} \int_{B_\epsilon(y)} \varphi_\epsilon(y-w) \mid \hat{X}(w) - \hat{X}(y) \mid \, dw$$

$$\le \max_{y\in\bar{B}} \max_{w\in\overline{B_\epsilon(y)}} \mid \hat{X}(w) - \hat{X}(y) \mid \longrightarrow 0 \qquad \text{for } \epsilon \searrow 0,$$

since $\hat{X}$ is uniformly continuous on $\overline{B_{1+\frac{\delta}{2}}(0)}$, which completes the proof.

◇

Next we prove a proposition due to McShane in [?], p. 719 (see also [?], p. 416):

Proposition 2.5 *Let $\varphi \in C^0(\partial B)$ be prescribed boundary values and $\{f^n\}$ a sequence in $C^0(\bar{B}) \cap H^{1,2}(B)$ with the following properties:*

$$f^n \mid_{\partial B} \longrightarrow \varphi \qquad \text{in } C^0(\partial B), \tag{2.36}$$

$$\operatorname{md}(f^n) \longrightarrow 0 \qquad \text{for } n \to \infty, \tag{2.37}$$

$$\mathcal{D}(f^n) \leq \text{const.} \qquad \forall\, n \in \mathbb{N}. \tag{2.38}$$

Then there exists a subsequence $\{f^{n_j}\}$ and a function $f^* \in C^0(\bar{B}) \cap H^{1,2}(B)$ such that $\operatorname{md}(f^*) = 0$ *and*

$$f^{n_j} \longrightarrow f^* \qquad \text{in } C^0(\bar{B}).$$

Proof: Let $\{p_i\}_{i \in \mathbb{N}}$ be a countable, dense subset of $\bar{B}$. From (**??**) and (**??**) we infer $\| f^n \|_{C^0(\bar{B})} \leq$ const. $\forall\, n \in \mathbb{N}$. Thus noting that $\{p_i\}$ is countable we obtain the existence of a subsequence $\{f^{n_j}\}$ such that $\{f^{n_j}(p_i)\}_{j \in \mathbb{N}}$ is convergent $\forall\, i \in \mathbb{N}$. We rename $\{f^{n_j}\}$ into $\{f^n\}$ and assume that this sequence would not converge uniformly on $\bar{B}$. Hence, $\{f^n\}$ would not be a Cauchy sequence in $C^0(\bar{B})$, i.e. there exists some $\epsilon > 0$ such that for any $n \in \mathbb{N}$ there exists a pair $k_n > j_n > n$ and a point $q_n \in \bar{B}$ with

$$\mid f^{k_n}(q_n) - f^{j_n}(q_n) \mid > \epsilon. \tag{2.39}$$

Let $q^* \in \bar{B}$ denote an accumulation point of $\{q_n\}$. As $\varphi \in C^0(\partial B)$ we can choose an $\eta_0 > 0$ such that

$$\operatorname{osc}_{\partial B \cap B_{\eta_0}(w)} \varphi < \frac{\epsilon}{36} \qquad \forall\, w \in \partial B. \tag{2.40}$$

Now let $\eta \in (0, \eta_0)$ be arbitrarily fixed. Then we can choose some arbitrarily large $\bar{n} \in \mathbb{N}$ such that $q_{\bar{n}} \in B_\eta(q^*)$. Now let p_l be a point in $\{p_i\} \cap B_\eta(q^*)$. As $\{f^n(p_l)\}$ is convergent there exists some $N \in \mathbb{N}$ such that

$$\mid f^t(p_l) - f^s(p_l) \mid < \frac{\epsilon}{2} \qquad \forall\, t, s > N. \tag{2.41}$$

If we choose $\bar{n} > N$ then we infer from (**??**) and (**??**):

$$\begin{aligned} &\mid f^{k_{\bar{n}}}(q_{\bar{n}}) - f^{k_{\bar{n}}}(p_l) \mid + \mid f^{j_{\bar{n}}}(p_l) - f^{j_{\bar{n}}}(q_{\bar{n}}) \mid \\ \geq &\mid f^{k_{\bar{n}}}(q_{\bar{n}}) - f^{j_{\bar{n}}}(q_{\bar{n}}) \mid - \mid f^{k_{\bar{n}}}(p_l) - f^{j_{\bar{n}}}(p_l) \mid > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}, \end{aligned} \tag{2.42}$$

as $k_{\bar{n}} > j_{\bar{n}} > \bar{n} > N$. Hence, either $\mid f^{k_{\bar{n}}}(p_l) - f^{k_{\bar{n}}}(q_{\bar{n}}) \mid$ or $\mid f^{j_{\bar{n}}}(p_l) - f^{j_{\bar{n}}}(q_{\bar{n}}) \mid$ has to be greater than $\frac{\epsilon}{4}$, let's assume

$$\mid f^{k_{\bar{n}}}(p_l) - f^{k_{\bar{n}}}(q_{\bar{n}}) \mid > \frac{\epsilon}{4}. \tag{2.43}$$

By (**??**) and (**??**) there exists some $\tilde{N} \in \mathbb{N}$ such that

$$\| f^m |_{\partial B} - \varphi \|_{C^0(\partial B)} < \frac{\epsilon}{36} \qquad \text{and} \qquad \text{md}(f^m) < \frac{\epsilon}{16} \qquad \forall\, m > \tilde{N}. \tag{2.44}$$

Moreover we note that for any $f \in C^0(\bar{B})$ there holds

$$\text{osc}_{\bar{G}} f - \text{osc}_{\partial G} f \leq 2\, \text{m}_G(f) \leq 2\, \text{md}(f) \qquad \forall \text{ open subsets } G \subseteq B \tag{2.45}$$

by Def. **??**. Hence, if we choose $\bar{n} > \max\{N, \tilde{N}\}$, set $q := q_{\bar{n}}$ and $k := k_{\bar{n}}$ and apply (**??**) to f^k and $B_\eta(q^*) \cap B$ we obtain due to $p_l, q \in \overline{B_\eta(q^*) \cap B}$, (**??**) and (**??**):

$$\text{osc}_{\partial(B_\eta(q^*) \cap B)} f^k \geq | f^k(p_l) - f^k(q) | - 2\, \text{md}(f^k) > \frac{\epsilon}{4} - 2\frac{\epsilon}{16} = \frac{\epsilon}{8}. \tag{2.46}$$

Now we firstly assume that $B_\eta(q^*) \cap \partial B \neq \emptyset$. Then we obtain for any two points $w', w'' \in B_\eta(q^*) \cap \partial B$ by (**??**) and (**??**):

$$| f^k(w') - f^k(w'') | \leq | f^k(w') - \varphi(w') | + | \varphi(w') - \varphi(w'') | + | \varphi(w'') - f^k(w'') |$$
$$< 3\frac{\epsilon}{36} = \frac{\epsilon}{12},$$

as $k > \bar{n} > \tilde{N}$. Thus we conclude:

$$\text{osc}_{\overline{B_\eta(q^*)} \cap \partial B} f^k \leq \frac{\epsilon}{12}.$$

Together with $\text{osc}_{\partial(B_\eta(q^*) \cap B)} f^k \leq \text{osc}_{\overline{B_\eta(q^*)} \cap \partial B} f^k + \text{osc}_{\partial B_\eta(q^*) \cap \bar{B}} f^k$ and (**??**) we obtain the existence of two points $b_1, b_2 \in \partial B_\eta(q^*) \cap \bar{B}$ that satisfy:

$$| f^k(b_1) - f^k(b_2) | \geq \text{osc}_{\partial(B_\eta(q^*) \cap B)} f^k - \text{osc}_{\overline{B_\eta(q^*)} \cap \partial B} f^k > \frac{\epsilon}{8} - \frac{\epsilon}{12} = \frac{\epsilon}{24}, \tag{2.47}$$

for any $\eta \in (0, \eta_0)$. If on the other hand $B_\eta(q^*) \cap \partial B = \emptyset$ then we have $\partial B_\eta(q^*) \cap \bar{B} = \partial(B_\eta(q^*) \cap B)$ and the statement of (**??**) follows immediately from (**??**). Furthermore since $f^k \in H^{1,2}(B)$ and $\mathcal{D}(f^k) \leq \text{const.} =: M$, by hypothesis, we may apply the Courant-Lebesgue-lemma which yields the existence of some $\eta \in (\delta, \sqrt{\delta})$, for any $\delta < \eta_0^2$, such that

$$| f^k(b_1) - f^k(b_2) | \leq \mathcal{L}(f^k |_{\partial B_\eta(q^*) \cap \bar{B}}) \leq \sqrt{\frac{8\pi\, M}{\log(\frac{1}{\delta})}} \longrightarrow 0 \qquad \text{for } \delta \searrow 0,$$

($\mathcal{L}$:=lenght) which contradicts (**??**) (in both considered cases) as ϵ was assumed to be a fixed positive number. Hence, we proved the existence of a subsequence $\{f^{n_j}\}$ satisfying

$$f^{n_j} \longrightarrow f^* \qquad \text{in } C^0(\bar{B}) \tag{2.48}$$

for some $f^* \in C^0(\bar{B})$. If we combine this with $\mathcal{D}(f^n) \leq M \;\; \forall\, n \in \mathbb{N}$, we infer in particular $\| f^{n_j} \|_{H^{1,2}(B)} \leq$ const., thus the existence of some further subsequence $\{f^{n_k}\}$ with

$$f^{n_k} \rightharpoonup f^* \in H^{1,2}(B).$$

Finally we conclude immediately by (**??**) and (**??**):

$$0 \longleftarrow \text{md}(f^{n_j}) \geq \text{m}_G(f^{n_j}) \longrightarrow \text{m}_G(f^*) \qquad \text{for } j \to \infty,$$

i.e. $\text{m}_G(f^*) = 0$ for every open subset $G \subseteq B$, which means that $\text{md}(f^*) = 0$.

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Finally we prove the following easy (see also [?], p. 548):

Proposition 2.6 *For any X, $X' \in H^{1,2}(B,\mathbb{R}^3)$ and any open subset $\Omega \subseteq B$ there holds:*

$$| \mathcal{J}_\Omega(X) - \mathcal{J}_\Omega(X') | \leq 2\,(m_2 + k)\big(\sqrt{\mathcal{D}_\Omega(X)} + \sqrt{\mathcal{D}_\Omega(X')}\big)\sqrt{\mathcal{D}_\Omega(X - X')}, \tag{2.49}$$

$$| \mathcal{I}_\Omega(X) - \mathcal{I}_\Omega(X') | \leq (2\,m_2 + k)\big(\sqrt{\mathcal{D}_\Omega(X)} + \sqrt{\mathcal{D}_\Omega(X')}\big)\sqrt{\mathcal{D}_\Omega(X - X')}. \tag{2.50}$$

Proof: Firstly we split up:

$$X_u \wedge X_v - X'_u \wedge X'_v = X_u \wedge (X_v - X'_v) + (X_u - X'_u) \wedge X'_v.$$

We estimate by the Hölder inequality on any open subset $\Omega \subseteq B$:

$$\int_\Omega | X_u \wedge (X_v - X'_v) | \; dudv \leq 2\sqrt{\mathcal{D}_\Omega(X)\,\mathcal{D}_\Omega(X - X')},$$

thus

$$\int_\Omega | X_u \wedge X_v - X'_u \wedge X'_v | \; dudv \leq 2\left(\sqrt{\mathcal{D}_\Omega(X)} + \sqrt{\mathcal{D}_\Omega(X')}\right)\sqrt{\mathcal{D}_\Omega(X - X')}. \tag{2.51}$$

In [?], p. 7, the Lipschitz continuity of the integrand F on $\mathbb{R}^3$, with Lip. const.$= m_2$, is derived from its required properties (??), (??) and (??). Together with (??) we arrive at

$$| \mathcal{F}_\Omega(X) - \mathcal{F}_\Omega(X') | \leq 2m_2\left(\sqrt{\mathcal{D}_\Omega(X)} + \sqrt{\mathcal{D}_\Omega(X')}\right)\sqrt{\mathcal{D}_\Omega(X - X')}. \tag{2.52}$$

Combining this again with (??) we obtain (??). Furthermore by the "triangle inequality" $| \sqrt{\mathcal{D}_\Omega(X)} - \sqrt{\mathcal{D}_\Omega(X')} | \leq \sqrt{\mathcal{D}_\Omega(X - X')}$ we have

$$\begin{aligned} | \mathcal{D}_\Omega(X) - \mathcal{D}_\Omega(X') | &= \left(\sqrt{\mathcal{D}_\Omega(X)} + \sqrt{\mathcal{D}_\Omega(X')}\right) | \sqrt{\mathcal{D}_\Omega(X)} - \sqrt{\mathcal{D}_\Omega(X')} | \\ &\leq \left(\sqrt{\mathcal{D}_\Omega(X)} + \sqrt{\mathcal{D}_\Omega(X')}\right) \sqrt{\mathcal{D}_\Omega(X - X')}. \end{aligned}$$

Hence, a combination of this with (??) yields (??).

◇

2.3 Levelling of $C_c^\infty(\mathbb{R}^2)$-functions

In this section we discuss the process of "levelling" a function $f \in C_c^\infty(\mathbb{R}^2)$ on the unit disc $\bar{B}$ for a given fineness $\delta > 0$ (see also [?], p. 553 and [?], p. 558). To this end let

$$\mathcal{Z} : \min_{\bar{B}} f = l_0 < l_1 < \ldots < l_N < l_{N+1} = \max_{\bar{B}} f$$

be a partition of the interval $[\min_{\bar{B}} f, \max_{\bar{B}} f]$ such that $\triangle\mathcal{Z} := \max_{i=1,\ldots,N+1} \{l_i - l_{i-1}\} < \delta$ and such that $l_1, \ldots, l_N$ are regular values of f, which is possible for any choice of δ by Sard's theorem (see [?], p. 205).
The levelling process starts on the level l_1. Since l_1 is a regular value of $f \in C_c^\infty(\mathbb{R}^2)$,

(especially $l_1 \neq 0$) the implicit function theorem (see [?], p. 303) exposes $f^{-1}([l_1, \infty))$ to be a compact 2-dimensional C^∞-manifold with boundary. Hence, $f^{-1}([l_1, \infty))$ is locally connected, in particular, and has therefore only a finite number of connected components. Now we consider the (disjoint) union $U_+^{l_1}$ of those connected components of $f^{-1}([l_1, \infty))$ that are contained in $\bar{B}$, in particular we have

$$f(w) > l_1 \qquad \forall\, w \in \mathring{U}_+^{l_1} \qquad \text{and} \qquad f(w) = l_1 \qquad \forall\, w \in \partial U_+^{l_1}, \tag{2.53}$$

as l_1 is a regular value of f and as f is continuous, and we set

$$f_+^{l_1}(w) := \begin{cases} l_1 & : \quad w \in U_+^{l_1} \\ f(w) & : \quad w \in \mathbb{R}^2 \setminus U_+^{l_1}. \end{cases} \qquad \star$$

We go on by considering the compact C^∞-manifold $f^{-1}((-\infty, l_1])$ which consists of only finitely many connected components again, and term $U_-^{l_1}$ the union of those connected components that are contained in $\bar{B}$. By (**??**) we infer $\mathring{U}_+^{l_1} \cap \mathring{U}_-^{l_1} = \emptyset$ and therefore

$$f_+^{l_1}(w) < l_1 \qquad \forall\, w \in \mathring{U}_-^{l_1} \qquad \text{and} \qquad f_+^{l_1}(w) = l_1 \qquad \forall\, w \in \partial U_-^{l_1},$$

again since l_1 is a regular value of f, by $\star$ and as f is continuous, and we set

$$f^{l_1}(w) := \begin{cases} l_1 & : \quad w \in U_-^{l_1} \\ f_+^{l_1}(w) & : \quad w \in \mathbb{R}^2 \setminus U_-^{l_1}. \end{cases} \qquad \star\star$$

Next we apply the same process to f^{l_1} on the level l_2 and note that for connected components P^1 of $U_\pm^{l_1}$ and P^2 of $U_+^{l_2}$ we have $P^1 \cap P^2 = \emptyset$ and for connected components P^1 of $U_\pm^{l_1}$ and P^2 of $U_-^{l_2}$ we have either $P^1 \cap P^2 = \emptyset$ or $P^1 \subset\subset P^2$. After that we apply the process to $(f^{l_1})^{l_2}$ on the level l_3 and so on, until we have performed the last levelling step on the level l_N. Thus after $2 \times N$ steps we arrive at a finite collection of "level sets" $U_\pm^{l_j}$, j=1,...,N, and at a function f^L on $\mathbb{R}^2$, that we term the "levelled" function of f, possessing the following properties:

Lemma 2.3 *Let $f \in C_c^\infty(\mathbb{R}^2)$ and a fineness δ be given arbitrarily. Firstly there holds $U_\pm^{l_j} \subset \bar{B}$ and $\mathring{U}_+^{l_j} \cap \mathring{U}_-^{l_j} = \emptyset$ for $j = 1, \ldots, N$. Secondly for connected components P^j of $U_\pm^{l_j}$ and P^i of $U_+^{l_i}$, with $j < i$, there holds $P^j \cap P^i = \emptyset$ and for connected components P^j of $U_\pm^{l_j}$ and P^i of $U_-^{l_i}$ ($j < i$) there holds either $P^j \cap P^i = \emptyset$ or $P^j \subset\subset P^i$. Furthermore $U_\pm^{l_j}$ are compact 2-dimensional C^∞-manifolds with boundary and $\partial U_\pm^{l_j}$ are closed 1-dimensional C^∞-manifolds. In particular, $U_\pm^{l_j}$ consist of only a finite number of connected components and $\partial U_\pm^{l_j}$ are Lebesgue-measurable with $\mathcal{L}^2(\partial U_\pm^{l_j}) = 0$. Moreover f^L satisfies:*

$$f^L \in C^0(\bar{B}) \cap H^{1,2}(B), \qquad f^L\,|_{\partial B} = f\,|_{\partial B}, \qquad \mathrm{md}(f^L\,|_{\bar{B}}) \leq \delta. \tag{2.54}$$

Proof: The assertions $U_\pm^{l_j} \subset \bar{B}$ and $\mathring{U}_+^{l_j} \cap \mathring{U}_-^{l_j} = \emptyset$ follow immediately from the definition of $U_\pm^{l_j}$ and as the l_j are regular values of f for $j = 1, \dots, N$. Next one obtains simultaneously $f^L \in C^0(\bar{B})$ and the relations between the connected components P^j of $U_\pm^{l_j}$ and P^i of $U_+^{l_i}$ resp. $U_-^{l_i}$, with $j < i$, by induction during the finite levelling process. As the levels l_j are regular values of $f \in C_c^\infty(\mathbb{R}^2)$ the implicit function theorem yields the assertions about the level sets $U_\pm^{l_j}$ and their boundaries $\partial U_\pm^{l_j}$ at once. Furthermore one has to note that manifolds M are locally connected, thus their connected components are open in M and compact manifolds can only consist of finitely many. Moreover $\mathcal{L}^2(\partial U_\pm^{l_j}) = 0$ follows immediately from the implicit function theorem and Prop. 8 of Section 1.11 in [?], p. 101. Furthermore by construction of the first levelling step we obtain $f_+^{l_1} \in H^{1,1}(B)$ due to Lemma A 6.9 in [?], p. 254, where we have to use that $\partial U_+^{l_1}$ is a closed C^∞-manifold, thus in particular a Lipschitz boundary. Moreover it is also clear that we have $\nabla f_+^{l_1} \in L^2(B, \mathbb{R}^2)$ as $f_+^{l_1} \equiv f$ on $\mathbb{R}^2 \setminus U_+^{l_1}$ and $\nabla f_+^{l_1} \equiv 0$ on $\mathring{U}_+^{l_1}$ and since $\partial U_+^{l_1}$ especially satisfies $\mathcal{L}^2(\partial U_+^{l_1}) = 0$. Hence, we have $f_+^{l_1} \in H^{1,2}(B)$. Now, using that $\partial U_-^{l_1}$ is a closed C^∞-manifold again, especially with $\mathcal{L}^2(\partial U_-^{l_1}) = 0$ the same reasoning as above yields that $f^{l_1} \in H^{1,2}(B)$ and again using that $\partial U_\pm^{l_2}$ is a C^∞-manifold just the same reasoning as above yields that $(f^{l_1})^{l_2} \in H^{1,2}(B)$. Hence, after $2 \times N$ steps we arrive at $f^L \in H^{1,2}(B)$. Next, if $U_+^{l_1} \cap \partial B = \emptyset$ we have $f_+^{l_1} \mid_{\partial B} \equiv f \mid_{\partial B}$, but if $U_+^{l_1} \cap \partial B \neq \emptyset$ we obtain by the construction of $f_+^{l_1}$:

$$f_+^{l_1} \equiv l_1 \equiv f \qquad \text{along} \quad \partial U_+^{l_1} \cap \partial B.$$

Since this argument holds true for each step of the levelling process we finally see that $f^L \mid_{\partial B} \equiv f \mid_{\partial B}$. If we suppose that there exists an open subset G of B such that $\max_{\bar{G}} f^L - \max_{\partial G} f^L > \delta$, then due to $\triangle \mathcal{Z} < \delta$ there would be some level $l_j \in \mathcal{Z}$ such that $\max_{\partial G} f^L < l_j$ but $\max_{\bar{G}} f^L > l_j$. Hence, together with the continuity of f^L we would have on a connected component G' $(\neq \emptyset)$ of $G \cap (f^L)^{-1}((l_j, \infty)) \subset\subset G$

$$f^L(w) > l_j \qquad \forall\, w \in G' \qquad \text{and} \qquad f^L(w) = l_j \qquad \forall\, w \in \partial G',$$

which implies that $f^L \equiv f$ on G' and $G' \subset U_+^{l_j}$. Therefore we must have $f^L \equiv l_i$ on G' for some $i \geq j$ by the construction of f^L and the second part of the assertion of the lemma, which is a contradiction. Similarily one proves that $\min_{\partial G} f^L - \min_{\bar{G}} f^L \leq \delta$ for all open subsets G of B again by the construction of f^L and the second part of the assertion of the lemma, hence $md(f^L \mid_{\bar{B}}) \leq \delta$.

◇

2.4 Levelling of the components of distorted surfaces $A\pi$

In this section we consider some smooth surface $\pi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ and its distortion $\tilde{\pi} := A\pi$, where $A := (a_1, a_2, a_3)^\top \in GL_3(\mathbb{R})$ is defined at the beginning of Section **??**. Its components satisfy

$$\tilde{\pi}_i = \langle a_i, \pi \rangle = (O_i\, \pi)_i = \pi_i'^{i} \tag{2.55}$$

for $i = 1,2,3$, where we termed $\pi'^i := O_i\,\pi$. We set $m := \min_{i=1,2,3}\{\min_{\bar{B}}\tilde{\pi}_i\}$ and $M := \max_{i=1,2,3}\{\max_{\bar{B}}\tilde{\pi}_i\}$ and choose a partition

$$\mathcal{Z} : m = l_0 < l_1 < \ldots < l_N < l_{N+1} = M$$

of the interval $[m, M]$ of fineness $\triangle\mathcal{Z} < \delta$, for an arbitrarily given $\delta > 0$, such that the levels l_j, $j = 1, \ldots, N$, are regular values of the three components $\tilde{\pi}_i$ simultaneously. At first we level the first component, i.e. $\tilde{\pi}_1 \mapsto (\tilde{\pi}_1)^L$, abbreviate $(\pi'^1)^L := ((\pi_1'^1)^L, \pi_2'^1, \pi_3'^1)$ and prove (see also (6.6) in [**?**])

Lemma 2.4 *For an arbitrary $\pi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ there holds:*

$$\mathcal{F}(\pi) \geq \mathcal{F}(O_1^{-1}(\pi'^1)^L). \tag{2.56}$$

Proof: We abbreviate $\pi' := \pi'^1 = O_1\pi$. It will suffice to consider only the first step of the levelling process on the level l_1 applied to $\pi_1' = \tilde{\pi}_1$. Let D be the open kernel of a connected component $\bar{D}$ of the level set $U_+^{l_1}$ which is a compact C^∞-manifold with boundary by Lemma **??**. Now we choose an Atlas $\mathcal{A} := \{(V_j, \psi_j)_{j=0,\ldots,k}\}$ of $\bar{D}$ such that $\partial D \subset \bigcup_{j=1}^k V_j$ and a subordinate partition of unity $\{\eta_j\}_{j=0,\ldots,k}$. Furthermore a careful examination of the implicit function theorem (see [**?**], p. 303) shows that we may arrange the charts $\psi_j : B_{r_j}^+(0) \overset{\cong}{\longrightarrow} V_j \cap \bar{D}$ such that $\gamma_j := \psi_j\,|_{[-r_j,r_j]}: [-r_j, r_j] \overset{\cong}{\longrightarrow} V_j \cap \partial D$ yields a parametrization of $V_j \cap \partial D$ with respect to its arc lenght, for $j = 1, \ldots, k$, implying that $((\gamma_j)_2', -(\gamma_j)_1')$ yields an outward pointing unit normal field ν_j along $V_j \cap \partial D$. Since we have $\pi_1' \equiv l_1$ along ∂D we infer:

$$\frac{d}{ds}\pi_1'(\gamma_j(s)) \equiv 0 \qquad \forall\, s \in [-r_j, r_j], \tag{2.57}$$

for $j = 1, \ldots, k$. Now we consider the vector field $h(x, y, z) := (-y, 0, 0)$ on $\mathbb{R}^3$. Firstly we note that $\operatorname{rot} h \equiv (0, 0, 1)$, thus setting $N := (N_1, N_2, N_3) := \pi_u' \wedge \pi_v'$ we have

$$N_3 = \langle rot\, h(\pi'), \pi_u' \wedge \pi_v' \rangle \qquad \text{on } \mathbb{R}^2. \tag{2.58}$$

Furthermore we set $w := (\langle h(\pi'), \pi_v' \rangle, -\langle h(\pi'), \pi_u' \rangle) \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Using $\pi_{uv}' = \pi_{vu}'$ due to Schwarz we calculate, as in the proof of Stokes' theorem (see p. 492 in [**?**]):

$$div\, w = \langle rot\, h(\pi'), \pi_u' \wedge \pi_v' \rangle \qquad \text{on } \mathbb{R}^2.$$

Now combining this with (**??**), the divergence theorem for Lipschitz boudaries (see [**?**], p. 252) and (**??**) we arrive at:

$$\int_D N_3\,dudv = \int_D div\,w\,dudv = \int_{\partial D}\langle w,\nu\rangle\,ds = \sum_{j=1}^k \int_{\partial D\cap V_j} \eta_j\,\langle w,\nu_j\rangle\,ds$$
$$= \sum_{j=1}^k \int_{-r_j}^{r_j} (\eta_j\,w_1)(\gamma_j(s))\,(\gamma_j)'_2 - (\eta_j\,w_2)(\gamma_j(s))\,(\gamma_j)'_1\,ds$$
$$= \sum_{j=1}^k \int_{-r_j}^{r_j} (\eta_j\,(-\pi'_2\,(\pi'_1)_v))(\gamma_j(s))\,(\gamma_j)'_2 - (\eta_j\,\pi'_2\,(\pi'_1)_u)(\gamma_j(s))\,(\gamma_j)'_1\,ds$$
$$= -\sum_{j=1}^k \int_{-r_j}^{r_j} (\eta_j\,\pi'_2)(\gamma_j(s))\,\frac{d}{ds}\pi'_1(\gamma_j(s))\,ds = 0. \qquad (2.59)$$

Moreover using $\tilde{h}(x,y,z) := (z,0,0)$, with $rot\,\tilde{h} = (0,1,0)$, we obtain analogously:

$$\int_D N_2\,dudv = \sum_{j=1}^k \int_{-r_j}^{r_j} (\eta_j\,\pi'_3)(\gamma_j(s))\,\frac{d}{ds}\pi'_1(\gamma_j(s))\,ds = 0, \qquad (2.60)$$

on account of (**??**). Furthermore, as we have $\nabla(\pi'_1)^{l_1}_+ \equiv 0$ on D we see:

$$N^{l_1} := \begin{pmatrix} N_1^{l_1} \\ N_2^{l_1} \\ N_3^{l_1} \end{pmatrix} := \begin{pmatrix} ((\pi'_1)^{l_1}_+)_u \\ (\pi'_2)_u \\ (\pi'_3)_u \end{pmatrix} \wedge \begin{pmatrix} ((\pi'_1)^{l_1}_+)_v \\ (\pi'_2)_v \\ (\pi'_3)_v \end{pmatrix}$$
$$= \begin{pmatrix} (\pi'_2)_u(\pi'_3)_v - (\pi'_2)_v(\pi'_3)_u \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} N_1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus by Lemma **??** we can conclude now:

$$F'(N) - F'(N^{l_1}) = F'(N_1,N_2,N_3) - F'(N_1,0,0) \geq c_y\,N_2 + c_z\,N_3.$$

Integration of this inequality over D yields

$$\int_D F'(N)\,dudv - \int_D F'(N^{l_1})\,dudv \geq c_y \int_D N_2\,dudv + c_z \int_D N_3\,dudv = 0,$$

where we used (**??**) and (**??**). Hence, as we have $(\pi'_1)^{l_1}_+ \equiv \pi'_1$ on $B\setminus U^{l_1}_+$ we obtain

$$\int_B F'(N)\,dudv \geq \int_B F'(N^{l_1})\,dudv.$$

Thus due to $O_1 \in SO(3)$ we finally achieve after $2\times N$ levelling steps:

$$\mathcal{F}(\pi) = \int_B F(O_1^{-1}(O_1\pi_u \wedge O_1\pi_v))\,dudv = \int_B F'(N)\,dudv$$
$$\geq \int_B F(O_1^{-1}((\pi')^L_u \wedge (\pi')^L_v))\,dudv = \mathcal{F}(O_1^{-1}(\pi')^L).$$

◇

Furthermore we shall also level the second and third component of $\tilde{\pi}$, i.e. $\tilde{\pi}_i \mapsto (\tilde{\pi}_i)^L$ for $i = 2, 3$. Abbreviating $(\pi'^2)^L := (\pi'^2_1, (\pi'^2_2)^L, \pi'^2_3)$ and $(\pi'^3)^L := (\pi'^3_1, \pi'^3_2, (\pi'^3_3)^L)$ we gain by (**??**) and (**??**) analogously for $i = 2, 3$:

$$\mathcal{F}(\pi) \geq \mathcal{F}(O_i^{-1}(\pi'^i)^L), \tag{2.61}$$

where one has to use the vector fields $h^2 := (0, -z, 0)$, $\tilde{h}^2 := (0, x, 0)$ for $i = 2$ and $h^3 := (0, 0, y)$, $\tilde{h}^3 := (0, 0, -x)$ for $i = 3$ to obtain the counterparts of the central equations (**??**) and (**??**). Next we prove (see also (6.7) in [**?**])

Lemma 2.5 *For an arbitrary* $\pi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ *there holds*

$$\mathcal{D}(\pi) - \mathcal{D}(O_i^{-1}(\pi'^i)^L) = \mathcal{D}(\pi'^i_i - (\pi'^i_i)^L), \tag{2.62}$$

for $i = 1, 2, 3$.
Proof: For $i = 1$ we abbreviate again $\pi' := \pi'^1$. We consider the union $\mathcal{L} := \bigcup_{j=1}^N \mathring{U}_\pm^{l_j}$ of all level sets that arise during the levelling process applied to $\tilde{\pi}_1 = \pi'_1$. Now combining the facts that π'_2 and π'_3 remain unchanged on B and that π'_1 remains unchanged on $B \setminus \mathcal{L}$, while we level π'_1, and that $\nabla \pi'_1 \equiv 0$ on $\mathcal{L}$ we infer:

$$\mathcal{D}(\pi') - \mathcal{D}((\pi')^L) = \mathcal{D}_{\mathcal{L}}(\pi'_1) - \mathcal{D}_{\mathcal{L}}((\pi'_1)^L) = \mathcal{D}_{\mathcal{L}}(\pi'_1) = \mathcal{D}_{\mathcal{L}}(\pi'_1 - (\pi'_1)^L) = \mathcal{D}(\pi'_1 - (\pi'_1)^L).$$

Together with the invariance of the Euclidean scalar product with respect to the action of $SO(3)$ we finally achieve the assertion (**??**) for $i = 1$. For $i = 2, 3$ the proof works analogously.

◇

A combination of (**??**), (**??**) and (**??**) yields

$$\mathcal{D}(\pi'^i_i - (\pi'^i_i)^L) \leq \frac{1}{k} (\mathcal{I}(\pi) - \mathcal{I}(O_i^{-1}(\pi'^i)^L)), \tag{2.63}$$

for $i = 1, 2, 3$. Furthermore we define $\tilde{\pi}^L := ((\tilde{\pi}_1)^L, (\tilde{\pi}_2)^L, (\tilde{\pi}_3)^L)$ and $\pi^L := A^{-1}\tilde{\pi}^L$ $(= A^{-1}(A\pi)^L)$ and state (see also Lemma 6.3 in [**?**])

Lemma 2.6 *The surface* π^L *has the following properties:*
(i) $\pi^L \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$, *(ii)* $\pi^L \mid_{\partial B} = \pi \mid_{\partial B}$, *(iii)* $md((A\pi^L)_i \mid_{\bar{B}}) \leq \delta$ *for* $i = 1, 2, 3$. *(iv) Using the matrix norm* $\parallel B \parallel := \sup_{x \in \mathbb{S}^2} \mid B\,x \mid$ *on* $Mat_{3,3}(\mathbb{R})$ *we have:*

$$\mathcal{D}(\pi^L - \pi) \leq \frac{\parallel A^{-1} \parallel^2}{k} \Big(\sum_{i=1}^3 \mathcal{I}(\pi) - \mathcal{I}(O_i^{-1}(\pi'^i)^L) \Big). \tag{2.64}$$

Proof: The points (i), (ii) and (iii) follow immediately from Lemma **??** and the definition of π^L. Moreover we calculate by (**??**) and (**??**):

$$\mathcal{D}(\pi^L - \pi) = \mathcal{D}(A^{-1}(\tilde{\pi}^L - \tilde{\pi})) \leq \parallel A^{-1} \parallel^2 \mathcal{D}(\tilde{\pi}^L - \tilde{\pi}) = \parallel A^{-1} \parallel^2 \Big(\sum_{i=1}^3 \mathcal{D}((\tilde{\pi}_i)^L - \tilde{\pi}_i) \Big)$$

$$= \parallel A^{-1} \parallel^2 \Big(\sum_{i=1}^3 \mathcal{D}((\pi'^i_i)^L - \pi'^i_i) \Big) \leq \frac{\parallel A^{-1} \parallel^2}{k} \Big(\sum_{i=1}^3 \mathcal{I}(\pi) - \mathcal{I}(O_i^{-1}(\pi'^i)^L) \Big).$$

◇

2.5 Proof of Theorems ?? and ??

Proof of Theorem **??**:
Now let $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{\frac{1}{2},2}(\partial B, \mathbb{R}^3)$ be prescribed boundary values. By Prop. **??** there exists a minimizing element $\{X^n\}$ for $\mathcal{I}$ in $M(\varphi)$, i.e. $\{X^n\} \in M(\varphi)$ satisfies

$$\lim_{n\to\infty} \mathcal{I}(X^n) = m(\varphi). \tag{2.65}$$

By Prop. **??** there exists a mollified sequence $\{\pi^n\} := \{X^n_{\epsilon_n}\} \subset C^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$ such that

$$\| \pi^n - X^n \|_{C^0(\bar{B})} + \| \pi^n - X^n \|_{H^{1,2}(B)} < \frac{1}{n} \qquad \forall\, n \in \mathbb{N}. \tag{2.66}$$

Firstly we infer from (**??**) and $\{X^n\} \in M(\varphi)$:

$$\| \pi^n \lfloor_{\partial B} - \varphi \|_{C^0(\partial B)} \leq \| \pi^n \lfloor_{\partial B} - X^n \lfloor_{\partial B} \|_{C^0(\partial B)} + \| X^n \lfloor_{\partial B} - \varphi \|_{C^0(\partial B)} \longrightarrow 0, \tag{2.67}$$

for $n \to \infty$, which shows that $\{\pi^n\} \in M(\varphi)$. Secondly a combination of (**??**) with Prop. **??** and (**??**) yields

$$| \mathcal{I}(\pi^n) - m(\varphi) | \leq | \mathcal{I}(\pi^n) - \mathcal{I}(X^n) | + | \mathcal{I}(X^n) - m(\varphi) | \longrightarrow 0, \tag{2.68}$$

where we also used that $\mathcal{D}(X^n) \leq \frac{1}{k}\mathcal{I}(X^n) \leq \text{const.}\ \forall\, n \in \mathbb{N}$ due to (**??**). Hence, $\{\pi^n\}$ is a minimizing element for $\mathcal{I}$ in $M(\varphi)$ again. Now we level the components of $\tilde{\pi}^n := A\,\pi^n$, i.e. $\tilde{\pi}^n \mapsto (\tilde{\pi}^n)^L$, with decreasing fineness $\delta_n \searrow 0$. Firstly by (**??**) and (**??**) we have

$$O_i^{-1}((\pi^n)'^i)^L \lfloor_{\partial B} = \pi^n \lfloor_{\partial B} \longrightarrow \varphi \qquad \text{in } C^0(\partial B, \mathbb{R}^3), \tag{2.69}$$

and therefore $\{O_i^{-1}((\pi^n)'^i)^L\} \in M(\varphi)$, for $i = 1, 2, 3$, having used (**??**) again. Furthermore by (**??**), (**??**) and (**??**) we obtain

$$\mathcal{I}(O_i^{-1}((\pi^n)'^i)^L) \leq \mathcal{I}(\pi^n) \qquad \forall\, n \in \mathbb{N},$$

for $i = 1, 2, 3$. Combining this with (**??**) and (**??**) we conclude:

$$m(\varphi) \leq \liminf_{n\to\infty} \mathcal{I}(O_i^{-1}((\pi^n)'^i)^L) \leq \limsup_{n\to\infty} \mathcal{I}(O_i^{-1}((\pi^n)'^i)^L) \leq \lim_{n\to\infty} \mathcal{I}(\pi^n) = m(\varphi),$$

implying that $\{O_i^{-1}((\pi^n)'^i)^L\}$ is a minimizing element for $\mathcal{I}$ in $M(\varphi)$, for $i = 1, 2, 3$. If we insert this and (**??**) into (**??**), applied to π^n, we obtain:

$$0 \leq \mathcal{D}((\pi^n)^L - \pi^n) \leq \frac{\| A^{-1} \|^2}{k} \Big(\sum_{i=1}^{3} \mathcal{I}(\pi^n) - \mathcal{I}(O_i^{-1}((\pi^n)'^i)^L) \Big) \longrightarrow 0, \tag{2.70}$$

for $n \to \infty$. Combining this with (**??**) and noting that $\{\mathcal{D}(\pi^n)\}$ and $\{\mathcal{D}((\pi^n)^L)\}$ are bounded due to (**??**) and (**??**) we arrive at:

$$| \mathcal{I}((\pi^n)^L) - m(\varphi) | \leq | \mathcal{I}((\pi^n)^L) - \mathcal{I}(\pi^n) | + | \mathcal{I}(\pi^n) - m(\varphi) | \longrightarrow 0, \tag{2.71}$$

for $n \to \infty$. Moreover by Lemma **??** (ii) and (**??**) we know that

$$(\pi^n)^L \mid_{\partial B} = \pi^n \mid_{\partial B} \longrightarrow \varphi \qquad \text{in } C^0(\partial B, \mathbb{R}^3).$$

Hence, together with Lemma **??** (i) and (**??**) we see that $\{(\pi^n)^L\}$ is a minimizing element for $\mathcal{I}$ in $M(\varphi)$. Now recalling Lemma **??** (iii) we gather the following facts about the sequence $\{A\,(\pi^n)^L\}$:

$$\begin{aligned} A\,(\pi^n)^L \mid_{\partial B} &\longrightarrow A\,\varphi \qquad \text{in } C^0(\partial B, \mathbb{R}^3),\\ md((A\,(\pi^n)^L)_i \mid_{\bar{B}}) &\leq \delta_n \searrow 0 \qquad \text{for } i = 1,2,3,\\ \mathcal{D}(A\,(\pi^n)^L) \leq \| A \|^2\, \mathcal{D}((\pi^n)^L) &\leq \text{const.} \qquad \forall\, n \in \mathbb{N}. \end{aligned}$$

Hence, we can apply Prop. **??** and obtain a subsequence $\{A\,(\pi^{n_j})^L\}$ and a surface $\pi^* \in C^0(\bar{B}, \mathbb{R}) \cap H^{1,2}(B, \mathbb{R})$ such that

$$A\,(\pi^{n_j})^L \mid_{\bar{B}} \longrightarrow \pi^* \qquad \text{in } C^0(\bar{B}, \mathbb{R}^3),$$

$md(\pi_i^*) = 0$, for $i = 1,2,3$, and $\pi^* \mid_{\partial B} \equiv A\,\varphi$. Thus, if we rename $\{A\,(\pi^{n_j})^L\}$ into $\{A\,(\pi^n)^L\}$ we conclude:

$$(\pi^n)^L \mid_{\bar{B}} \longrightarrow A^{-1}\pi^* \qquad \text{in } C^0(\bar{B}, \mathbb{R}^3), \tag{2.72}$$

with $A^{-1}\pi^* \mid_{\partial B} \equiv \varphi$. As we already know $\mathcal{D}((\pi^n)^L) \leq$ const. this entails in particular $\| (\pi^n)^L \|_{H^{1,2}(B)} \leq$ const., $\forall\, n \in \mathbb{N}$, implying the existence of a further subsequence $\{(\pi^{n_j})^L\}$ with

$$(\pi^{n_j})^L \mid_B \rightharpoonup A^{-1}\pi^* \qquad \text{in } H^{1,2}(B, \mathbb{R}^3).$$

We set $X^* := A^{-1}\pi^*$. Now using the weak lower semicontinuity of $\mathcal{I}$ due to [**?**], Theorem II.4, (see also [**?**], p. 12) we conclude together with (**??**) and (**??**):

$$j(\varphi) := \inf_{H^{1,2}_\varphi(B) \cap C^0(\bar{B})} \mathcal{I} \leq \mathcal{I}(X^*) \leq \liminf_{j \to \infty} \mathcal{I}((\pi^{n_j})^L) = m(\varphi) \leq j(\varphi). \tag{2.73}$$

Moreover in [**?**], p. 34, it is proved that the (unique) minimizer Y of $\mathcal{I}$ within the class $H^{1,2}_\varphi(B, \mathbb{R}^3)$ lies already in $C^0(\bar{B}, \mathbb{R}^3)$, if $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{\frac{1}{2},2}(\partial B, \mathbb{R}^3)$, which implies

$$\mathcal{I}(Y) = \inf_{H^{1,2}_\varphi(B)} \mathcal{I} \leq \inf_{H^{1,2}_\varphi(B) \cap C^0(\bar{B})} \mathcal{I} \leq \mathcal{I}(Y).$$

Combining this with (**??**) we finally obtain:

$$\mathcal{I}(X^*) = j(\varphi) = \inf_{H^{1,2}_\varphi(B)} \mathcal{I},$$

with $md((A\,X^*)_i) = md(\pi_i^*) = 0$, for $i = 1,2,3$.

$\diamond$

Proof of Theorem **??**:
Firstly by hypothesis we have the equicontinuity and uniform boundedness of the distorted boundary values $\{A\,X^n\,|_{\partial B}\}$, thus we gain a convergent subsequence $\{A\,X^{n_j}\,|_{\partial B}\}$ in $C^0(\partial B,\mathbb{R}^3)$ by Arzelà-Ascoli's theorem, which we rename $\{A\,X^n\,|_{\partial B}\}$ again. Now we infer by Theorem **??** that $\{A\,X^n\}\subset C^0(\bar{B},\mathbb{R}^3)\cap H^{1,2}(B,\mathbb{R}^3)$ satisfies

$$md((A\,X^n)_i)=0 \qquad \text{for } i=1,2,3, \quad \forall\, n\in\mathbb{N}.$$

Hence, together with $\mathcal{D}(A\,X^n)\leq\parallel A\parallel^2\,\mathcal{D}(X^n)\leq$ const. we see that Prop. **??** implies the existence of a further subsequence $\{A\,X^{n_j}\}$ and some surface $Y\in C^0(\bar{B},\mathbb{R}^3)\cap H^{1,2}(B,\mathbb{R}^3)$ such that

$$A\,X^{n_j}\longrightarrow Y \qquad \text{in } C^0(\bar{B},\mathbb{R}^3)$$

and $\text{md}(Y_i)=0$ for $i=1,2,3$. Thus the subsequence $\{X^{n_j}\}$ converges uniformly to $\bar{X}:=A^{-1}Y\in C^0(\bar{B},\mathbb{R}^3)\cap H^{1,2}(B,\mathbb{R}^3)$ and $\text{md}((A\,\bar{X})_i)=0$ for $i=1,2,3$. Together with the required boundedness of $\{\mathcal{D}(X^n)\}$ we obtain $\parallel X^{n_j}\parallel_{H^{1,2}(B)}\leq$ const., $\forall\, j\in\mathbb{N}$, and therefore the asserted weak $H^{1,2}$-convergence in (**??**) for a further subsequence.

$\diamond$

3 Compactness resp. closedness of the set of $\mathcal{I}$-surfaces in $H^{1,2}_{loc}(B,\mathbb{R}^3)$ resp. $C^0(\bar{B},\mathbb{R}^3)$

In this chapter we prove Theorems 10.2 and 10.3 of [?], pp. 558–561, whose proofs in [?] are rather sketchy. Throughout the paper we will use the notations $Z := X_u \wedge X_v$, $\delta Z := X_u \wedge \varphi_v + \varphi_u \wedge X_v$ and $\delta^2 Z := \varphi_u \wedge \varphi_v$ for any $X, \varphi \in H^{1,2}(B,\mathbb{R}^3)$,

$$\mathcal{R} := \mathcal{R}(X) := \{(u,v) \in B \mid (X_u \wedge X_v)(u,v) \neq 0\},$$

$$\mathcal{S} := \mathcal{S}(X) := B \setminus \mathcal{R}(X)$$

and $C_{r_1 r_2} := B_{r_2}(0) \setminus \overline{B_{r_1}(0)}$ for $r_1 < r_2 \in (0,1)$. Firstly we prove (see p. 560 in [?])

Proposition 3.1 *Let $\{Y^n\}$ be a sequence in $H^{1,2}(B,\mathbb{R}^3)$ with $\mathcal{D}(Y^n) \leq const.$ and let $\{\delta_n\} \subset \mathbb{R}_{>0}$ be some sequence with $\delta_n \to 0$. Setting $r_n := r + \delta_n$ for every $r \in (0,1)$ we prove that*

$$m(r) := \liminf_{n\to\infty} \mathcal{D}_{C_{r r_n}}(Y^n) = 0 \qquad \text{for a.e. } r \in (0,1). \tag{3.1}$$

Proof: We assume that there is some $\epsilon_0 > 0$ such that $P_\epsilon := \{r \in (0,1) \mid m(r) \geq \epsilon\}$ is non-empty for $\epsilon \in (0,\epsilon_0]$, otherwise we are done. We choose some $\epsilon \in (0,\epsilon_0]$ arbitrarily and a collection of finitely many different radii $r^1,\dots,r^q$ in P_ϵ, where $q \leq \text{card}(P_\epsilon)$ is arbitrarily fixed (which means that we choose $q \in \mathbb{N}$ arbitrarily if P_ϵ should have infinitely many elements). Firstly due to $\delta_n \to 0$ there exists a number N_1 such that $C_{r^i r^i_n} \cap C_{r^j r^j_n} = \emptyset \quad \forall\, i \neq j, \forall\, n > N_1$, which implies that

$$\sum_{i=1}^{q} \mathcal{D}_{C_{r^i r^i_n}}(Y^n) \leq \mathcal{D}(Y^n) \leq const. =: M, \tag{3.2}$$

$\forall\, n > N_1$. Furthermore we can determine a number $N_2 \geq N_1$ such that $\mathcal{D}_{C_{r^i r^i_n}}(Y^n) \geq \frac{m(r^i)}{2} \geq \frac{\epsilon}{2}$ $\forall\, n > N_2$ and for $i = 1,\dots,q$ simultaneously. Hence, together with (**??**) we see that $q\frac{\epsilon}{2} \leq M$, i.e. $q \leq \frac{2M}{\epsilon}$. This shows that $\text{card}(P_\epsilon) \leq \frac{2M}{\epsilon}$. Now every $r \in (0,1)$ with $m(r) > 0$ lies in some set $P_{\frac{1}{n}}$ for some $n > \frac{1}{m(r)}$, i.e. $\mathcal{B} := \{r \in (0,1) \mid m(r) > 0\} \subset \bigcup_{n\in\mathbb{N}} P_{\frac{1}{n}}$ which is a countable set on account of $\text{card}(P_{\frac{1}{n}}) \leq 2Mn$, $\forall\, n > \frac{1}{\epsilon_0}$, thus in particular $\mathcal{L}^1(\mathcal{B}) = 0$, which proves the assertion.

$\diamond$

For the readers convenience we recall here that we have by Proposition 3.3, Lemma 4.1 and (8) in [?]:

$$\delta^+\mathcal{I}(X,\varphi) = \delta\mathcal{F}_{\mathcal{R}}(X,\varphi) + \delta^+\mathcal{F}_{\mathcal{S}}(X,\varphi) + k\,\delta\mathcal{D}(X,\varphi)$$
$$= \int_{\mathcal{R}} \langle \nabla F(Z), \delta Z\rangle\, dudv + \int_{\mathcal{S}} F(\delta Z)\, dudv + k\int_B DX\cdot D\varphi\, dudv \tag{3.3}$$

for any $X,\varphi \in H^{1,2}(B,\mathbb{R}^3)$.

Theorem 3.1 *Let $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces with $\mathcal{D}(X^n) \le const.$, $\forall\, n\in\mathbb{N}$, and with equicontinuous and uniformly bounded boundary values, as in Theorem **??**. Then for every $r\in(0,1)$ there exists a subsequence $\{X^{n_k}\}$ such that*

$$\| X^{n_k} - \bar{X} \|_{H^{1,2}(B_r(0))} \longrightarrow 0 \qquad \text{for } k\to\infty, \tag{3.4}$$

*for the surface $\bar{X}$ of Theorem **??**.*

Proof: From Theorem **??** we infer the existence of a subsequence $\{X^{n_j}\}$ such that $\| \bar{X} - X^{n_j} \|_{C^0(\bar{B})} \to 0$. Without loss of generality we may assume that $\| \bar{X} - X^{n_j} \|_{C^0(\bar{B})} > 0\ \forall j\in\mathbb{N}$. We rename $\{n_j\}$ into $\{n\}$, choose some $r\in(0,1)$ arbitrarily such that (**??**) holds for $Y^n := \bar{X} - X^n$ and $\delta_n := \| \bar{X} - X^n \|_{C^0(\bar{B})}$ and consider the sequence $\{r_n\}$ given by $r_n := r + \delta_n$ (as in (**??**)). Without loss of generality we may assume that $\{r_n\}\subset (r,1)\ \forall n\in\mathbb{N}$. By Lemma 2 of Section 2.5 in [?], p. 23, the $\mathcal{I}$-surfaces X^n are characterized by the variational inequality

$$\delta^+\mathcal{I}(X^n,\varphi) \ge 0 \qquad \forall\,\varphi\in \mathring{H}^{1,2}(B,\mathbb{R}^3), \tag{3.5}$$

(see (**??**)) which we are going to test now by

$$\varphi^n(w) := \begin{cases} \bar{X}(w) - X^n(w) & : \quad w\in B_r(0) \\ \frac{r^n - |w|}{r^n - r}(\bar{X}(w) - X^n(w)) & : \quad w\in C_{rr^n} \\ 0 & : \quad w\in C_{r^n 1}. \end{cases}$$

Knowing that $X^n, \bar{X}\in H^{1,2}(B,\mathbb{R}^3)$ one easily checks that $\varphi^n\in \mathring{H}^{1,2}(B,\mathbb{R}^3)$, $\forall n\in\mathbb{N}$, on account of Lemma A 6.9 in [?], p. 254, and by the estimate

$$| D\varphi^n | \le \frac{r_n - |w|}{r_n - r} | D(\bar{X} - X^n) | + \frac{|\bar{X} - X^n|}{r_n - r} \le | D(\bar{X} - X^n) | + 1 \quad on\ C_{rr_n}, \tag{3.6}$$

where we inserted the definition of $\{r_n\}$. We will use the following abbreviations as in Section 4 of [?]:

$$Z^n := X^n_u \wedge X^n_v, \qquad \delta Z^n := X^n_u \wedge \varphi^n_v + \varphi^n_u \wedge X^n_v, \qquad \delta^2 Z^n := \varphi^n_u \wedge \varphi^n_v, \tag{3.7}$$

and we observe that $Z := \bar{X}_u \wedge \bar{X}_v$ can be expressed as

$$Z = Z^n + \delta Z^n + \delta^2 Z^n \quad on\ B_r(0). \tag{3.8}$$

Furthermore we define $\mathcal{R}^n := \mathcal{R}(X^n)$ and $\mathcal{S}^n := \mathcal{S}(X^n)$. Firstly we note that

$$\int_{B_\rho(0)} DX^n \cdot D(\bar{X} - X^n)\, dudv = \mathcal{D}_{B_\rho(0)}(\bar{X}) - \mathcal{D}_{B_\rho(0)}(X^n) - \mathcal{D}_{B_\rho(0)}(\bar{X} - X^n)$$

$\forall\, \rho \in (0,1]$. Now combining this with (**??**), (**??**) and $F(0) = 0$ we arrive at:

$$\begin{aligned} 0 \le \delta^+ \mathcal{I}(X^n, \varphi^n) = \int_{\mathcal{R}^n \cap B_r(0)} \langle \nabla F(Z^n), \delta Z^n \rangle\, dudv + \int_{\mathcal{R}^n \cap C_{rr_n}} \langle \nabla F(Z^n), \delta Z^n \rangle\, dudv \\ + \int_{\mathcal{S}^n \cap B_r(0)} F(\delta Z^n)\, dudv + \int_{\mathcal{S}^n \cap C_{rr_n}} F(\delta Z^n)\, dudv \quad (3.9) \\ + k\left(\mathcal{D}_{B_r(0)}(\bar{X}) - \mathcal{D}_{B_r(0)}(X^n) - \mathcal{D}_{B_r(0)}(\bar{X} - X^n)\right) + k \int_{C_{rr_n}} DX^n \cdot D\varphi^n\, dudv. \end{aligned}$$

As in (9) and (11) of [?] we gain by (**??**), the convexity of $F \in C^1(\mathbb{R}^3 \setminus \{0\})$, $\mid \nabla F \mid \le m_2$ on $\mathbb{R}^3 \setminus \{0\}$ and $\mid \delta^2 Z^n \mid \le \frac{1}{2} \mid D\varphi^n \mid^2$:

$$\begin{aligned} \mathcal{F}_{\mathcal{R}^n \cap B_r(0)}(\bar{X}) - \mathcal{F}_{\mathcal{R}^n \cap B_r(0)}(X^n) \ge \int_{\mathcal{R}^n \cap B_r(0)} \langle \nabla F(Z^n), \delta Z^n \rangle\, dudv \\ - m_2\, \mathcal{D}_{\mathcal{R}^n \cap B_r(0)}(\varphi^n), \end{aligned} \tag{3.10}$$

and together with $F \ge 0$ on $\mathbb{R}^3$ and $F(0) = 0$, using that $Z^n \equiv 0$ on $\mathcal{S}^n$:

$$\mathcal{F}_{\mathcal{S}^n \cap B_r(0)}(\bar{X}) - \mathcal{F}_{\mathcal{S}^n \cap B_r(0)}(X^n) \ge \int_{\mathcal{S}^n \cap B_r(0)} F(\delta Z^n)\, dudv - m_2\, \mathcal{D}_{\mathcal{S}^n \cap B_r(0)}(\varphi^n). \tag{3.11}$$

Now combining (**??**) and (**??**) with (**??**) and noting that $k > m_2$ we obtain:

$$\begin{aligned} &\mathcal{I}_{B_r(0)}(\bar{X}) - \mathcal{I}_{B_r(0)}(X^n) \\ &\ge \int_{\mathcal{R}^n \cap B_r(0)} \langle \nabla F(Z^n), \delta Z^n \rangle\, dudv + \int_{\mathcal{S}^n \cap B_r(0)} F(\delta Z^n)\, dudv \\ &\quad - m_2\, \mathcal{D}_{B_r(0)}(\varphi^n) + k\left(\mathcal{D}_{B_r(0)}(\bar{X}) - \mathcal{D}_{B_r(0)}(X^n)\right) \\ &\ge - \int_{\mathcal{R}^n \cap C_{rr_n}} \langle \nabla F(Z^n), \delta Z^n \rangle\, dudv - \int_{\mathcal{S}^n \cap C_{rr_n}} F(\delta Z^n)\, dudv \\ &\quad + (k - m_2)\, \mathcal{D}_{B_r(0)}(\varphi^n) - k \int_{C_{rr_n}} DX^n \cdot D\varphi^n\, dudv \\ &\ge - \int_{\mathcal{R}^n \cap C_{rr_n}} \langle \nabla F(Z^n), \delta Z^n \rangle\, dudv - \int_{\mathcal{S}^n \cap C_{rr_n}} F(\delta Z^n)\, dudv \\ &\quad - k \int_{C_{rr_n}} DX^n \cdot D\varphi^n\, dudv = -\delta^+ \mathcal{I}_{C_{rr_n}}(X^n, \varphi^n). \end{aligned} \tag{3.12}$$

Next combining (**??**) with $\mathcal{L}^2(C_{rr_n}) \le 2\pi\,(r_n - r)$ we gain by Cauchy-Schwarz' inequality:

$$\mathcal{D}_{C_{rr_n}}(\varphi^n) \le 2\, \mathcal{D}_{C_{rr_n}}(\bar{X} - X^n) + 2\pi\,(r_n - r), \tag{3.13}$$

$\forall\, n \in \mathbb{N}$. Moreover by Proposition **??** and our choice of $r \in (0,1)$ we obtain an increasing sequence $\{n_k\} \subset \mathbb{N}$ with

$$\mathcal{D}_{C_{rr_{n_k}}}(\bar{X} - X^{n_k}) \longrightarrow 0 \qquad \textit{for } k \to \infty. \tag{3.14}$$

Combining this with (**??**), $\mathcal{D}(X^{n_k}) \le$ const. by hypothesis, $r_n \to r$ and the Hölder inequality we arrive at

$$\mid \int_{C_{rr_{n_k}}} DX^{n_k} \cdot D\varphi^{n_k}\, dudv \mid \le const. \sqrt{\mathcal{D}_{C_{rr_{n_k}}}(\varphi^{n_k})} \longrightarrow 0 \qquad \textit{for } k \to \infty. \tag{3.15}$$

Moreover by (**??**) we estimate $\delta Z^{n_k} = X^{n_k}_u \wedge \varphi^{n_k}_v + \varphi^{n_k}_u \wedge X^{n_k}_v$ on $C_{rr_{n_k}}$ by

$$\mid \delta Z^{n_k} \mid \le 2 \mid DX^{n_k} \mid\mid D\varphi^{n_k} \mid \le 2 \mid DX^{n_k} \mid (\mid D(\bar{X} - X^{n_k}) \mid +1),$$

which implies by the Hölder inequality, (**??**), $\mathcal{D}(X^{n_k}) \le$ const. and $r_{n_k} \to r$:

$$\int_{C_{rr_{n_k}}} \mid \delta Z^{n_k} \mid dudv \le const. \sqrt{\mathcal{D}_{C_{rr_{n_k}}}(\bar{X} - X^{n_k})} + const. \sqrt{r_{n_k} - r} \longrightarrow 0. \tag{3.16}$$

Hence by $\mid \nabla F \mid \le m_2$ on $\mathbb{R}^3 \setminus \{0\}$ and $F(z) \le m_2 \mid z \mid \; \forall z \in \mathbb{R}^3$ we arrive at

$$\mid \int_{\mathcal{R}^{n_k} \cap C_{rr_{n_k}}} \langle \nabla F(Z^{n_k}), \delta Z^{n_k} \rangle\, dudv \mid \le m_2 \int_{C_{rr_{n_k}}} \mid \delta Z^{n_k} \mid dudv \longrightarrow 0 \tag{3.17}$$

$$\textit{and} \qquad \mid \int_{\mathcal{S}^{n_k} \cap C_{rr_{n_k}}} F(\delta Z^{n_k})\, dudv \mid \le m_2 \int_{C_{rr_{n_k}}} \mid \delta Z^{n_k} \mid dudv \longrightarrow 0. \tag{3.18}$$

Now combining (**??**), (**??**) and (**??**) with (**??**) we gain

$$\liminf_{k \to \infty} (\mathcal{I}_{B_r(0)}(\bar{X}) - \mathcal{I}_{B_r(0)}(X^{n_k})) \ge 0. \tag{3.19}$$

On the other hand we have $X^{n_k} \rightharpoonup \bar{X}$ in $H^{1,2}(B, \mathbb{R}^3)$ by Theorem **??**, hence by the weak lower semicontinuity of $\mathcal{I}_{B_r(0)}$ and (**??**) we finally obtain

$$\limsup_{k \to \infty} \mathcal{I}_{B_r(0)}(X^{n_k}) \le \mathcal{I}_{B_r(0)}(\bar{X}) \le \liminf_{k \to \infty} \mathcal{I}_{B_r(0)}(X^{n_k}).$$

Due to this result and the weak convergence of $\{X^{n_k}\}$ to $\bar{X}$ we infer from Lemma 6 in Chapter 4 of [?]:

$$\mathcal{D}_{B_r(0)}(X^{n_k}) \longrightarrow \mathcal{D}_{B_r(0)}(\bar{X}) \qquad \textit{for } k \to \infty,$$

which again combined with the weak convergence in $H^{1,2}(B, \mathbb{R}^3)$ and the convergence in $C^0(\bar{B}, \mathbb{R}^3)$ of $\{X^{n_k}\}$ to $\bar{X}$ finally yields the assertion in (**??**) for a.e. $r \in (0,1)$, thus for every $r \in (0,1)$.

$\diamond$

Now combining the above theorem with Lemma 2 of Section 2.5 in [?], p. 23, we prove Theorem 10.3 of [?], p. 560.

Theorem 3.2 *The surface $\bar{X}$ of Theorem **??** is an $\mathcal{I}$-surface again.*

Proof: Let $r \in (0,1)$ be arbitrarily chosen. We rename the sequence $\{n_k\}$ in (**??**) into $\{n\}$ and define $\mathcal{S}_r^n := \mathcal{S}(X^n) \cap B_r(0)$, $\mathcal{R}_r^n := \mathcal{R}(X^n) \cap B_r(0)$,

$$\sigma^n := \mathcal{S}_r^n \setminus \mathcal{S}_r = \mathcal{R}_r \setminus \mathcal{R}_r^n \quad \textit{and} \quad \tau^n := \mathcal{S}_r \setminus \mathcal{S}_r^n = \mathcal{R}_r^n \setminus \mathcal{R}_r$$

and moreover $Z := \bar{X}_u \wedge \bar{X}_v$, $Z^n := X_u^n \wedge X_v^n$, $\delta Z := \bar{X}_u \wedge \varphi_v + \varphi_u \wedge \bar{X}_v$ and $\delta Z^n := X_u^n \wedge \varphi_v + \varphi_u \wedge X_v^n$ for some arbitrarily chosen $\varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$. The decisive step consists of the proof of

$$\delta^+ \mathcal{F}_{B_r(0)}(\bar{X}, \varphi) \geq \liminf_{n \to \infty} \delta^+ \mathcal{F}_{B_r(0)}(X^n, \varphi) \tag{3.20}$$

$\forall \varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$. Firstly we estimate:

$$\begin{aligned} | Z^n - Z | = | X_u^n \wedge X_v^n - \bar{X}_u \wedge \bar{X}_v | &= | X_u^n \wedge (X_v^n - \bar{X}_v) - \bar{X}_v \wedge (X_u^n - \bar{X}_u) | \\ &\leq (| DX^n | + | D\bar{X} |) \, | D(X^n - \bar{X}) | \, . \end{aligned}$$

From this we infer by the Hölder inequality and (**??**):

$$\int_{B_r(0)} | Z^n - Z | \ dudv \leq 2 \left(\sqrt{\mathcal{D}_{B_r(0)}(X^n)} + \sqrt{\mathcal{D}_{B_r(0)}(\bar{X})} \right) \sqrt{\mathcal{D}_{B_r(0)}(X^n - \bar{X})} \longrightarrow 0. \tag{3.21}$$

Next we estimate:

$$| \delta Z^n - \delta Z | = | (X_u^n - \bar{X}_u) \wedge \varphi_v + \varphi_u \wedge (X_v^n - \bar{X}_v) | \leq 2 \ | D\varphi || D(X^n - \bar{X}) |, \tag{3.22}$$

which implies again by (**??**):

$$\int_{B_r(0)} | \delta Z^n - \delta Z | \ dudv \leq 4 \sqrt{\mathcal{D}_{B_r(0)}(\varphi) \ \mathcal{D}_{B_r(0)}(X^n - \bar{X})} \longrightarrow 0. \tag{3.23}$$

Next we split up the integrals on the sets $\mathcal{R}_r^n$ and $\mathcal{R}_r$ occuring in (**??**) resp. (**??**):

$$\begin{aligned} &\int_{\mathcal{R}_r^n} \langle \nabla F(Z^n), \delta Z^n \rangle \, dudv - \int_{\mathcal{R}_r} \langle \nabla F(Z), \delta Z \rangle \, dudv \\ &= \int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n), \delta Z^n \rangle + \chi_{\tau^n} \langle \nabla F(Z^n), \delta Z^n \rangle \\ &\qquad - \chi_{\mathcal{R}_r \cap \mathcal{R}_r^n} \langle \nabla F(Z), \delta Z \rangle - \chi_{\sigma^n} \langle \nabla F(Z), \delta Z \rangle \, dudv \\ &= \int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n), \delta Z^n - \delta Z \rangle + \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n) - \nabla F(Z), \delta Z \rangle \, dudv \\ &\qquad - \int_{B_r(0)} \chi_{\sigma^n} \langle \nabla F(Z), \delta Z \rangle \, dudv + \int_{B_r(0)} \chi_{\tau^n} \langle \nabla F(Z^n), \delta Z^n \rangle \, dudv. \end{aligned} \tag{3.24}$$

For the first integral in (**??**) we have by $\mid \nabla F \mid \leq m_2$ on $\mathbb{R}^3 \setminus \{0\}$ and (**??**):

$$\mid \int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n), \delta Z^n - \delta Z \rangle \, dudv \mid \leq m_2 \int_{B_r(0)} \mid \delta Z^n - \delta Z \mid \, dudv \longrightarrow 0. \quad (3.25)$$

Now we are going to examine the second integral in (**??**). By (**??**) we obtain a subsequence $\{Z^{n_k}\}$ for which

$$Z^{n_k}(w) \longrightarrow Z(w) \qquad \textit{for a.e. } w \in B_r(0). \quad (3.26)$$

We rename $\{n_k\}$ into $\{n\}$ again and shall consider this sequence henceforth. Now we choose some point $w \in B_r(0) \setminus \mathcal{N}$ arbitrarily, where $\mathcal{N} \subset B_r(0)$ is the subset of $\mathcal{L}^2$-measure zero on which (**??**) does not hold and δZ does not exist, and distinguish between the following two cases:
Case (1): There holds $w \in \mathcal{R}_r^{n_j} \cap \mathcal{R}_r$ for an increasing sequence $\{n_j\} \subset \mathbb{N}$. Then we obtain by (**??**) and the continuity of ∇F on $\mathbb{R}^3 \setminus \{0\}$:

$$\nabla F(Z^{n_j})(w) \longrightarrow \nabla F(Z)(w) \qquad \textit{for } j \to \infty.$$

As we have $w \notin \mathcal{R}_r^n \cap \mathcal{R}_r$, i.e. $\chi_{\mathcal{R}_r^n \cap \mathcal{R}_r}(w) = 0$, for $n \in \mathbb{N} \setminus \{n_j\}$ we can conclude:

$$\chi_{\mathcal{R}_r^n \cap \mathcal{R}_r}(w) \left(\nabla F(Z^n)(w) - \nabla F(Z)(w) \right) \delta Z(w) \longrightarrow 0 \qquad \textit{for } n \to \infty. \quad (3.27)$$

Case (2): There exists some number $N \in \mathbb{N}$ such that $w \notin \mathcal{R}_r^n \cap \mathcal{R}_r$, i.e. $\chi_{\mathcal{R}_r^n \cap \mathcal{R}_r}(w) = 0$, $\forall\, n > N$. In this case we obtain (**??**) immediately.
Hence, we gain (**??**) for a.e. $w \in B_r(0)$. Furthermore we see due to $\mid \nabla F \mid \leq m_2$ on $\mathbb{R}^3 \setminus \{0\}$:

$$\mid \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \left(\nabla F(Z^n) - \nabla F(Z) \right) \delta Z \mid \leq 2 m_2 \mid \delta Z \mid \in L^1(B_r(0)),$$

$\forall\, n \in \mathbb{N}$. Therefore the Lebesgue convergence theorem finally implies that

$$\int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \left(\nabla F(Z^n) - \nabla F(Z) \right) \delta Z \, dudv \longrightarrow 0. \quad (3.28)$$

Now we examine the third integral in (**??**). We have $Z^n \equiv 0$ a.e. on $\sigma^n = \mathcal{S}_r^n \setminus \mathcal{S}_r$. Hence, we obtain by (**??**):

$$\int_{B_r(0)} \chi_{\sigma^n} \mid Z \mid \, dudv = \int_{B_r(0)} \chi_{\sigma^n} \mid Z - Z^n \mid \, dudv \longrightarrow 0.$$

Thus we gain an increasing sequence $\{n_k\}$ such that

$$\chi_{\sigma^{n_k}}(w) \mid Z(w) \mid \longrightarrow 0 \qquad \textit{for a.e. } w \in B_r(0).$$

Renaming $\{n_k\}$ into $\{n\}$ again and noticing that $\mid Z \mid > 0$ on $\sigma^n \subset \mathcal{R}_r$, $\forall\, n \in \mathbb{N}$, we arrive at $\chi_{\sigma^n}(w) \to 0$ for a.e. $w \in B_r(0)$, i.e.

$$\mathcal{L}^2(\sigma^n) \longrightarrow 0 \qquad \textit{for } n \to \infty. \quad (3.29)$$

As we know $\langle \nabla F(Z), \delta Z \rangle \in L^1(\mathcal{R}_r)$ due to $| \nabla F | \leq m_2$ on $\mathbb{R}^3 \setminus \{0\}$ we infer from the absolute continuity of the Lebesgue integral that

$$\int_{B_r(0)} \chi_{\sigma^n} \langle \nabla F(Z), \delta Z \rangle \, dudv \longrightarrow 0 \qquad \textit{for } n \to \infty. \tag{3.30}$$

Now the fourth integral in (**??**) has to be examined simultaneously with the remaining integrals on the sets $\mathcal{S}_r^n$ and $\mathcal{S}_r$ occuring in (**??**) resp. (**??**), which we also split up:

$$\begin{aligned} \int_{\mathcal{S}_r^n} F(\delta Z^n) \, dudv - \int_{\mathcal{S}_r} F(\delta Z) \, dudv = \int_{\mathcal{S}_r^n \cap \mathcal{S}_r} F(\delta Z^n) \, dudv + \int_{\sigma^n} F(\delta Z^n) \, dudv \\ - \int_{\mathcal{S}_r \cap \mathcal{S}_r^n} F(\delta Z) \, dudv - \int_{\tau^n} F(\delta Z) \, dudv. \end{aligned} \tag{3.31}$$

Since F is Lipschitz continuous with Lip.-const.$= m_2$ by Lemma 3.2 in [**?**] we firstly obtain together with (**??**) that

$$\int_{B_r(0)} | F(\delta Z^n) - F(\delta Z) | \, dudv \leq m_2 \int_{B_r(0)} | \delta Z^n - \delta Z | \, dudv \longrightarrow 0, \tag{3.32}$$

which estimates the difference of the first and third integral in (**??**) in particular. Now (**??**) yields a subsequence $\{\delta Z^{n_k}\}$ such that $F(\delta Z^{n_k})(w) \to F(\delta Z)(w)$ for a.e. $w \in B_r(0)$ and by Vitali's theorem we know that $\forall \, \epsilon > 0$ there exists some $\delta(\epsilon)$ such that

$$\int_E F(\delta Z^{n_k}) \, dudv < \epsilon, \qquad \textit{if } \mathcal{L}^2(E) < \delta(\epsilon) \tag{3.33}$$

uniformly $\forall \, k \in \mathbb{N}$. Again we rename $\{n_k\}$ into $\{n\}$. As (**??**) means that for any given $\delta > 0$ there is some $N(\delta)$ with $\mathcal{L}^2(\sigma^n) < \delta \quad \forall \, n > N(\delta)$ we conclude together with (**??**) that

$$\int_{\sigma^n} F(\delta Z^n) \, dudv \longrightarrow 0 \qquad \textit{for } n \to \infty. \tag{3.34}$$

Now there only remain the fourth integrals in (**??**) and (**??**). On $\tau^n = \mathcal{R}_r^n \setminus \mathcal{R}_r$ we obtain by the convexity of $F \in C^1(\mathbb{R}^3 \setminus \{0\})$ and its positive homogeneity:

$$\begin{aligned} \langle \nabla F(Z^n), \delta Z^n \rangle &\leq F(\delta Z^n) - F(Z^n) + \langle \nabla F(Z^n), Z^n \rangle \\ &= F(\delta Z^n) - F(Z^n) + F(Z^n) = F(\delta Z^n). \end{aligned}$$

Hence we obtain together with (**??**):

$$\int_{\tau^n} \langle \nabla F(Z^n), \delta Z^n \rangle - F(\delta Z) \, dudv \leq \int_{\tau^n} F(\delta Z^n) - F(\delta Z) \, dudv \longrightarrow 0. \tag{3.35}$$

Now terming $\{n_j\} \subset \mathbb{N}$ the resulting increasing sequence, having selected subsequences several times after (**??**), and collecting (**??**), (**??**), (**??**), (**??**), (**??**), (**??**), (**??**), (**??**) and

(**??**) we finally conclude:

$$
\begin{aligned}
&\liminf_{n\to\infty} (\delta^+\mathcal{F}_{B_r(0)}(X^n,\varphi) - \delta^+\mathcal{F}_{B_r(0)}(\bar{X},\varphi)) \\
&\leq \liminf_{j\to\infty} (\delta^+\mathcal{F}_{B_r(0)}(X^{n_j},\varphi) - \delta^+\mathcal{F}_{B_r(0)}(\bar{X},\varphi)) \\
&= \liminf_{j\to\infty} \int_{\mathcal{T}^{n_j}} \langle \nabla F(Z^{n_j}), \delta Z^{n_j}\rangle - F(\delta Z)\, dudv \leq 0
\end{aligned}
$$

$\forall\, \varphi \in \mathring{H}^{1,2}(B_r(0),\mathbb{R}^3)$, which proves (**??**). Moreover we obtain immediately by (**??**) (for the same sequence as in (**??**)):

$$
\delta\mathcal{D}_{B_r(0)}(X^n,\varphi) = \int_{B_r(0)} DX^n \cdot D\varphi\, dudv \longrightarrow \int_{B_r(0)} D\bar{X}\cdot D\varphi\, dudv = \delta\mathcal{D}_{B_r(0)}(\bar{X},\varphi).
$$

Hence, together with (**??**) and (**??**) we arrive at

$$
\delta^+\mathcal{I}_{B_r(0)}(\bar{X},\varphi) \geq \liminf_{n\to\infty}\ \delta^+\mathcal{I}_{B_r(0)}(X^n,\varphi) \geq 0, \tag{3.36}
$$

$\forall\, \varphi \in \mathring{H}^{1,2}(B_r(0),\mathbb{R}^3)$, where we used that the $\mathcal{I}$-surfaces X^n satisfy $\delta^+\mathcal{I}_{B_r(0)}(X^n,\varphi) \geq 0$ $\forall\, \varphi \in \mathring{H}^{1,2}(B_r(0),\mathbb{R}^3)$ by Lemma 2 in Section 2.5 in [**?**] and $F(0) = 0$. Moreover for any $\varphi \in C_c^\infty(B,\mathbb{R}^3)$ we have $\mathrm{supp}(\varphi) \subset\subset B_r(0)$ for some $r \in (0,1)$, hence we gain by (**??**) and $F(0) = 0$:

$$
\delta^+\mathcal{I}(\bar{X},\varphi) \geq 0 \qquad \forall\, \varphi \in C_c^\infty(B,\mathbb{R}^3). \tag{3.37}
$$

Now we consider some arbitrarily fixed $\varphi \in \mathring{H}^{1,2}(B,\mathbb{R}^3)$ and some approximating sequence $\{\varphi^j\} \subset C_c^\infty(B,\mathbb{R}^3)$, i.e.

$$
\varphi^j \longrightarrow \varphi \qquad \textit{in}\ \mathring{H}^{1,2}(B,\mathbb{R}^3). \tag{3.38}
$$

We set $\delta Z^j := \bar{X}_u \wedge \varphi_v^j + \varphi_u^j \wedge \bar{X}_v$ and estimate as in (**??**):

$$
\mid \delta Z^j - \delta Z \mid \leq 2 \mid D\bar{X} \mid\mid D(\varphi^j - \varphi) \mid, \tag{3.39}
$$

which implies by (**??**):

$$
\int_B \mid \delta Z^j - \delta Z \mid\ dudv \leq 4\sqrt{\mathcal{D}(\bar{X})\ \mathcal{D}(\varphi^j - \varphi)} \longrightarrow 0.
$$

Therefore we obtain as in (**??**):

$$
\mid \int_{\mathcal{R}} \langle \nabla F(Z), \delta Z^j - \delta Z\rangle\, dudv \mid \leq m_2 \int_{\mathcal{R}} \mid \delta Z^j - \delta Z \mid\ dudv \longrightarrow 0, \tag{3.40}
$$

and as in (**??**):

$$
\mid \int_{\mathcal{S}} F(\delta Z^j) - F(\delta Z)\, dudv \mid \leq m_2 \int_{\mathcal{S}} \mid \delta Z^j - \delta Z \mid\ dudv \longrightarrow 0. \tag{3.41}
$$

Moreover we have

$$\int_B D\bar{X} \cdot D\varphi^j \, dudv \longrightarrow \int_B D\bar{X} \cdot D\varphi \, dudv \tag{3.42}$$

immediately by (**??**). Hence, recalling (**??**) and combining (**??**), (**??**) and (**??**) with (**??**) we finally arrive at

$$\delta^+\mathcal{I}(\bar{X}, \varphi) = \lim_{j\to\infty} \delta^+\mathcal{I}(\bar{X}, \varphi^j) \geq 0$$

$\forall\, \varphi \in \mathring{H}^{1,2}(B, \mathbb{R}^3)$, which exposes $\bar{X}$ to be an $\mathcal{I}$-surface by Lemma 2 in Section 2.5 in [**?**].

$\diamond$

4 Continuity theorems for $\mathcal{A}$, $\mathcal{J}$ and $\mathcal{I}$

The aim of this chapter are precise proofs of the "continuity theorems" 11.1 and 12.2 in [**?**] for the functionals $\mathcal{J}$ and $\mathcal{I}$ in application to sequences of $\mathcal{I}$-surfaces that converge in $C^0(\bar{B}, \mathbb{R}^3)$, see Theorem **??** and Corollary **??** below. In fact these results are easily derived from a deep "continuity theorem" for the area functional $\mathcal{A}$ applied to harmonic surfaces on ring regions $C_{\rho 1} = B_1(0) \setminus \overline{B_\rho(0)}$ with convergent boundary values in $(C^0 \cap BV)(\partial C_{\rho 1}, \mathbb{R}^3)$ due to Morse and Tompkins in [**?**].

4.1 Continuity theorem for $\mathcal{A}$ by Morse and Tompkins

In this section we present a detailed proof of Morse's and Tompkin's "continuity theorem" for $\mathcal{A}$ applied to harmonic surfaces on ring regions in [**?**] which is precisely

Theorem 4.1 *Let* $\{\varphi_1^n\} \subset (C^0 \cap H^{\frac{1}{2},2} \cap BV)(\partial B_1(0), \mathbb{R}^3)$ *and* $\{\varphi_\rho^n\} \subset (C^0 \cap H^{\frac{1}{2},2} \cap BV)(\partial B_\rho(0), \mathbb{R}^3)$ *be prescribed boundary values on* $\partial C_{\rho 1} = \partial B_1(0) \cup \partial B_\rho(0)$ *for some* $\rho \in (0,1)$ *such that*

$$\varphi_1^n \longrightarrow \varphi_1 \quad \text{in } C^0(\partial B_1(0), \mathbb{R}^3) \quad \text{and} \quad \mathcal{L}(\varphi_1^n) \longrightarrow \mathcal{L}(\varphi_1), \tag{4.1}$$

$$\varphi_\rho^n \longrightarrow \varphi_\rho \quad \text{in } C^0(\partial B_\rho(0), \mathbb{R}^3) \quad \text{and} \quad \mathcal{L}(\varphi_\rho^n) \longrightarrow \mathcal{L}(\varphi_\rho). \tag{4.2}$$

($\mathcal{L}$:=lenght) for some functions $\varphi_1 \in (C^0 \cap H^{\frac{1}{2},2} \cap BV)(\partial B_1(0), \mathbb{R}^3)$ *and* $\varphi_\rho \in (C^0 \cap H^{\frac{1}{2},2} \cap BV)(\partial B_\rho(0), \mathbb{R}^3)$*. Then we prove for the harmonic extensions* H^n *resp.* H *of the boundary values* $(\varphi_1^n, \varphi_\rho^n)$ *resp.* $(\varphi_1, \varphi_\rho)$ *onto* $\bar{C}_{\rho 1}$ *that*

$$\mathcal{A}_{C_{\rho 1}}(H^n) \longrightarrow \mathcal{A}_{C_{\rho 1}}(H) \qquad \text{for } n \to \infty. \tag{4.3}$$

Before giving the proof we need several fundamental formulas based on the Poisson representation (in polar coordinates) of harmonic surfaces on discs $B_s(0)$, for $s \in (0,1]$.

Proposition 4.1 *Let* h *denote the harmonic extension of some prescribed boundary values* $\varphi \in (C^0 \cap BV)(\partial B_s(0), \mathbb{R}^3)$ *onto the disc* $\bar{B}_s(0)$*, for some* $s \in (0,1]$*, then we have the following Poisson formulas:*

$$h(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\alpha)\frac{s^2-r^2}{s^2-2sr\cos(\alpha-\theta)+r^2}\,d\alpha, \tag{4.4}$$

$$h_\theta(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2-r^2}{s^2-2sr\cos(\alpha-\theta)+r^2}\,d\varphi(\alpha), \tag{4.5}$$

$$h_r(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} \frac{s\,\sin(\alpha-\theta)}{s^2-2rs\cos(\alpha-\theta)+r^2}\,d\varphi(\alpha) \tag{4.6}$$

$$\text{and} \qquad \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2-r^2}{s^2-2sr\cos(\theta-\alpha)+r^2}\,d\alpha \equiv 1 \tag{4.7}$$

$\forall\, r \in (0,s),\ \forall\, \theta \in [0,2\pi]$.

Proof: (**??**) is well known. (**??**) follows by means of (**??**), commuting $\frac{\partial}{\partial\theta}$ with $\int_0^{2\pi}\dots d\alpha$ by [**?**], p. 146, transforming the Lebesgue integral into a Stieltjes integral by [**?**], p. 177, and integration by parts by [**?**], p. 161:

$$\begin{aligned} h_\theta(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(\alpha)\frac{\partial}{\partial\theta}\Big(\frac{s^2-r^2}{s^2-2sr\cos(\alpha-\theta)+r^2}\Big)\,d\alpha \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \varphi(\alpha)\frac{\partial}{\partial\alpha}\Big(\frac{s^2-r^2}{s^2-2sr\cos(\alpha-\theta)+r^2}\Big)\,d\alpha \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \varphi(\alpha)\,d\Big(\frac{s^2-r^2}{s^2-2sr\cos(\alpha-\theta)+r^2}\Big) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2-r^2}{s^2-2sr\cos(\alpha-\theta)+r^2}\,d\varphi(\alpha). \end{aligned}$$

(**??**): Firstly we introduce polar coordinates on $B_s(0)$ by $\phi(r,\theta) := r\,e^{i\theta}$ and remember the Cauchy-Riemann DE in polar coordinates of the components u^1, u^2 of a holomorphic function u on $B_s(0)$ (terming $u \circ \phi$ again u):

$$r\,u^1_r = u^2_\theta, \qquad u^1_\theta = -r\,u^2_r \tag{4.8}$$

$\forall\, r \in (0,s)$, $\forall\, \theta \in [0,2\pi]$. In particular, u^1 and u^2 are harmonic on $B_s(0)$, i.e.

$$\triangle_\phi(u^i) := \frac{1}{r}(r\,u^i_r)_r + \frac{1}{r^2}u^i_{\theta\theta} \equiv 0, \tag{4.9}$$

$i = 1,2$. Vice versa, given a harmonic function u on $B_s(0)$ the functions $r\,u_r, -u_\theta$ are conjugate to each other, i.e. satisfy (**??**) due to

$$r\,(r\,u_r)_r \overset{(??)}{=} (-u_\theta)_\theta \qquad \text{and} \qquad (r\,u_r)_\theta = r\,u_{r\theta} = -r\,(-u_\theta)_r. \tag{4.10}$$

Now we show that the functions

$$k^1(r,\theta) := -\frac{s^2-r^2}{s^2-2sr\cos(\alpha-\theta)+r^2}, \qquad k^2(r,\theta) := \frac{2rs\,\sin(\alpha-\theta)}{s^2-2sr\cos(\alpha-\theta)+r^2} \tag{4.11}$$

are conjugate to each other, where α is arbitrarily fixed in $[0, 2\pi]$. To this end we set $\Omega_s(r, \alpha, \theta) := s^2 - 2sr\cos(\alpha - \theta) + r^2$ and calculate:

$$-k_r^1 = \frac{\Omega_s(r,\alpha,\theta)(-2r) - (-2s\,\cos(\alpha-\theta) + 2r)(s^2 - r^2)}{\Omega_s(r,\alpha,\theta)^2}$$
$$= \frac{2r^2 s\cos(\alpha-\theta) - 4rs^2 + 2s^3\,\cos(\alpha-\theta)}{\Omega_s(r,\alpha,\theta)^2}$$

and

$$k_\theta^2 = \frac{\Omega_s(r,\alpha,\theta)(-2rs\,\cos(\alpha-\theta)) - (-(2rs\,\sin(\alpha-\theta))^2)}{\Omega_s(r,\alpha,\theta)^2}$$
$$= \frac{4r^2s^2 - 2rs^3\cos(\alpha-\theta) - 2r^3 s\,\cos(\alpha-\theta)}{\Omega_s(r,\alpha,\theta)^2} = r\,k_r^1.$$

Furthermore we have

$$k_r^2 = \frac{\Omega_s(r,\alpha,\theta)\,2s\,\sin(\alpha-\theta) - (-2s\,\cos(\alpha-\theta) + 2r)\,2rs\,\sin(\alpha-\theta)}{\Omega_s(r,\alpha,\theta)^2}$$
$$= \frac{2s^3\sin(\alpha-\theta) + 2r^2 s\,\sin(\alpha-\theta) - 4r^2 s\,\sin(\alpha-\theta)}{\Omega_s(r,\alpha,\theta)^2} = \frac{2(s^2-r^2)s\,\sin(\alpha-\theta)}{\Omega_s(r,\alpha,\theta)^2}$$

and

$$k_\theta^1 = -\frac{2rs\,\sin(\alpha-\theta)(s^2-r^2)}{\Omega_s(r,\alpha,\theta)^2} = -r\,k_r^2$$

$\forall\, r \in (0, s)$, $\forall\, \theta, \alpha \in [0, 2\pi]$. Thus together with (**??**) we see that

$$K := \frac{1}{2\pi}\int_0^{2\pi} k^2\,d\varphi(\alpha) \qquad \text{and} \qquad -h_\theta = \frac{1}{2\pi}\int_0^{2\pi} k^1\,d\varphi(\alpha)$$

are conjugate to each other, where we used that $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial r}$ commute with $\int_0^{2\pi}\ldots d\varphi(\alpha)$ by [**?**], p. 146. On the other hand, since h is harmonic on $B_s(0)$ we know by (**??**) that $r\,h_r$ and $-h_\theta$ are conjugate to each other, as well. Hence, recalling (**??**) it follows that $\nabla_{(r,\theta)}(r\,h_r) \equiv \nabla_{(r,\theta)}K \;\; \forall\, r \in (0, s)$, $\forall\, \theta \in [0, 2\pi]$, implying $r\,h_r \equiv K$+const.. Furthermore as h_r is bounded on a punctured neighborhood $B_\epsilon(0) \setminus \{0\}$ of 0, $\epsilon < s$, we have $r\,h_r(r,\theta) \longrightarrow 0$ for $r \searrow 0$, and since $k^2(r,\theta) \longrightarrow 0$ for $r \searrow 0$ and

$$|\, k^2(r,\theta) \,| \le const.(\epsilon) \qquad \forall\, r \in (0,\epsilon),\ \epsilon < s, \qquad \forall\, \alpha, \theta \in [0, 2\pi]$$

we infer by the theorem of dominated convergence for Stieltjes integrals in [**?**], p. 146, that $K(r,\theta) \longrightarrow 0$ for $r \searrow 0$. Hence, we arrive at $r\,h_r \equiv K \;\; \forall\, r \in (0, s)$, $\forall\, \theta \in [0, 2\pi]$, which yields (**??**) by (**??**).
Finally we obtain (**??**) by substitution and by the periodicity of cos:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{s^2 - r^2}{s^2 - 2sr\cos(\theta-\alpha) + r^2}\,d\alpha = -\frac{1}{2\pi}\int_\theta^{-2\pi+\theta} \frac{s^2 - r^2}{s^2 - 2sr\cos(z) + r^2}\,dz$$
$$= -\frac{1}{2\pi}\int_0^{-2\pi} \frac{s^2 - r^2}{s^2 - 2sr\cos(x+\theta) + r^2}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2 - r^2}{s^2 - 2sr\cos(x+\theta) + r^2}\,dx.$$

Now applying (**??**) to $\varphi \equiv 1$ we see by the maximum principle for harmonic functions that the last integral yields the value of the constant function $H \equiv 1$ in the point $(r, -\theta)$, which proves the assertion.

$\diamond$

Using these formulas and several ideas of [**?**] Courant proved in [**?**], pp. 134–139:

Proposition 4.2 *Let* $\{\varphi^n\} \subset (C^0 \cap H^{\frac{1}{2},2} \cap BV)(\partial B, \mathbb{R}^3)$ *be prescribed boundary values on* ∂B *such that*

$$\varphi^n \longrightarrow \varphi \qquad \text{in } C^0(\partial B, \mathbb{R}^3) \qquad \text{and} \qquad \mathcal{L}(\varphi^n) \longrightarrow \mathcal{L}(\varphi) \tag{4.12}$$

($\mathcal{L}$:=lenght) for some function $\varphi \in (C^0 \cap BV)(\partial B, \mathbb{R}^3)$*. Then we prove for the harmonic extensions* h^n *of the boundary values* φ^n *onto* $\bar{B}$ *that for any* $\epsilon > 0$ *there is some* $R(\epsilon) \in (0,1)$ *such that*

$$\mathcal{A}_{C_{\varrho 1}}(h^n) < \epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.13}$$

if $\varrho \in (R(\epsilon), 1)$.

Proof: Firstly we infer from (**??**) and (**??**) for $s = 1$:

$$(h^n_r \wedge h^n_\theta)(r,\theta) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{(1-r^2)\, 2\sin(\alpha-\theta)}{\Omega_1(r,\alpha,\theta)\, \Omega_1(r,\beta,\theta)}\, d\varphi^n(\alpha) \wedge d\varphi^n(\beta), \tag{4.14}$$

$\forall\, n \in \mathbb{N}$, $\forall\, (r,\theta) \in (0,1) \times [0, 2\pi]$. Now interchanging the variables α and β in (**??**) and noting that $d\varphi^n(\alpha) \wedge d\varphi^n(\beta) = -d\varphi^n(\beta) \wedge d\varphi^n(\alpha)$ we obtain:

$$(h^n_r \wedge h^n_\theta)(r,\theta) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{(1-r^2)\,(\sin(\alpha-\theta) - \sin(\beta-\theta))}{\Omega_1(r,\alpha,\theta)\, \Omega_1(r,\beta,\theta)}\, d\varphi^n(\alpha) \wedge d\varphi^n(\beta). \tag{4.15}$$

If we use $\sin\zeta - \sin\xi = 2\cos\left(\frac{\zeta+\xi}{2}\right) \sin\left(\frac{\zeta-\xi}{2}\right)$ for $\zeta := \alpha - \theta$ and $\xi := \beta - \theta$ we gain $|\sin(\alpha-\theta) - \sin(\beta-\theta)| \leq 2\,|\sin\left(\frac{\alpha-\beta}{2}\right)|$. Hence, applying the "triangle inequality" for Stieltjes integrals (see [**?**], p. 144) we derive from this and (**??**):

$$|\,(h^n_r \wedge h^n_\theta)(r,\theta)\,| \leq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{2\,(1-r^2)\;|\sin\left(\frac{\alpha-\beta}{2}\right)|}{\Omega_1(r,\alpha,\theta)\, \Omega_1(r,\beta,\theta)}\, d\,|\,\varphi^n(\alpha)\,|\;d\,|\,\varphi^n(\beta)\,|\,. \tag{4.16}$$

By the invariance of $\mathcal{A}$ with respect to diffeomorphic transformations of its parameter domain, (**??**) and Fubini's theorem for Stieltjes integrals (see [**?**], p. 151) we obtain the estimate

$$\mathcal{A}_{C_{\varrho\tilde{\varrho}}}(h^n) = \int_0^{2\pi} \int_\varrho^{\tilde{\varrho}} |\,h^n_r \wedge h^n_\theta\,|\; dr d\theta \tag{4.17}$$

$$\leq \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_\varrho^{\tilde{\varrho}} \int_0^{2\pi} \frac{(1-r^2)\;|\sin\left(\frac{\alpha-\beta}{2}\right)|}{\Omega_1(r,\alpha,\theta)\, \Omega_1(r,\beta,\theta)}\, d\theta dr\, d\,|\,\varphi^n(\alpha)\,|\;d\,|\,\varphi^n(\beta)\,|,$$

for any $\varrho < \tilde{\varrho} \in (0,1)$ and $\forall\, n \in \mathbb{N}$. Now we calculate the inner integral $\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{\Omega_1(r,\alpha,\theta)\,\Omega_1(r,\beta,\theta)}\, d\theta$. To this end we fix some $r \in (0,1)$ and $\alpha, \beta \in [0, 2\pi]$ arbitrarily and interpret $b(\theta) := \Omega_1(r,\beta,\theta)^{-1}$ as boundary values along ∂B of the unique harmonic extension $u(se^{i\theta})$ onto $\bar{B}$ whose evaluation in $re^{i\alpha}$ is given by its Poisson representation which is just the considered integral. Noting that

$$0 < (s-r)^2 = s^2 - 2sr + r^2 \leq s^2 - 2sr\cos(\beta - \theta) + r^2 \tag{4.18}$$

for $s \neq r$ we know that the Poisson kernel

$$k(s,\theta) := \frac{s^2 - r^2}{s^2 - 2sr\cos(\beta-\theta) + r^2} = \frac{s^2 - r^2}{\Omega_s(r,\beta,\theta)}$$

is a harmonic function, i.e. satisfies (**??**), especially for $s > r$. One easily calculates for its Kelvin-transform

$$k^*(s,\theta) := -k\Big(\frac{1}{s}, \theta\Big) \qquad \textit{for}\;\; s \in (0,1] \tag{4.19}$$

by the Beltrami-Laplace operator in (**??**):

$$\triangle_\phi k^*(s,\theta) = -\frac{1}{s^4}\,\triangle_\phi k\Big(\frac{1}{s},\theta\Big) \equiv 0 \qquad \forall\, s \in (0,1),\;\; \forall\, \theta \in [0, 2\pi]. \tag{4.20}$$

Hence, the function

$$u(se^{i\theta}) := -\frac{1}{1-r^2}\, k^*(s,\theta) = \frac{1}{1-r^2}\,\frac{\frac{1}{s^2} - r^2}{\frac{1}{s^2} - \frac{2}{s}\, r\, \cos(\beta-\theta) + r^2}$$

is harmonic on $B \setminus \{0\}$ and satisfies $u(w) \longrightarrow \frac{1}{1-r^2}$ for $|\, w \,| \searrow 0$, which guarantees that u possesses a harmonic continuation onto B by setting $u(0) := \frac{1}{1-r^2}$. Furthermore we see that $u(e^{i\theta}) \equiv \Omega_1(r,\beta,\theta)^{-1} \quad \forall\, \theta \in [0, 2\pi]$, hence, as explained above its evaluation in $re^{i\alpha}$ yields the unknown integral:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{\Omega_1(r,\alpha,\theta)\,\Omega_1(r,\beta,\theta)}\, d\theta &= u(re^{i\alpha}) = \frac{1}{1-r^2}\,\frac{\frac{1}{r^2} - r^2}{\frac{1}{r^2} - \frac{2}{r}\, r\, \cos(\beta-\alpha) + r^2} \\
&= \frac{1+r^2}{(1-r^2)(1+r^2)}\,\frac{1-r^4}{1-2r^2\,\cos(\beta-\alpha)+r^4} = \frac{1+r^2}{1-2r^2\,\cos(\beta-\alpha)+r^4},
\end{aligned}$$

$\forall\, r \in (0,1),\;\; \forall\, \alpha \in [0, 2\pi]$ and any $\beta \in [0, 2\pi]$. Now we can calculate the integral

$$S(\alpha,\beta) := \frac{1}{2\pi^2}\int_\varrho^{\tilde{\varrho}}\int_0^{2\pi} \frac{1-r^2}{\Omega_1(r,\alpha,\theta)\,\Omega_1(r,\beta,\theta)}\, d\theta dr = \frac{1}{\pi}\int_\varrho^{\tilde{\varrho}} u(re^{i\alpha})\, dr \tag{4.21}$$

as follows. We have:

$$\begin{aligned}
&\frac{1}{1-2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2} + \frac{1}{1+2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2} \\
&= \frac{(1+2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2) + (1-2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2)}{(1+2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2)\,(1-2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2)} = \frac{2+2r^2}{1+2r^2-4r^2\,\cos^2\left(\frac{\beta-\alpha}{2}\right)+r^4} \\
&= \frac{2+2r^2}{1+2r^2-4r^2\,\frac{1}{2}(1+\cos(\beta-\alpha))+r^4} = 2\,\frac{1+r^2}{1-2r^2\,\cos(\beta-\alpha)+r^4} = 2\,u(re^{i\alpha}).
\end{aligned}$$

Hence, using integration formulas for rational functions we arrive at

$$
\begin{aligned}
S(\alpha,\beta) &= \frac{1}{2\pi}\int_{\varrho}^{\tilde\varrho} \frac{1}{1-2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2} + \frac{1}{1+2r\,\cos\left(\frac{\beta-\alpha}{2}\right)+r^2}\,dr \\
&= \frac{1}{2\pi}\Big(\frac{2}{\sqrt{4-4\,\cos^2\left(\frac{\beta-\alpha}{2}\right)}}\arctan\Big(\frac{2r-2\,\cos\left(\frac{\beta-\alpha}{2}\right)}{\sqrt{4-4\,\cos^2\left(\frac{\beta-\alpha}{2}\right)}}\Big) \\
&\quad + \frac{2}{\sqrt{4-4\,\cos^2\left(\frac{\beta-\alpha}{2}\right)}}\arctan\Big(\frac{2r+2\,\cos\left(\frac{\beta-\alpha}{2}\right)}{\sqrt{4-4\,\cos^2\left(\frac{\beta-\alpha}{2}\right)}}\Big)\Big)\ |_{\varrho}^{\tilde\varrho}
\end{aligned}
$$

$$
= \frac{1}{2\pi\ \mid \sin\left(\frac{\beta-\alpha}{2}\right)\mid}\Big(\arctan\Big(\frac{r-\cos\left(\frac{\beta-\alpha}{2}\right)}{\mid\sin\left(\frac{\beta-\alpha}{2}\right)\mid}\Big)+\arctan\Big(\frac{r+\cos\left(\frac{\beta-\alpha}{2}\right)}{\mid\sin\left(\frac{\beta-\alpha}{2}\right)\mid}\Big)\Big)\ |_{\varrho}^{\tilde\varrho}\ .
$$

If we also use $\arctan x+\arctan y=\arctan\left(\frac{x+y}{1-xy}\right)$ and calculate

$$
\frac{\Big(\big(r-\cos\left(\frac{\beta-\alpha}{2}\right)\big)+\big(r+\cos\left(\frac{\beta-\alpha}{2}\right)\big)\Big)\frac{1}{|\sin\left(\frac{\beta-\alpha}{2}\right)|}}{1-\big(r-\cos\left(\frac{\beta-\alpha}{2}\right)\big)\big(r+\cos\left(\frac{\beta-\alpha}{2}\right)\big)\frac{1}{|\sin\left(\frac{\beta-\alpha}{2}\right)|^2}}
$$

$$
= \frac{\frac{2r}{|\sin\left(\frac{\beta-\alpha}{2}\right)|}}{1-\big(r^2-\cos^2\left(\frac{\beta-\alpha}{2}\right)\big)\frac{1}{|\sin\left(\frac{\beta-\alpha}{2}\right)|^2}} = \frac{2r\ \mid\sin\left(\frac{\beta-\alpha}{2}\right)\mid}{1-r^2}
$$

we finally obtain

$$
S(\alpha,\beta) = \frac{1}{2\pi\ \mid\sin\left(\frac{\beta-\alpha}{2}\right)\mid}\arctan\Big(\frac{2r\ \mid\sin\left(\frac{\beta-\alpha}{2}\right)\mid}{1-r^2}\Big)\ |_{\varrho}^{\tilde\varrho}\ .
$$

Thus combining this with (**??**) we achieve in (**??**):

$$
\mathcal{A}_{C_{\varrho\tilde\varrho}}(h^n) \le \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\arctan\Big(\frac{2r\ \mid\sin\left(\frac{\beta-\alpha}{2}\right)\mid}{1-r^2}\Big)\ |_{\varrho}^{\tilde\varrho}\ d\mid\varphi^n(\alpha)\mid\ d\mid\varphi^n(\beta)\mid \qquad (4.22)
$$

for any $\varrho<\tilde\varrho\in(0,1)$ and $\forall\, n\in\mathbb{N}$. Now we denote

$$
\eta(l) := \sup\{\ Var_{\vartheta}^{\vartheta+l}(\varphi^n)\mid\vartheta\in[0,2\pi]/(0\sim 2\pi),\ \ n\in\mathbb{N}\}
$$

for $l\in(0,\pi)$ and prove that

$$
\eta(l)\longrightarrow 0 \qquad \textit{for}\ l\searrow 0. \qquad (4.23)
$$

We assume the contrary. By (**??**) we know in particular that $\mathcal{L}(\varphi^n)\le$ const., hence there would have to exist sequences $l_j\searrow 0$, $\{\vartheta_j\}\subset[0,2\pi]/(0\sim 2\pi)$, a subsequence $\{\varphi^{n_j}\}$ and some $\epsilon>0$ such that

$$
\mid Var_{\vartheta_j}^{\vartheta_j+l_j}(\varphi^{n_j})-\epsilon\mid\le\mid Var_{\vartheta_j}^{\vartheta_j+l_j}(\varphi^{n_j})-\eta(l_j)\mid+\mid\eta(l_j)-\epsilon\mid\longrightarrow 0 \qquad (4.24)
$$

for $j \to \infty$. By the compactness of $[0,2\pi]/(0 \sim 2\pi)$ there exists a convergent subsequence $\vartheta_{j_k} \to \vartheta^*$. We rename $\{\vartheta_{j_k}\}$ into $\{\vartheta_j\}$ and term γ_j the arc on $\mathbb{S}^1$ that corresponds to $(\vartheta_j, \vartheta_j + l_j)$ via $\exp(i\,\cdot\,)$ and $\kappa_j := \mathbb{S}^1 \setminus \gamma_j$. Due to $\varphi \in (C^0 \cap BV)(\mathbb{S}^1, \mathbb{R}^3)$ we can choose some $\delta > 0$ such that on $\gamma^* := B_\delta(e^{i\vartheta^*}) \cap \mathbb{S}^1$ there holds $\mathcal{L}(\varphi\,|_{\gamma^*}) < \epsilon$ (see [?], p. 250). Furthermore we have by (??)

$$\varphi^{n_j}\,|_{\mathbb{S}^1\setminus\gamma^*} \longrightarrow \varphi\,|_{\mathbb{S}^1\setminus\gamma^*} \qquad \textit{in } C^0(\mathbb{S}^1 \setminus \gamma^*, \mathbb{R}^3).$$

Hence, by the lower semicontinuity of $\mathcal{L}$ w. r. to C^0-convergence (see [?], p. 15) we obtain:

$$\mathcal{L}(\varphi\,|_{\mathbb{S}^1\setminus\gamma^*}) \le \liminf_{j\to\infty} \mathcal{L}(\varphi^{n_j}\,|_{\mathbb{S}^1\setminus\gamma^*}) \le \liminf_{j\to\infty} \mathcal{L}(\varphi^{n_j}\,|_{\kappa_j}),$$

where we used that $\mathbb{S}^1 \setminus \gamma^* \subset \kappa_j$ for sufficiently large j. Thus we obtain together with (??) and (??):

$$\begin{aligned}\mathcal{L}(\varphi) = \lim_{j\to\infty} \mathcal{L}(\varphi^{n_j}) = \lim_{j\to\infty} (\mathcal{L}(\varphi^{n_j}\,|_{\kappa_j}) + \mathcal{L}(\varphi^{n_j}\,|_{\gamma_j})) = \liminf_{j\to\infty} \mathcal{L}(\varphi^{n_j}\,|_{\kappa_j}) + \lim_{j\to\infty} \mathcal{L}(\varphi^{n_j}\,|_{\gamma_j}) \\ = \liminf_{j\to\infty} \mathcal{L}(\varphi^{n_j}\,|_{\kappa_j}) + \epsilon > \mathcal{L}(\varphi\,|_{\mathbb{S}^1\setminus\gamma^*}) + \mathcal{L}(\varphi\,|_{\gamma^*}) = \mathcal{L}(\varphi),\end{aligned}$$

which is a contradiction and proves (??). Now we fix some $l \in (0,\pi)$ arbitrarily and split up $([0,2\pi]/(0 \sim 2\pi))^2$ into the sets of pairs of angles $D_1(l) := \{(\alpha,\beta) \mid\mid \alpha - \beta \mid < \frac{l}{2}\}$ and $D_2(l) := \{(\alpha,\beta) \mid\mid \alpha - \beta \mid \ge \frac{l}{2}\}$, where $\mid \alpha - \beta \mid$ means the shorter distance in $[0,2\pi]/(0 \sim 2\pi)$. Now by the definition of $\eta(l)$ and $D_1(l)$ and $\mathcal{L}(\varphi^n) \le$ const.=: L we can estimate on account of Fubini's theorem for Stieltjes integrals:

$$\int_{D_1} d \mid \varphi^n(\alpha) \mid \, d \mid \varphi^n(\beta) \mid \le \int_0^{2\pi} \Big(\sup_{\vartheta\in[0,2\pi],\, n\in\mathbb{N}} \int_\vartheta^{\vartheta+l} d \mid \varphi^n(\alpha) \mid \Big)\, d \mid \varphi^n(\beta) \mid \le \eta(l)\, L \qquad (4.25)$$

$\forall\, n \in \mathbb{N}$. Moreover by $\arctan\colon \mathbb{R} \xrightarrow{\cong} (-\frac{\pi}{2}, \frac{\pi}{2})$ we see that

$$0 < \arctan\Big(\frac{2r \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-r^2}\Big)\, \Big|_\varrho^{\tilde\varrho} < \pi \qquad \textit{for } \varrho < \tilde\varrho \in (0,1), \quad \forall\,(\alpha,\beta) \in [0,2\pi]^2. \quad (4.26)$$

Hence, combining this with (??) we conclude:

$$\frac{1}{2\pi} \int_{D_1} \arctan\Big(\frac{2r \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-r^2}\Big)\, \Big|_\varrho^{\tilde\varrho}\, d \mid \varphi^n(\alpha) \mid \, d \mid \varphi^n(\beta) \mid < \frac{1}{2}\, \eta(l)\, L \qquad (4.27)$$

$\forall\, \varrho < \tilde\varrho \in (0,1)$ and $\forall\, n \in \mathbb{N}$. Furthermore on account of (??) and $\varphi^n \in BV(\mathbb{S}^1, \mathbb{R}^3)$ we may use the theorem of dominated convergence for Stieltjes integrals (see [?], p. 146) which yields in (??) for $\tilde\varrho \nearrow 1$:

$$\begin{aligned}\frac{1}{2\pi} \int_{D_1} \arctan\Big(\frac{2r \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-r^2}\Big)\, \Big|_\varrho^{\tilde\varrho}\, d \mid \varphi^n(\alpha) \mid \, d \mid \varphi^n(\beta) \mid \\ \longrightarrow \frac{1}{2\pi} \int_{D_1} \arctan\Big(\frac{2r \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-r^2}\Big)\, \Big|_\varrho^{1}\, d \mid \varphi^n(\alpha) \mid \, d \mid \varphi^n(\beta) \mid \le \frac{1}{2}\, \eta(l)\, L \qquad (4.28)\end{aligned}$$

$\forall\, n \in \mathbb{N},\ \forall\, \varrho \in (0,1)$. Moreover on $D_2(l)$ we have $\frac{l}{4} \leq \mid \frac{\beta-\alpha}{2} \mid \leq \frac{\pi}{2}$ implying that

$$\arctan\Big(\frac{2r \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-r^2}\Big) \longrightarrow \frac{\pi}{2} \qquad \textit{for } r \to 1, \quad \forall\,(\alpha,\beta) \in D_2(l). \tag{4.29}$$

Hence, again on account of (**??**) we may use the theorem of dominated convergence for Stieltjes integrals yielding for $\tilde{\varrho} \nearrow 1$:

$$\frac{1}{2\pi}\int_{D_2} \arctan\Big(\frac{2r \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-r^2}\Big) \Big|_{\varrho}^{\tilde{\varrho}}\, d \mid \varphi^n(\alpha) \mid\, d \mid \varphi^n(\beta) \mid$$

$$\longrightarrow \frac{1}{2\pi}\int_{D_2} \frac{\pi}{2} - \arctan\Big(\frac{2\varrho \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-\varrho^2}\Big)\, d \mid \varphi^n(\alpha) \mid\, d \mid \varphi^n(\beta) \mid \tag{4.30}$$

$\forall\, n \in \mathbb{N},\ \forall\, \varrho \in (0,1)$. Furthermore one easily derives from the Taylor expansion of sin:

$$\sin x > x - \frac{x^3}{6} > \frac{x}{2} \qquad \textit{for } x \in (0, \sqrt{3}).$$

Hence, due to $\frac{l}{4} < \frac{\pi}{4} < \sqrt{3}$ we obtain

$$\mid \sin\Big(\frac{\beta-\alpha}{2}\Big) \mid \geq \sin\frac{l}{4} > \frac{l}{8} \qquad \textit{on } D_2(l),$$

which yields together with the monotonicity of arctan:

$$\arctan\Big(\frac{2\varrho \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-\varrho^2}\Big) > \arctan\Big(\frac{\varrho\, l}{4\,(1-\varrho^2)}\Big) \qquad \textit{on } D_2(l)$$

and $\forall\, \varrho \in (0,1)$. Hence, combining this with $\mathcal{L}(\varphi^n) \leq L$ and $\tan x = \cot\big(\frac{\pi}{2} - x\big)$ we obtain

$$\frac{1}{2\pi}\int_{D_2} \frac{\pi}{2} - \arctan\Big(\frac{2\varrho \mid \sin\big(\frac{\beta-\alpha}{2}\big) \mid}{1-\varrho^2}\Big)\, d \mid \varphi^n(\alpha) \mid\, d \mid \varphi^n(\beta) \mid$$

$$< \frac{1}{2\pi}\Big(\frac{\pi}{2} - \arctan\Big(\frac{\varrho\, l}{4\,(1-\varrho^2)}\Big)\Big)\, L^2 = \frac{1}{2\pi}\Big(\operatorname{arccot}\Big(\frac{\varrho\, l}{4\,(1-\varrho^2)}\Big)\Big)\, L^2 \qquad \forall\, n \in \mathbb{N} \tag{4.31}$$

and $\forall\, \varrho \in (0,1)$. Finally we have by hypothesis $\varphi^n \in H^{\frac{1}{2},2}(\partial B, \mathbb{R}^3)$, i.e. $\mathcal{D}(h^n) < \infty$. Thus combining now (**??**), (**??**) and (**??**) we finally achieve:

$$\mathcal{A}_{C_{\varrho 1}}(h^n) = \lim_{\tilde{\varrho} \nearrow 1} \mathcal{A}_{C_{\varrho\tilde{\varrho}}}(h^n) < \frac{1}{2}\,\eta(l)\, L + \frac{1}{2\pi}\Big(\operatorname{arccot}\Big(\frac{\varrho\, l}{4\,(1-\varrho^2)}\Big)\Big)\, L^2 \tag{4.32}$$

$\forall\, n \in \mathbb{N},\ \forall\, \varrho \in (0,1)$ and $\forall\, l \in (0,\pi)$. By (**??**) there exists for any fixed $\epsilon > 0$ an $l^* > 0$ such that $\eta(l^*)\, L < \epsilon$. After that we use $\operatorname{arccot} x \longrightarrow 0$ for $x \to \infty$, which guarantees the existence of some $R(\epsilon) \in (0,1)$ such that $\frac{1}{\pi}\Big(\operatorname{arccot}\Big(\frac{\varrho\, l^*}{4\,(1-\varrho^2)}\Big)\Big)\, L^2 < \epsilon \ \ \forall\, \varrho \in (R(\epsilon), 1)$. Thus we finally conclude by (**??**) that

$$\mathcal{A}_{C_{\varrho 1}}(h^n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall\, n \in \mathbb{N}$$

if $\varrho \in (R(\epsilon), 1)$, which proves the assertion of the proposition.

$\diamond$

Remark 4.1 *From (??) one can immediately derive an isoperimetric inequality for harmonic surfaces on $\bar{B}$ which will be stated in Section ?? (see [?], p. 138).*

Proof of Theorem ??: Firstly we obtain by the weak maximum principle for harmonic functions, (??) and (??):

$$\| H^n - H \|_{C^0(\bar{C}_{\rho 1})} \leq \sqrt{3} \, \| (H^n - H) \, |_{\partial C_{\rho 1}} \|_{C^0(\partial C_{\rho 1})} \longrightarrow 0. \tag{4.33}$$

Together with Cauchy estimates this yields

$$\| DH^n - DH \|_{C^0(S)} \longrightarrow 0 \qquad \forall \text{compact } S \subset\subset C_{\rho 1}, \tag{4.34}$$

which especially implies:

$$\mathcal{A}_S(H^n) \longrightarrow \mathcal{A}_S(H) \qquad \forall \text{compact } S \subset\subset C_{\rho 1}. \tag{4.35}$$

Hence, in view of (??) we have to estimate the areas $\mathcal{A}_{C_{\rho\sigma}}(H^n)$ resp. $\mathcal{A}_{C_{\varrho 1}}(H^n)$ on small boundary strips about $\partial B_\rho(0)$ resp. $\partial B_1(0)$.
Part I) Firstly we examine $\mathcal{A}_{C_{\varrho 1}}(H^n)$ for $\varrho \in (\rho, 1)$:
We consider the harmonic extensions h^n of the boundary values φ_1^n onto the whole disc $\bar{B}$, the harmonic differences $\omega^n := H^n - h^n$ on $\bar{C}_{\rho 1}$ and their Kelvin-transforms

$$(\omega^n)^*(w) := -\omega^n\Big(\frac{w}{| w |^2}\Big) \qquad \textit{for } w \in \bar{C}_{1\frac{1}{\rho}}. \tag{4.36}$$

As already stated in (??) one easily calculates:

$$\triangle(\omega^n)^*(w) = -\frac{1}{| w |^4}\triangle(\omega^n)\Big(\frac{w}{| w |^2}\Big) = 0 \qquad \forall\, w \in C_{1\frac{1}{\rho}}. \tag{4.37}$$

Now on account of $(\omega^n)^* \, |_{\partial B_1(0)} = -\omega^n \, |_{\partial B_1(0)} \equiv 0$ Schwarz' reflection principle for spheres confirms that the composed functions

$$\bar{\omega}^n(w) := \begin{cases} \omega^n(w) & : \; w \in \bar{C}_{\rho 1} \\ (\omega^n)^*(w) & : \; w \in \bar{C}_{1\frac{1}{\rho}} \end{cases} \qquad \star$$

are harmonic continuations of ω^n onto $\bar{C}_{\rho\frac{1}{\rho}}$. Using the maximum principle again we have $\max_{\bar{B}} | h^n | \leq \sqrt{3} \max_{\partial B} | \varphi_1^n | \leq$ const. by (??), thus together with (??), (??) and $\star$ we see that

$$\max_{\bar{C}_{\rho\frac{1}{\rho}}} | \bar{\omega}^n | = \max_{\bar{C}_{\rho 1}} | \omega^n | \leq \max_{\bar{C}_{\rho 1}} | H^n | + \max_{\bar{C}_{\rho 1}} | h^n | \leq \textit{const.} \qquad \forall\, n \in \mathbb{N}. \tag{4.38}$$

Now on account of $\star$ we may apply Cauchy estimates to ω^n on a ring region $C_{\varrho 1}$ for any $\varrho \in (\rho, 1)$:

$$\sup_{C_{\varrho 1}} | D\omega^n | \leq \textit{const.}(\varrho - \rho) \max_{\bar{C}_{\rho\frac{1}{\rho}}} | \bar{\omega}^n | \leq \textit{const.} \qquad \forall\, n \in \mathbb{N}. \tag{4.39}$$

By the invariance of $\mathcal{A}$ with respect to diffeomorphic transformations of its parameter domain we have

$$\mathcal{A}_{C_{\varrho 1}}(H^n) = \int_0^{2\pi} \int_\varrho^1 \mid H_r^n \wedge H_\theta^n \mid \, dr d\theta, \tag{4.40}$$

and by the definition of ω^n we see:

$$H_r^n \wedge H_\theta^n = \omega_r^n \wedge \omega_\theta^n + \omega_r^n \wedge h_\theta^n + h_r^n \wedge \omega_\theta^n + h_r^n \wedge h_\theta^n. \tag{4.41}$$

Firstly by (**??**) we gain immediately that for any $\epsilon > 0$ there exists an $R(\epsilon) \in (\rho, 1)$ such that

$$\int_0^{2\pi} \int_\varrho^1 \mid \omega_r^n \wedge \omega_\theta^n \mid \, dr d\theta < \epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.42}$$

if $\varrho \in (R(\epsilon), 1)$. Furthermore on account of (**??**) we may apply Proposition **??** to the harmonic extensions h^n of φ_1^n onto the whole disc $\bar{B}$, thus for any $\epsilon > 0$ there exists an $R(\epsilon) \in (\rho, 1)$ such that

$$\int_0^{2\pi} \int_\varrho^1 \mid h_r^n \wedge h_\theta^n \mid \, dr d\theta < \epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.43}$$

if $\varrho \in (R(\epsilon), 1)$. In view of (**??**) we now estimate $\int_0^{2\pi} \int_\varrho^1 \mid \omega_r^n \mid\mid h_\theta^n \mid \, dr d\theta$. Combining (**??**), (**??**), (**??**) (below) and (**??**) we obtain by Fubini's theorem and the "triangle inequality" for Stieltjes integrals (see [**?**], p. 151 and p. 144):

$$\begin{aligned}
\int_0^{2\pi} \int_\varrho^1 \mid \omega_r^n \mid\mid h_\theta^n \mid \, dr d\theta &\leq \frac{C}{2\pi} \int_0^{2\pi} \int_\varrho^1 \Big| \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2} \, d\varphi_1^n(\alpha) \Big| \, dr d\theta \\
&\leq \frac{C}{2\pi} \int_0^{2\pi} \int_\varrho^1 \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2} \, d\theta \, dr \, d \mid \varphi_1^n(\alpha) \mid \\
&= C \int_0^{2\pi} \int_\varrho^1 1 \, dr \, d \mid \varphi_1^n(\alpha) \mid = C\, \mathcal{L}(\varphi_1^n)\, (1 - \varrho).
\end{aligned}$$

Now by $\mathcal{L}(\varphi_1^n) \leq$ const.=:L, $\forall\, n \in \mathbb{N}$, due to (**??**) we conclude that for any $\epsilon > 0$ there exists an $R(\epsilon) \in (\rho, 1)$ such that

$$\int_0^{2\pi} \int_\varrho^1 \mid \omega_r^n \mid\mid h_\theta^n \mid \, dr d\theta < \epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.44}$$

whenever $R(\epsilon) < \varrho < 1$. Now we estimate the remaining integral $\int_0^{2\pi} \int_\varrho^1 \mid h_r^n \mid\mid \omega_\theta^n \mid \, dr d\theta$. Combining (**??**), (**??**) and

$$0 < (1-r)^2 = 1 - 2r + r^2 \leq 1 - 2r \mid \cos(\alpha - \theta) \mid + r^2 \leq 1 - 2r\cos(\alpha - \theta) + r^2, \tag{4.45}$$

$\forall\, r \in (0, 1)$, $\forall\, \theta, \alpha \in [0, 2\pi]$, we obtain by Fubini's theorem and the "triangle inequality" for Stieltjes integrals (see [**?**], p. 151 and p. 144):

$$J(n,\varrho,\tilde{\varrho}) := \int_0^{2\pi}\int_{\varrho}^{\tilde{\varrho}} \mid h_r^n \mid\mid \omega_\theta^n \mid \, dr d\theta \tag{4.46}$$
$$\leq \frac{C}{\pi}\int_0^{2\pi}\int_{\varrho}^{\tilde{\varrho}} \mid \int_0^{2\pi} \frac{\sin(\alpha-\theta)}{1-2r\cos(\alpha-\theta)+r^2}\, d\varphi_1^n(\alpha) \mid \, dr d\theta$$
$$\leq \frac{C}{\pi}\int_0^{2\pi}\int_0^{2\pi}\int_{\varrho}^{\tilde{\varrho}} \frac{\mid \sin(\alpha-\theta)\mid}{1-2r\mid\cos(\alpha-\theta)\mid+r^2}\, dr\ d\theta d \mid \varphi_1^n(\alpha)\mid$$

$\forall\, \varrho < \tilde{\varrho} \in (\rho, 1)$. Furthermore one easily verifies that

$$\mid \sin(\alpha-\theta)\mid \int_{\varrho}^{\tilde{\varrho}} \frac{1}{1-2r\mid\cos(\alpha-\theta)\mid+r^2}\, dr = \arctan\Big(\frac{r-\mid\cos(\alpha-\theta)\mid}{\mid\sin(\alpha-\theta)\mid}\Big) \mid_{\varrho}^{\tilde{\varrho}}$$
$$=: W(\theta,\alpha,\varrho,\tilde{\varrho}) \tag{4.47}$$

$\forall\, \varrho < \tilde{\varrho} \in (\rho,1)$, $\forall\,(\theta,\alpha) \in [0,2\pi]^2 \setminus \Theta$, where $\Theta := \{(\theta,\alpha)\in[0,2\pi]^2 \mid\mid \alpha-\theta \mid\in \{0,\pi\}\}$. Noting that $\sin(\alpha-\theta) = 0\ \ \forall\,(\theta,\alpha)\in\Theta$, (**??**) and that W can be extended continuously onto Θ by setting $W(\,\cdot\,,\,\cdot\,,\varrho,\tilde{\varrho}) \equiv 0$ on Θ, $\forall\,\varrho<\tilde{\varrho}\in(\rho,1)$, we arrive at:

$$J(n,\varrho,\tilde{\varrho}) \leq \frac{C}{\pi}\int_0^{2\pi}\int_0^{2\pi} W(\theta,\alpha,\varrho,\tilde{\varrho})\, d\theta d \mid \varphi_1^n(\alpha)\mid . \tag{4.48}$$

Now noting that W is periodic in θ with period π and that $W(\,\cdot\,,\,\cdot\,,\varrho,\tilde{\varrho})$ only depends on the difference $\alpha-\theta$ we may rearrange (**??**) into

$$J(n,\varrho,\tilde{\varrho}) \leq \frac{2C}{\pi}\int_0^{2\pi}\int_{\alpha-\frac{\pi}{2}}^{\alpha+\frac{\pi}{2}} W(\theta,\alpha,\varrho,\tilde{\varrho})\, d\theta d \mid \varphi_1^n(\alpha)\mid . \tag{4.49}$$

Since arctan: $\mathbb{R} \xrightarrow{\cong} (-\frac{\pi}{2},\frac{\pi}{2})$ is monotonic we have $0 \leq W(\theta,\alpha,\varrho,\tilde{\varrho}) < \pi\ \ \forall\,\varrho<\tilde{\varrho}\in(\rho,1)$, $\forall\,(\theta,\alpha)\in[0,2\pi]^2$. Moreover we set $I(\alpha,\delta) := (\alpha-\frac{\pi}{2},\alpha+\frac{\pi}{2})\setminus(\alpha-\delta,\alpha+\delta)$ for an arbitrarily chosen $\delta\in(0,\frac{\pi}{2})$ and split up the right hand side of (**??**):

$$J(n,\varrho,\tilde{\varrho}) < 2C\int_0^{2\pi}\int_{\alpha-\delta}^{\alpha+\delta} d\theta d \mid \varphi_1^n(\alpha)\mid \tag{4.50}$$
$$+\frac{2C}{\pi}\int_0^{2\pi}\int_{I(\alpha,\delta)} W(\theta,\alpha,\varrho,\tilde{\varrho})\, d\theta d\mid\varphi_1^n(\alpha)\mid .$$

The first integral in (**??**) can immediately be estimated by $\mathcal{L}(\varphi_1^n) \leq \mathrm{L}$, $\forall\, n\in\mathbb{N}$:

$$2C\int_0^{2\pi}\int_{\alpha-\delta}^{\alpha+\delta} d\theta d\mid\varphi_1^n(\alpha)\mid \leq 4CL\,\delta \qquad \forall\, n\in\mathbb{N}. \tag{4.51}$$

Furthermore, as we have $\delta \leq\mid \alpha-\theta\mid< \frac{\pi}{2}\ \forall\,\theta\in I(\alpha,\delta)$ we obtain the estimate

$$\frac{1}{\mid\sin(\alpha-\theta)\mid} \leq \frac{1}{\mid\sin(\delta)\mid} \qquad \forall\,\theta\in I(\alpha,\delta),\ \ \forall\,\alpha\in[0,2\pi].$$

Hence, we see that for any $\epsilon > 0$ there exists an $R(\epsilon, \delta) \in (\rho, 1)$ such that

$$0 < \frac{\tilde{\varrho}- \mid \cos(\alpha-\theta) \mid}{\mid \sin(\alpha-\theta) \mid} - \frac{\varrho- \mid \cos(\alpha-\theta) \mid}{\mid \sin(\alpha-\theta) \mid} = \frac{\tilde{\varrho}-\varrho}{\mid \sin(\alpha-\theta) \mid} \leq \frac{\tilde{\varrho}-\varrho}{\mid \sin(\delta) \mid} < \epsilon$$

$\forall\, \theta \in I(\alpha, \delta)$, $\forall\, \alpha \in [0, 2\pi]$, whenever $R(\epsilon, \delta) < \varrho < \tilde{\varrho} < 1$. Together with the uniform continuity of arctan on $\mathbb{R}$ and (**??**) we conclude that for any $\varepsilon > 0$ there exists an $R(\varepsilon, \delta) \in (\rho, 1)$ such that

$$0 < W(\theta, \alpha, \varrho, \tilde{\varrho}) < \varepsilon \qquad \forall\, \theta \in I(\alpha, \delta),\ \forall\, \alpha \in [0, 2\pi],$$

whenever $R(\varepsilon, \delta) < \varrho < \tilde{\varrho} < 1$. Combining this with $\mathcal{L}(\varphi_1^n) \leq L$, $\forall\, n \in \mathbb{N}$, $\mid I(\alpha, \delta) \mid < \pi$ and choosing now $\varepsilon = \delta$ we arrive at:

$$\frac{2C}{\pi} \int_0^{2\pi} \int_{I(\alpha,\delta)} W(\theta, \alpha, \varrho, \tilde{\varrho})\, d\theta d \mid \varphi_1^n(\alpha) \mid < 2CL\,\delta \qquad \forall\, n \in \mathbb{N}, \tag{4.52}$$

whenever $R(\delta) < \varrho < \tilde{\varrho} < 1$. Hence, together with (**??**) and (**??**) we achieve:

$$\int_0^{2\pi} \int_{\varrho}^{\tilde{\varrho}} \mid h_r^n \mid\mid \omega_\theta^n \mid\, drd\theta = J(n, \varrho, \tilde{\varrho}) < 2CL\,\delta + 4CL\,\delta = 6CL\,\delta \qquad \forall\, n \in \mathbb{N}, \tag{4.53}$$

whenever $R(\delta) < \varrho < \tilde{\varrho} < 1$, where $\delta \in (0, \frac{\pi}{2})$ was arbitrary. Now by (**??**) and $\nabla h^n \in L^2(B, \mathbb{R}^6)$ due to $\varphi_1^n \in H^{\frac{1}{2},2}(\partial B, \mathbb{R}^3)$ we obtain for $\tilde{\varrho} \nearrow 1$:

$$\int_0^{2\pi} \int_{\varrho}^{1} \mid h_r^n \mid\mid \omega_\theta^n \mid\, drd\theta = \lim_{\tilde{\varrho} \nearrow 1} J(n, \varrho, \tilde{\varrho}) \leq 6CL\,\delta \qquad \forall\, n \in \mathbb{N},$$

if $\varrho \in (R(\delta), 1)$, $\forall\, \delta \in (0, \frac{\pi}{2})$. Hence, for any $\epsilon > 0$ there exists an $R(\epsilon) \in (\rho, 1)$ such that

$$\int_0^{2\pi} \int_{\varrho}^{1} \mid h_r^n \mid\mid \omega_\theta^n \mid\, drd\theta < \epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.54}$$

if $\varrho \in (R(\epsilon), 1)$. Now combining (**??**), (**??**), (**??**) and (**??**) with (**??**) and (**??**) we finally infer that for any $\epsilon > 0$ there exists an $R(\epsilon) \in (\rho, 1)$ such that

$$\mathcal{A}_{C_{\varrho 1}}(H^n) = \int_0^{2\pi} \int_{\varrho}^{1} \mid H_r^n \wedge H_\theta^n \mid\, drd\theta < 4\epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.55}$$

whenever $R(\epsilon) < \varrho < 1$.
Part II) Now we are going to examine $\mathcal{A}_{C_{\rho\sigma}}(H^n)$ for $\sigma \in (\rho, 1)$:
To this end we consider the scaled Kelvin-transforms of H^n, given by

$$(\tilde{H^n})^*(w) := (H^n)^*(\rho\, w) := -H^n(\rho \frac{w}{\mid w \mid^2}) \qquad on\ \bar{C}_{\rho 1}. \tag{4.56}$$

One easily verifies that $\triangle(\tilde{H^n})^*(w) = -\frac{\rho^2}{|w|^4} \triangle(H^n)(\rho \frac{w}{|w|^2}) = 0$ $\forall\, w \in C_{\rho 1}$. Hence, $(\tilde{H^n})^*$ are the unique harmonic extensions of the boundary values $-\varphi_\rho^n(\rho\,\cdot)$ on ∂B and

$-\varphi_1^n(\frac{1}{\rho}\,\cdot)$ on $\partial B_\rho(0)$ onto $\bar{C}_{\rho 1}$. Therefore we may replace the H^n in Part I of the proof by the $(\tilde{H^n})^*$ and infer from (**??**) that for any $\epsilon > 0$ there is an $R(\epsilon) \in (\rho, 1)$ such that

$$\mathcal{A}_{C_{\varrho 1}}((\tilde{H^n})^*) < 4\epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.57}$$

whenever $R(\epsilon) < \varrho < 1$. Furthermore by the invariance of $\mathcal{A}$ with respect to the reflection $\phi : C_{\rho 1} \xrightarrow{\cong} C_{\rho^2 \rho}$, $\phi(w) := \rho^2 \frac{w}{|w|^2}$, and w. r. to scaling we have:

$$\mathcal{A}_{C_{\varrho 1}}((\tilde{H^n})^*) = \mathcal{A}_{C_{\rho \frac{\rho\varrho}{\varrho^2}}}(-H^n) = \mathcal{A}_{C_{\rho \frac{\rho}{\varrho}}}(H^n). \tag{4.58}$$

Hence, setting $\sigma := \frac{\rho}{\varrho}$ we conclude by (**??**) and (**??**) that for any $\epsilon > 0$ there is an $R(\epsilon) \in (\rho, 1)$ (near ρ) such that

$$\mathcal{A}_{C_{\rho\sigma}}(H^n) = \int_0^{2\pi} \int_\rho^\sigma \mid (H^n)_r \wedge (H^n)_\theta \mid \, dr d\theta < 4\epsilon \qquad \forall\, n \in \mathbb{N}, \tag{4.59}$$

if $\sigma \in (\rho, R(\epsilon))$. Hence, combining the estimates (**??**) and (**??**) with (**??**) we see that for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ and an $N(\epsilon) \in \mathbb{N}$ with

$$\begin{aligned} \mid \mathcal{A}_{C_{\rho 1}}(H) - \mathcal{A}_{C_{\rho 1}}(H^n) \mid \leq \mid \mathcal{A}_{C_{\sigma\varrho}}(H) - \mathcal{A}_{C_{\sigma\varrho}}(H^n) \mid + \mathcal{A}_{C_{\rho\sigma}}(H) + \mathcal{A}_{C_{\rho\sigma}}(H^n) \\ + \mathcal{A}_{C_{\varrho 1}}(H) + \mathcal{A}_{C_{\varrho 1}}(H^n) < 17\,\epsilon \end{aligned}$$

$\forall\, n > N$, if we choose $\sigma - \rho < \delta$ and $1 - \varrho < \delta$, which proves the theorem.

$\diamond$

4.2 Continuity theorems for $\mathcal{J}$ and $\mathcal{I}$

In this section we prove the "continuity theorems" 11.1 and 12.2 in [**?**], see Theorem **??** and Corollary **??** below, by combining Theorem **??** with the following estimate, Lemma 8.1 in [**?**], which is gained by "harmonic substitution".

Lemma 4.1 *Let X be an $\mathcal{I}$-surface and $\Omega \subset B$ any open subset with a Lipschitz boundary. Then for the harmonic extension H of the boundary values $X\mid_{\partial\Omega}$ we have:*

$$\mathcal{F}_\Omega(X) \leq \mathcal{F}_\Omega(H) - k\,\mathcal{D}_\Omega(X - H). \tag{4.60}$$

Remark 4.2 *We note that for any open bounded subset Ω of $\mathbb{R}^2$ with a Lipschitz boundary and any $\varphi \in H^{\frac{1}{2},2}(\partial\Omega, \mathbb{R}^3)$ there exists a unique harmonic surface H in the boundary value class $H^{1,2}_\varphi(\Omega, \mathbb{R}^3)$ which satisfies*

$$\mathcal{D}_\Omega(H) \leq \mathcal{D}_\Omega(Y) \qquad \forall\, Y \in H^{1,2}_\varphi(\Omega, \mathbb{R}^3). \tag{4.61}$$

To see this one has to consider a minimizing sequence $\{X_n\}$ for $\mathcal{D}_\Omega$ in $H^{1,2}_\varphi(\Omega, \mathbb{R}^3)$ $(\neq \emptyset)$ yielding a weakly convergent subsequence

$$X_{n_k} \rightharpoonup H \qquad \text{in } H^{1,2}_\varphi(\Omega, \mathbb{R}^3)$$

*for some weak limit $H \in H^{1,2}_{\varphi}(\Omega, \mathbb{R}^3)$ (see [?], p. 223). By the weak lower semicontinuity of $\mathcal{D}_\Omega$ one confirms (**??**). As usual this implies:*

$$0 = \delta \mathcal{D}_\Omega(H, \eta) = \int_\Omega DH \cdot D\eta \, dw \qquad \forall \, \eta \in \mathring{H}^{1,2}(\Omega, \mathbb{R}^3). \tag{4.62}$$

*Hence, we conclude by the weak maximum principle of the L^2-Theory that H is the unique solution of (**??**) in $H^{1,2}_{\varphi}(\Omega, \mathbb{R}^3)$ and by Weyl's lemma that H is in fact harmonic on Ω, i.e. φ possesses a unique harmonic extension H in $H^{1,2}_{\varphi}(\Omega, \mathbb{R}^3)$ satisfying (**??**).*

Proof of Lemma **??**: We consider the composed surface

$$X'(w) := \begin{cases} H(w) & : \quad w \in \Omega \\ X(w) & : \quad w \in B \setminus \Omega. \end{cases}$$

By the above remark we have $H \in H^{1,2}(\Omega, \mathbb{R}^3)$. Together with $H \mid_{\partial\Omega} \equiv X \mid_{\partial\Omega}$ and as $\partial\Omega$ is required to be a Lipschitz boundary Lemma A 6.9 in [?], p. 254, yields that $X' \in H^{1,2}(B, \mathbb{R}^3)$. Furthermore as X is an $\mathcal{I}$-surface and $X' \mid_{\partial B} \equiv X \mid_{\partial B}$ we infer $\mathcal{I}(X) \leq \mathcal{I}(X')$, which implies together with $X' \mid_{B\setminus\Omega} \equiv X \mid_{B\setminus\Omega}$:

$$\mathcal{F}_\Omega(X) + k \, \mathcal{D}_\Omega(X) = \mathcal{I}_\Omega(X) \leq \mathcal{I}_\Omega(H) = \mathcal{F}_\Omega(H) + k \, \mathcal{D}_\Omega(H). \tag{4.63}$$

Testing (**??**) with $X - H \in \mathring{H}^{1,2}(\Omega, \mathbb{R}^3)$ we obtain $\int_\Omega DH \cdot D(X - H) \, dw = 0$, thus

$$\mathcal{D}_\Omega(X - H) = \mathcal{D}_\Omega(X) + \mathcal{D}_\Omega(H) - \int_\Omega DH \cdot DX \, dw$$

$$= \mathcal{D}_\Omega(X) + \mathcal{D}_\Omega(H) - \int_\Omega DH \cdot (DH + D(X - H)) \, dw = \mathcal{D}_\Omega(X) - \mathcal{D}_\Omega(H).$$

Combining this with (**??**) we gain (**??**).

◇

Theorem 4.2 *Let $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces with $X^n \mid_{\partial B} \in (C^0 \cap BV)(\partial B, \mathbb{R}^3)$, $\mathcal{D}(X^n) \leq$const. $\forall \, n \in \mathbb{N}$ and*

$$X^n \longrightarrow \bar{X} \qquad \text{in } C^0(\bar{B}, \mathbb{R}^3), \qquad \mathcal{L}(X^n \mid_{\partial B}) \longrightarrow \mathcal{L}(\bar{X} \mid_{\partial B}) \tag{4.64}$$

for an $\mathcal{I}$-surface $\bar{X}$ with $\bar{X} \mid_{\partial B} \in (C^0 \cap BV)(\partial B, \mathbb{R}^3)$. Then there holds:

$$\mathcal{J}(X^n) \longrightarrow \mathcal{J}(\bar{X}) \qquad \text{for } n \to \infty. \tag{4.65}$$

Proof: Let $\epsilon > 0$ be given arbitrarily. By the absolute continuity of the Lebesgue integral there exists some $\rho' \in (0, 1)$ such that

$$\mathcal{D}_{C_{\rho 1}}(\bar{X}) < \epsilon \qquad \forall \, \rho \in [\rho', 1]. \tag{4.66}$$

By Theorem **??** we obtain for every $\rho'' \in (\rho', 1)$ a subsequence $\{X^{n_k}\}$ with

$$\mathcal{D}_{B_{\rho''}(0)}(X^{n_k} - \bar{X}) \longrightarrow 0 \qquad \textit{for } k \to \infty. \tag{4.67}$$

We fix such a ρ'' arbitrarily, rename the corresponding subsequence into $\{X^n\}$ again and show firstly that there is a further subsequence $\{X^{n_k}\}$ which satisfies

$$\mathcal{L}(X^{n_k} \mid_{\partial B_r(0)}) \longrightarrow \mathcal{L}(\bar{X} \mid_{\partial B_r(0)}) \qquad \textit{for a.e. } r \in (\rho', \rho''). \tag{4.68}$$

We set $S^n(\,\cdot\,) := \frac{1}{2}\int_0^{2\pi} \mid (X^n - \bar{X})_\theta(\,\cdot\,, \theta) \mid^2 \, d\theta \in L^1((\rho', \rho''))$ and see by (**??**)

$$0 \leq \int_{\rho'}^{\rho''} S^n(r)\, dr \leq p'' \int_{\rho'}^{\rho''} S^n(r)\, \frac{1}{r}\, dr \leq p''\, \mathcal{D}_{C_{\rho'\rho''}}(X^n - \bar{X}) \longrightarrow 0.$$

Hence, there exists a subsequence $\{S^{n_k}\}$ such that $S^{n_k}(r) \longrightarrow 0$ for a.e. $r \in (\rho', \rho'')$, which implies

$$\mathcal{L}((X^{n_k} - \bar{X}) \mid_{\partial B_r(0)}) = \int_0^{2\pi} \mid (X^{n_k} - \bar{X})_\theta(r, \theta) \mid \, d\theta \leq \sqrt{4\pi\, S^{n_k}(r)} \longrightarrow 0$$

for a.e. $r \in (\rho', \rho'')$, thus (**??**). We rename $\{X^{n_k}\}$ into $\{X^n\}$ again, fix some $\rho \in (\rho', \rho'')$ for which holds (**??**) and consider the harmonic extensions H^n resp. H of the boundary values $(X^n \mid_{\partial B_\rho(0)}, X^n \mid_{\partial B})$ resp. $(\bar{X} \mid_{\partial B_\rho(0)}, \bar{X} \mid_{\partial B})$ onto $\bar{C}_{\rho 1}$, which exist by Remark **??**. From (**??**) and (**??**) we infer:

$$\mathcal{A}_{C_{\rho 1}}(H) \leq \mathcal{D}_{C_{\rho 1}}(H) \leq \mathcal{D}_{C_{\rho 1}}(\bar{X}) < \epsilon. \tag{4.69}$$

Now on account of (**??**) and (**??**) and recalling that the $\mathcal{I}$-surfaces X^n and $\bar{X}$ lie in $H^{1,2}(B, \mathbb{R}^3) \cap C^0(\bar{B}, \mathbb{R}^3)$ we may apply the continuity theorem **??** yielding:

$$\mathcal{A}_{C_{\rho 1}}(H^n) \longrightarrow \mathcal{A}_{C_{\rho 1}}(H) \qquad \textit{for } n \to \infty.$$

Thus together with (**??**) we infer that there exists an $N(\epsilon) \in \mathbb{N}$ such that

$$\mathcal{A}_{C_{\rho 1}}(H^n) < 2\epsilon \qquad \forall\, n > N(\epsilon). \tag{4.70}$$

Together with $m_1 \mid z \mid \leq F(z) \leq m_2 \mid z \mid$ and (**??**) we arrive at

$$m_1 \mathcal{A}_{C_{\rho 1}}(X^n) \leq \mathcal{F}_{C_{\rho 1}}(X^n) \leq \mathcal{F}_{C_{\rho 1}}(H^n) \leq m_2\, \mathcal{A}_{C_{\rho 1}}(H^n) < 2 m_2\, \epsilon \qquad \forall\, n > N(\epsilon),$$

which finally implies for $\mathcal{J} = \mathcal{F} + k\, \mathcal{A}$:

$$\mathcal{J}_{C_{\rho 1}}(X^n) < 2\left(m_2 + \frac{k\, m_2}{m_1}\right)\epsilon \qquad \forall\, n > N(\epsilon). \tag{4.71}$$

Furthermore by (**??**) we have $\mathcal{A}_{C_{\rho 1}}(\bar{X}) \leq \mathcal{D}_{C_{\rho 1}}(\bar{X}) < \epsilon$, hence

$$\mathcal{J}_{C_{\rho 1}}(\bar{X}) < (m_2 + k)\, \epsilon. \tag{4.72}$$

Moreover by (**??**), $\rho < \rho''$ and $\mathcal{D}(X^n) \leq$const. one obtains by Proposition **??** that there exists an $N(\epsilon) \in \mathbb{N}$ such that

$$| \mathcal{J}_{B_\rho(0)}(X^n) - \mathcal{J}_{B_\rho(0)}(\bar{X}) | < \epsilon \qquad \forall\, n > N(\epsilon).$$

Now combining this with (**??**) and (**??**) we see that there exists an $N(\epsilon) \in \mathbb{N}$ such that

$$| \mathcal{J}(X^n) - \mathcal{J}(\bar{X}) | \leq | \mathcal{J}_{B_\rho(0)}(X^n) - \mathcal{J}_{B_\rho(0)}(\bar{X}) | + \mathcal{J}_{C_{\rho 1}}(X^n) + \mathcal{J}_{C_{\rho 1}}(\bar{X})$$
$$< \epsilon + 2\left(m_2 + \frac{k\, m_2}{m_1}\right)\epsilon + (m_2 + k)\,\epsilon = \left(1 + 3m_2 + \frac{2m_2 + m_1}{m_1}k\right)\epsilon \qquad \forall\, n > N(\epsilon).$$

Since we selected several subsequences we can firstly only conclude that there is a subsequence $\{X^{n_j}\}$ of the original sequence $\{X^n\}$ for which holds the assertion (**??**). But then we achieve (**??**) for the whole sequence $\{X^n\}$ due to the "principle of subsequences".

◇

The above theorem immediately implies Theorem 12.2 in [**?**]:

Corollary 4.1 *Let* $\{X^n\}$ *be a sequence of* $\mathcal{I}$*-surfaces as in Theorem* **??** *that are additionally (a.e.) conformally parametrized on* B*. Then firstly there holds*

$$\mathcal{I}(X^n) \longrightarrow \mathcal{I}(\bar{X}) \qquad \text{for } n \to \infty, \tag{4.73}$$

where $\bar{X}$ *is the limit* $\mathcal{I}$*-surface as in Theorem* **??***, and secondly* $\bar{X}$ *proves to be (a.e.) conformally parametrized on* B*.*

Proof: Applying Theorem **??** to the conformally parametrized $\mathcal{I}$-surfaces X^n we have

$$\mathcal{J}(\bar{X}) = \lim_{n\to\infty} \mathcal{J}(X^n) = \lim_{n\to\infty} \mathcal{I}(X^n) \tag{4.74}$$

(see (6) in [**?**]). Moreover we infer from our hypothesises that $\| X^n \|_{H^{1,2}(B,\mathbb{R}^3)} \leq$ const. $\forall\, n \in \mathbb{N}$, hence there exists a subsequence $\{X^{n_k}\}$ which satisfies $X^{n_k} \rightharpoonup \bar{X}$ weakly in $H^{1,2}(B,\mathbb{R}^3)$. Thus the weak lower semicontinuity of $\mathcal{I}$ and (**??**) imply:

$$\mathcal{J}(\bar{X}) \leq \mathcal{I}(\bar{X}) \leq \liminf_{k\to\infty} \mathcal{I}(X^{n_k}) = \lim_{n\to\infty} \mathcal{I}(X^n) = \mathcal{J}(\bar{X}).$$

This proves simultaneously the assertion (**??**) and $\mathcal{J}(\bar{X}) = \mathcal{I}(\bar{X})$, i.e. $\mathcal{A}(\bar{X}) = \mathcal{D}(\bar{X})$ yielding the second assertion of the corollary (see (6) in [**?**]).

◇

4.3 Isoperimetric inequalities for $\mathcal{A}$ and $\mathcal{J}$

In this section we prove Theorem 9.1 in [**?**]. As already mentioned in Remark **??** we can derive the following isoperimetric inequality for harmonic surfaces on $\bar{B}$ (see [**?**], p. 138):

Theorem 4.3 *Let* $\varphi \in (C^0 \cap H^{\frac{1}{2},2} \cap BV)(\partial B, \mathbb{R}^3)$ *and* h *the harmonic extension of* φ *onto* $\bar{B}$*, then there holds:*

$$\mathcal{A}(h) \le \frac{1}{4}\,\mathcal{L}(\varphi)^2. \tag{4.75}$$

Proof: Considering the constant sequence $h^n \equiv h$ in Proposition **??** we achieve as in the proof of (**??**) the estimate

$$\mathcal{A}_{C_{\varrho 1}}(h) < \frac{1}{2}\,\eta(l)\,\mathcal{L}(\varphi) + \frac{1}{2\pi}\Big(\text{arccot}\Big(\frac{\varrho\, l}{4\,(1-\varrho^2)}\Big)\Big)\,\mathcal{L}(\varphi)^2$$

$\forall\,\varrho \in (0,1)$ and for any $l \in (0,\pi)$. On account of $\mathcal{D}(h) < \infty$ due to $\varphi \in H^{\frac{1}{2},2}(\partial B, \mathbb{R}^3)$, (**??**) and $\text{arccot}\, 0 = \frac{\pi}{2}$ we gain the assertion of the theorem by letting $\varrho \searrow 0$ and $l \searrow 0$.

◇

Combining this result with Lemma **??** one easily obtains

Corollary 4.2 *For an* $\mathcal{I}$*-surface* X *with* $X\,|_{\partial B} \in (C^0 \cap BV)(\partial B, \mathbb{R}^3)$ *there holds:*

$$\mathcal{J}(X) \le \Big(1 + \frac{k}{m_1}\Big)\frac{m_2}{4}\,\mathcal{L}(X\,|_{\partial B})^2. \tag{4.76}$$

Proof: Let h denote the harmonic extension of the boundary values $X\,|_{\partial B}$ onto $\bar{B}$. Lemma **??** yields for $\Omega := B$ in particular $\mathcal{F}(X) \le \mathcal{F}(h)$. Hence, together with $m_1 \mid z \mid \le F(z) \le m_2 \mid z \mid$ and (**??**) we see:

$$m_1\,\mathcal{A}(X) \le \mathcal{F}(X) \le \mathcal{F}(h) \le m_2\,\mathcal{A}(h) \le \frac{m_2}{4}\,\mathcal{L}(X\,|_{\partial B})^2.$$

Thus by $\mathcal{J} = \mathcal{F} + k\,\mathcal{A}$ we obtain the assertion of the corollary.

◇

5 Combination with the results of [?]

In this chapter we combine all results that we have achieved so far in this paper and in [**?**] with a special continuity theorem for $\mathcal{I}$, Prop. **??**, and a compactness result for boundary values, Prop. **??**, in order to prove the main result, Theorem **??**.

5.1 Preliminary definitions and propositions

5.1.1 Approximation of closed rectifiable Jordan curves by polygons

In this subsection we prove a technical approximation lemma which is also stated in [**?**], Lemma 5 (without proof), a compactness result for boundary values due to Nitsche ([**?**], p. 208) and a crucial continuity theorem which enables us to apply the results of [**?**] to the proof of the main result, Theorem **??**, and which is proved similarily as Lemma 6 in [**?**]. Firstly we need the following

Definition 5.1 *i) Let Γ be an arbitrary closed rectifiable Jordan curve in $\mathbb{R}^3$. Then we term a simple closed polygon $\tilde{\Gamma} \subset \mathbb{R}^3$ a polygonal approximation of Γ if all vertices $\tilde{A}_1, \ldots, \tilde{A}_M$ $(M > 3)$ of $\tilde{\Gamma}$ lie on Γ and if the arc on Γ between any two adjacent points $\tilde{A}_m$, $\tilde{A}_{m+1}$, which does not contain the remaining vertices of $\tilde{\Gamma}$, is indeed the shorter one $\Gamma\mid_{(\tilde{A}_m, \tilde{A}_{m+1})}$ connecting $\tilde{A}_m$ and $\tilde{A}_{m+1}$.*
ii) For a polygonal approximation $\tilde{\Gamma}$ of Γ with vertices $\tilde{A}_1, \ldots, \tilde{A}_M$ we define its fineness $\triangle(\tilde{\Gamma})$ by $\triangle(\tilde{\Gamma}) := \max_{j=1,\ldots,M} \mid \tilde{A}_j - \tilde{A}_{j-1} \mid$, with $\tilde{A}_0 := \tilde{A}_M$.
iii) Let Γ', Γ'' be two polygonal approximations of Γ. Then their common refinement $\Gamma^* := \Gamma' \vee \Gamma''$ is defined to be the polygonal approximation of Γ whose set of vertices consists of the vertices of Γ' and Γ''.

Definition 5.2 *A closed rectifiable Jordan curve Γ in $\mathbb{R}^3$ meets a chord-arc condition if there is a constant C such that*

$$\mathcal{L}(\Gamma\mid_{(P^1,P^2)}) \leq C \mid P^1 - P^2 \mid \qquad \forall P^1, P^2 \in \Gamma, \tag{5.1}$$

where $\Gamma\mid_{(P^1,P^2)}$ denotes the shorter arc on Γ connecting P^1 and P^2.

Proposition 5.1 *Let Γ be an arbitrary closed rectifiable Jordan curve in $\mathbb{R}^3$ which satisfies a chord-arc condition (**??**). Then there exists a sequence $\{\Gamma^n\}$ of polygonal approximations of Γ and homeomorphisms $\varphi^n : \Gamma \xrightarrow{\cong} \Gamma^n$ that satisfy:*

$$\mathcal{L}(\Gamma^n) \longrightarrow \mathcal{L}(\Gamma), \tag{5.2}$$

$$\triangle(\Gamma^n) \longrightarrow 0, \tag{5.3}$$

$$\max_{P\in\Gamma} \mid P - \varphi^n(P) \mid \longrightarrow 0 \qquad \text{for } n \to \infty, \tag{5.4}$$

$$\mid \varphi^n(P^1) - \varphi^n(P^2) \mid \leq \mathcal{L}(\Gamma\mid_{(P^1,P^2)}) \qquad \forall P^1, P^2 \in \Gamma, \quad \forall n \in \mathbb{N}. \tag{5.5}$$

Finally the φ^n keep the vertices of the Γ^n fixed.

Firstly we need the following elementary

Lemma 5.1 *Let Γ be an arbitrary closed rectifiable Jordan curve in $\mathbb{R}^3$ that satisfies a chord-arc condition (**??**). Then for any $\epsilon > 0$ there exists some $\delta > 0$, depending on ϵ and Γ, such that for any polygonal approximation $\tilde{\Gamma}$ of Γ with $\triangle(\tilde{\Gamma}) < \delta$ there holds $0 \leq \mathcal{L}(\Gamma) - \mathcal{L}(\tilde{\Gamma}) < \epsilon$.*

Proof: We choose an arbitrary $\epsilon > 0$ and some arbitrary polygonal approximation Γ^* of Γ with l vertices $A_1^*, \ldots, A_l^*$ and with

$$0 \leq \mathcal{L}(\Gamma) - \mathcal{L}(\Gamma^*) < \frac{\epsilon}{2}. \tag{5.6}$$

Moreover we consider an arbitrary polygonal approximation $\tilde{\Gamma}$ of Γ with the vertices $\tilde{A}_1, \ldots, \tilde{A}_M$ and with $\triangle(\tilde{\Gamma}) < \delta$, where δ will be determined later. Now we work with their common refinement $\Gamma' := \Gamma^* \vee \tilde{\Gamma}$. The summands in the expressions of $\mathcal{L}(\Gamma')$ and $\mathcal{L}(\tilde{\Gamma})$ only differ if there are vertices $A_j^*, \ldots, A_{j+i}^*$, $i \geq 0$, of Γ^* on some open arc $\mathring{\Gamma}\mid_{(\tilde{A}_m,\tilde{A}_{m+1})}$, $m \in \{0, \ldots, M-1\}$ ($\tilde{A}_0 := \tilde{A}_M$). We can estimate the respective contribution to $\mathcal{L}(\Gamma')$ by the chord-arc condition (**??**) imposed on Γ as follows:

$$\begin{aligned} \mid \tilde{A}_m - A_j^* \mid + \mid A_j^* - A_{j+1}^* \mid + \ldots + \mid A_{j+i}^* - \tilde{A}_{m+1} \mid &\leq \mathcal{L}(\Gamma\mid_{(\tilde{A}_m,\tilde{A}_{m+1})}) \\ &\leq C \mid \tilde{A}_m - \tilde{A}_{m+1} \mid \leq C\,\triangle(\tilde{\Gamma}) < C\delta. \end{aligned}$$

As Γ^* has l vertices and as Γ' is a refinement of Γ^* we are led to the rough estimate

$$0 \leq \mathcal{L}(\Gamma) - \mathcal{L}(\tilde{\Gamma}) \leq \mathcal{L}(\Gamma) - \mathcal{L}(\Gamma') + lC\delta \leq \mathcal{L}(\Gamma) - \mathcal{L}(\Gamma^*) + lC\delta < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where we used (**??**) and set $\delta := \frac{\epsilon}{2lC}$ which in fact only depends on ϵ and Γ.

◇

Proof of Proposition **??**: Firstly we note that Lemma **??** guarantees the existence of a sequence $\{\Gamma^n\}$ of polygonal approximations of Γ which satisfies (**??**) and (**??**). Moreover let

$$(P_0^n, A_1^n, \ldots, A_{l_n}^n; P_1^n; A_{l_n+1}^n, \ldots, A_{m_n}^n; P_2^n; A_{m_n+1}^n, \ldots, A_{N_n}^n) \tag{5.7}$$

denote the vertices of Γ^n, where we may assume that the three points $\{P_k^n\}$ of the three-point-condition in $\mathcal{C}^*(\Gamma^n)$ satisfy $P_k^n \equiv P_k$, $k = 0, 1, 2$, (see (**??**)) and where $0 \leq l_n \leq m_n \leq N_n$ are fixed for each $n \in \mathbb{N}$. Now we define the homeomorphisms φ^n. To this end we consider a parametrization $\gamma : \mathbb{S}^1 \xrightarrow{\cong} \Gamma$ of Γ with $\gamma(e^{i\psi_k}) = P_k$, for $\psi_k = \frac{2\pi k}{3}$, $k = 0,1,2$, and some fixed Γ^n, which yields unique angles

$$0 = \psi_0 < \theta_1^n < \ldots < \theta_{l_n}^n < \psi_1 < \theta_{l_n+1}^n < \ldots < \theta_{m_n}^n < \psi_2 < \theta_{m_n+1}^n < \ldots < \theta_{N_n}^n < 2\pi, \tag{5.8}$$

such that $\gamma(e^{i\theta_j^n}) = A_j^n$ for $j = 1, \ldots, N_n$. Now we fix some interval $[\theta_{j-1}^n, \theta_j^n]$ that does not contain any angle ψ_k, $k = 0,1,2$, and set

$$Q_j^n(t) := \frac{\mathcal{L}(\gamma \mid_{[\theta_{j-1}^n, t]})}{\mathcal{L}(\gamma \mid_{[\theta_{j-1}^n, \theta_j^n]})} \qquad \textit{and} \qquad f_j^n(t) := \theta_{j-1}^n + Q_j^n(t)\,(\theta_j^n - \theta_{j-1}^n)$$

for $t \in [\theta_{j-1}^n, \theta_j^n]$ and furthermore

$$\gamma_j^n(e^{it}) := \frac{\theta_j^n - f_j^n(t)}{\theta_j^n - \theta_{j-1}^n} A_{j-1}^n + \frac{f_j^n(t) - \theta_{j-1}^n}{\theta_j^n - \theta_{j-1}^n} A_j^n \qquad \textit{for } t \in [\theta_{j-1}^n, \theta_j^n]. \tag{5.9}$$

These terms are defined analogously on intervals like $[\theta_{l_n}^n, \psi_1]$ and $[\psi_1, \theta_{l_n+1}^n]$ and so on. Hence, the collection of functions in (**??**) yields a homeomorphism $\gamma^n : \mathbb{S}^1 \xrightarrow{\cong} \Gamma^n$, mapping the arcs $[e^{i\theta_{j-1}^n}, e^{i\theta_j^n}]$ resp. $[e^{i\theta_{l_n}^n}, e^{i\psi_1}]$, and so on, onto the line segments $[A_{j-1}^n, A_j^n]$ resp. $[A_{l_n}^n, P_1]$ of Γ^n. Now the compositions $\varphi^n := \gamma^n \circ \gamma^{-1} : \Gamma \xrightarrow{\cong} \Gamma^n$, $n \in \mathbb{N}$, will turn out to have the required properties. Due to $\triangle(\Gamma^n) \longrightarrow 0$ and (**??**) there is for every $\epsilon > 0$ an $N(\epsilon) \in \mathbb{N}$ such that for two arbitrary points $P^1, P^2 \in \Gamma \mid_{(A_{j-1}^n, A_j^n)}$, for any $j \in \{1, \ldots, N_n\}$, ($A_0^n := A_{N_n}^n$) there holds:

$$\mid P^1 - P^2 \mid \leq \mathcal{L}(\Gamma \mid_{(A_{j-1}^n, A_j^n)}) \leq C \mid A_{j-1}^n - A_j^n \mid \leq 4C \, \triangle(\Gamma^n) < \frac{\epsilon}{2} \qquad \forall\, n > N(\epsilon).$$

Hence, we obtain for any point $P \in \Gamma \mid_{(A_{j-1}^n, A_j^n)}$ and any $j \in \{1, \ldots, N_n\}$:

$$\mid P - \varphi^n(P) \mid \leq \mid P - A_j^n \mid + \mid A_j^n - \varphi^n(P) \mid \leq \mid P - A_j^n \mid + \mid A_j^n - A_{j-1}^n \mid < 2\,\frac{\epsilon}{2} = \epsilon,$$

$\forall\, n > N(\epsilon)$, which proves the assertion (**??**). Now for some fixed $n \in \mathbb{N}$, some interval $[\theta_{j-1}^n, \theta_j^n]$ that does not contain any angle ψ_k and for any two angles $\vartheta_1 < \vartheta_2 \in [\theta_{j-1}^n, \theta_j^n]$ we consider the quotient

$$Q_j^n(\vartheta_1, \vartheta_2) := \frac{\mathcal{L}(\gamma \mid_{[\vartheta_1, \vartheta_2]})}{\mathcal{L}(\Gamma \mid_{(A_{j-1}^n, A_j^n)})} = Q_j^n(\vartheta_2) - Q_j^n(\vartheta_1). \tag{5.10}$$

For the corresponding two points $P^1 = \gamma(e^{i\vartheta_1})$, $P^2 = \gamma(e^{i\vartheta_2})$ we show:

$$\mid \varphi^n(P^2) - \varphi^n(P^1) \mid = Q_j^n(\vartheta_1, \vartheta_2) \mid A_{j-1}^n - A_j^n \mid . \tag{5.11}$$

To this end we calculate by (**??**):

$$\begin{aligned} \gamma_j^n(e^{i\vartheta_l}) - A_{j-1}^n &= \frac{\theta_j^n - (\theta_{j-1}^n + Q_j^n(\vartheta_l)\,(\theta_j^n - \theta_{j-1}^n))}{\theta_j^n - \theta_{j-1}^n} A_{j-1}^n + \\ &\qquad \frac{\theta_{j-1}^n + Q_j^n(\vartheta_l)\,(\theta_j^n - \theta_{j-1}^n) - \theta_{j-1}^n}{\theta_j^n - \theta_{j-1}^n} A_j^n - A_{j-1}^n \\ &= (1 - Q_j^n(\vartheta_l))\, A_{j-1}^n + Q_j^n(\vartheta_l)\, A_j^n - A_{j-1}^n = Q_j^n(\vartheta_l)\,(A_j^n - A_{j-1}^n), \end{aligned}$$

for $l=1,2$. Thus we obtain together with (??):

$$
\begin{aligned}
\mid \varphi^n(P^2)-\varphi^n(P^1) \mid &= \mid \gamma_j^n(e^{i\vartheta_2})-\gamma_j^n(e^{i\vartheta_1}) \mid = \mid \gamma_j^n(e^{i\vartheta_2})-A_{j-1}^n \mid - \mid \gamma_j^n(e^{i\vartheta_1})-A_{j-1}^n \mid \\
&= Q_j^n(\vartheta_2) \mid A_j^n - A_{j-1}^n \mid -Q_j^n(\vartheta_1) \mid A_j^n - A_{j-1}^n \mid = Q_j^n(\vartheta_1,\vartheta_2) \mid A_j^n - A_{j-1}^n \mid,
\end{aligned}
$$

which is (??) and which implies by (??) the estimate

$$
\mid \varphi^n(P^2)-\varphi^n(P^1) \mid \leq Q_j^n(\vartheta_1,\vartheta_2)\,\mathcal{L}(\Gamma \mid_{(A_{j-1}^n,A_j^n)}) = \mathcal{L}(\gamma \mid_{[\vartheta_1,\vartheta_2]}), \tag{5.12}
$$

which proves (??) in the special case $P^1,P^2 \in \Gamma \mid_{(A_{j-1}^n,A_j^n)}$, where $\Gamma \mid_{(A_{j-1}^n,A_j^n)}$ does not contain any of the points $\{P_k\}_{k=0,1,2}$. Now for the general case it suffices to consider the situation $P^1 \in \Gamma \mid_{(A_{j-1}^n,A_j^n)}$, $P^2 \in \Gamma \mid_{(A_{l-1}^n,A_l^n)}$, for some fixed n and $j \leq l-1 \in \{2,\ldots,N_n-1\}$, such that the shorter arc $\Gamma \mid_{(A_{j-1}^n,A_l^n)}$ connecting A_{j-1}^n and A_l^n on Γ coincides with image$(\gamma \mid_{[\theta_{j-1}^n,\theta_l^n]})$ and such that $\psi_k \notin [\theta_{j-1}^n,\theta_l^n]$, $k=0,1,2$. Then setting again $P^l=\gamma(e^{i\vartheta_l})$, $l=1,2$, we infer by $A_j^n=\varphi^n(A_j^n)=\varphi^n(\gamma(e^{i\theta_j^n}))$ and (??):

$$
\begin{aligned}
&\mid \varphi^n(P^1)-\varphi^n(P^2) \mid \leq \mid \varphi^n(P^1)-A_j^n \mid + \mathcal{L}(\Gamma \mid_{(A_j^n,A_{l-1}^n)}) + \mid A_{l-1}^n - \varphi^n(P^2) \mid \\
&\leq \mathcal{L}(\gamma \mid_{[\vartheta_1,\theta_j^n]}) + \mathcal{L}(\gamma \mid_{[\theta_j^n,\theta_{l-1}^n]}) + \mathcal{L}(\gamma \mid_{[\theta_{l-1}^n,\vartheta_2]}) = \mathcal{L}(\gamma \mid_{[\vartheta_1,\vartheta_2]}) = \mathcal{L}(\Gamma \mid_{(P^1,P^2)}),
\end{aligned}
$$

which proves the assertion (??). The last statement about the φ^n is clear by their construction.

$\diamond$

Now let Γ be a fixed, closed rectifiable Jordan curve in $\mathbb{R}^3$ meeting a chord-arc condition (??) and $\{\Gamma^n\}$ a fixed sequence of polygonal approximations as in Prop. ?? with vertices as in (??). We consider some arbitrarily chosen $\mathcal{I}$-surface $X \in \mathcal{C}^*(\Gamma)$ and the sequence of boundary values $\varphi^n(X \mid_{\partial B}) : \mathbb{S}^1 \longrightarrow \Gamma^n$ which by their surjectivity give rise to a sequence of angles

$$
0=\psi_0<\tau_1^n<\ldots<\tau_{l_n}^n<\psi_1<\tau_{l_n+1}^n<\ldots<\tau_{m_n}^n<\psi_2<\tau_{m_n+1}^n<\ldots<\tau_{N_n}^n<2\pi, \tag{5.13}
$$

with $\psi_k=\frac{2\pi k}{3}$, for every $n \in \mathbb{N}$ such that

$$
\varphi^n(X \mid_{\partial B})(e^{i\tau_j^n}) = A_j^n \qquad \text{for } j=1,\ldots,N_n, \tag{5.14}
$$

$$
\text{resp.} \qquad \varphi^n(X \mid_{\partial B})(e^{i\psi_k}) \equiv P_k \qquad \text{for } k=0,1,2. \tag{5.15}
$$

Hence, we obtain a sequence of tuples $\tau^n \in T^n \subset (0,2\pi)^{N_n}$ (see Def. 6.1 in [?]) which yield the unique minimizers $X(\tau^n)$ of $\mathcal{I}$ in the sets $\mathcal{U}(\Gamma^n,\tau^n)$ (see (4), (5) and Def. 6.2, 6.3 in [?]). We are going to prove the crucial

Proposition 5.2 *There holds*

$$
X(\tau^n) \longrightarrow X \qquad \text{in } C^0(\bar{B},\mathbb{R}^3), \tag{5.16}
$$

$$
\mathcal{I}(X(\tau^n)) \longrightarrow \mathcal{I}(X) \qquad \text{for } n \to \infty. \tag{5.17}
$$

Proof: We set $Z^n := \varphi^n(X\mid_{\partial B})$ and $\eta^n := Z^n - X\mid_{\partial B}$ and consider the harmonic extensions h resp. h^n of $X\mid_{\partial B}$ resp. η^n onto $\bar{B}$. By (**??**) and (**??**) we derive the estimate

$$\begin{aligned} &\mid \eta^n(e^{i\alpha}) - \eta^n(e^{i\beta}) \mid \leq \mid X(e^{i\alpha}) - X(e^{i\beta}) \mid + \mid Z^n(e^{i\alpha}) - Z^n(e^{i\beta}) \mid \\ &\leq \mid X(e^{i\alpha}) - X(e^{i\beta}) \mid + \mathcal{L}(\Gamma \mid_{(X(e^{i\alpha}),X(e^{i\beta}))}) \leq (1+C) \mid X(e^{i\alpha}) - X(e^{i\beta}) \mid \end{aligned} \tag{5.18}$$

$\forall\, \alpha,\beta \in [0,2\pi]$. Now combining this with Douglas' formula (2.23) in [?] (see [?], p. 277, for a proof) and (**??**) we infer:

$$\begin{aligned} \mathcal{A}_0(\eta^n) &:= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\mid \eta^n(e^{i\alpha}) - \eta^n(e^{i\beta}) \mid^2}{4\sin^2(\frac{\alpha-\beta}{2})}\, d\alpha d\beta \\ &\leq \frac{(1+C)^2}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\mid X(e^{i\alpha}) - X(e^{i\beta}) \mid^2}{4\sin^2(\frac{\alpha-\beta}{2})}\, d\alpha d\beta \\ &= (1+C)^2\,\mathcal{A}_0(X\mid_{\partial B}) = (1+C)^2\,\mathcal{D}(h) \leq (1+C)^2\,\mathcal{D}(X). \end{aligned}$$

Hence, $(1+C)^2\,\frac{|X(e^{i\alpha})-X(e^{i\beta})|^2}{4\sin^2(\frac{\alpha-\beta}{2})}$ yields a Lebesgue dominant for the integrands $\frac{|\eta^n(e^{i\alpha})-\eta^n(e^{i\beta})|^2}{4\sin^2(\frac{\alpha-\beta}{2})}$ on $[0,2\pi]^2$. Moreover by (**??**) we see that

$$\eta^n = \varphi^n(X\mid_{\partial B}) - X\mid_{\partial B} \longrightarrow 0 \qquad in\ C^0(\partial B,\mathbb{R}^3). \tag{5.19}$$

Hence, using Douglas' formula again we can infer by Lebesgue's convergence theorem:

$$\mathcal{D}(h^n) = \mathcal{A}_0(\eta^n) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\mid \eta^n(e^{i\alpha}) - \eta^n(e^{i\beta}) \mid^2}{4\sin^2(\frac{\alpha-\beta}{2})}\, d\alpha d\beta \longrightarrow 0, \tag{5.20}$$

and by the weak maximum principle for harmonic functions:

$$h^n \longrightarrow 0 \qquad in\ C^0(\bar{B},\mathbb{R}^3). \tag{5.21}$$

Furthermore we consider the surfaces $X^n := X + h^n$ on $\bar{B}$. By (**??**) we have that $\mathcal{D}(X^n - X) = \mathcal{D}(h^n) \longrightarrow 0$, hence, together with $\mathcal{D}(X^n) \leq 2\,(\mathcal{D}(X) + \mathcal{D}(h^n)) \leq$const. Prop. **??** yields

$$\mid \mathcal{I}(X^n) - \mathcal{I}(X) \mid \leq const.\sqrt{\mathcal{D}(X^n - X)} \longrightarrow 0 \qquad for\ n \to \infty. \tag{5.22}$$

Moreover we see $X^n\mid_{\partial B} = X\mid_{\partial B} + \eta^n = X\mid_{\partial B} + Z^n - X\mid_{\partial B} = \varphi^n(X\mid_{\partial B})$. Hence, since $\varphi^n(X\mid_{\partial B}) : \mathbb{S}^1 \longrightarrow \Gamma^n$ yields a weakly monotonic continuous map satisfying (**??**) and (**??**) and since $h^n \in H^{1,2}(B,\mathbb{R}^3)$ by (**??**) and (**??**) we obtain that $X^n \in \mathcal{U}(\Gamma^n,\tau^n)$, $\forall\, n \in \mathbb{N}$ (see (4), (5) and Def. 6.2 in [?]). Thus we conclude for the unique minimizer $X(\tau^n)$ of $\mathcal{I}$ in $\mathcal{U}(\Gamma^n,\tau^n)$ $\mathcal{I}(X(\tau^n)) \leq \mathcal{I}(X^n)$, $\forall\, n \in \mathbb{N}$, which implies together with (**??**):

$$\limsup_{n\to\infty} \mathcal{I}(X(\tau^n)) \leq \limsup_{n\to\infty} \mathcal{I}(X^n) = \lim_{n\to\infty} \mathcal{I}(X^n) = \mathcal{I}(X), \tag{5.23}$$

especially

$$\mathcal{D}(X(\tau^n)) \leq const. \qquad \forall\, n \in \mathbb{N}. \tag{5.24}$$

Moreover using that both $X(\tau^n), X^n \in \mathcal{U}(\Gamma^n, \tau^n)$ we gain together with (**??**) and (**??**):

$$\begin{aligned} \mid (X(\tau^n) - X) \mid_{\partial B} \mid \leq \mid (X(\tau^n) - X^n) \mid_{\partial B} \mid + \mid (X^n - X) \mid_{\partial B} \mid \\ \leq \triangle(\Gamma^n) + \mid \eta^n \mid \longrightarrow 0 \qquad in\ C^0(\partial B). \end{aligned} \tag{5.25}$$

Now recalling that the $X(\tau^n)$ are $\mathcal{I}$-surfaces in particular (see Def. 2.1 and 6.3 in [**?**]) we infer by (**??**) and (**??**) that we may apply Theorems **??** and **??** which yield a subsequence $X(\tau^{n_j})$ satisfying

$$X(\tau^{n_j}) \longrightarrow \bar{X} \qquad in\ C^0(\bar{B}, \mathbb{R}^3), \tag{5.26}$$

for some $\mathcal{I}$-surface $\bar{X}$. Again by (**??**) we conclude that $\bar{X} \mid_{\partial B} = X \mid_{\partial B}$. Thus as we required X to be an $\mathcal{I}$-surface the uniqueness of $\mathcal{I}$-surfaces, by Theorem 4.3 in [**?**], yields $\bar{X} = X$. Hence, we gain the assertion (**??**) by (**??**) and the "principle of subsequences". Now combining this with Theorem **??** again we arrive at

$$X(\tau^{n_j}) \rightharpoonup X \qquad in\ H^{1,2}(B, \mathbb{R}^3).$$

Hence, on account of the weak lower semicontinuity of $\mathcal{I}$ and (**??**) we finally achieve:

$$\limsup_{j\to\infty} \mathcal{I}(X(\tau^{n_j})) \leq \limsup_{n\to\infty} \mathcal{I}(X(\tau^n)) \leq \mathcal{I}(X) \leq \liminf_{j\to\infty} \mathcal{I}(X(\tau^{n_j})).$$

Thus we obtain the assertion (**??**) again by the "principle of subsequences".

◇

Finally we need a compactness result which is also proved in [**?**], p. 208:

Proposition 5.3 *Let Γ and $\{\Gamma^n\}$ be as in Proposition **??** and $X^n \in \mathcal{C}^*(\Gamma^n)$, $n \in \mathbb{N}$, a sequence of surfaces with $\mathcal{D}(X^n) \leq const.$, $\forall\, n \in \mathbb{N}$, satisfying the three-point-condition $X^n(e^{i\psi_k}) = P_k \in \Gamma \ \ \forall\, n \in \mathbb{N}$ (see (**??**) and (**??**)). Then there exists a subsequence $\{X^{n_k}\}$ whose boundary values satisfy:*

$$X^{n_k} \mid_{\partial B} \longrightarrow \beta \qquad \text{in } C^0(\partial B, \mathbb{R}^3),$$

where $\beta : \mathbb{S}^1 \longrightarrow \Gamma$ is a continuous, weakly monotonic map onto Γ, with $\beta(e^{i\psi_k}) = P_k$.

Proof: We consider a fixed parametrization $\gamma : \mathbb{S}^1 \xrightarrow{\cong} \Gamma$ of Γ and the weakly monotonic maps $(\varphi^n)^{-1} \circ X^n \mid_{\partial B}: \partial B \longrightarrow \Gamma$ onto Γ. For each $n \in \mathbb{N}$ there exist non-decreasing maps $\sigma^n : [0, 2\pi] \longrightarrow [0, 4\pi]$, with $\sigma^n(2\pi) = \sigma^n(0) + 2\pi$, such that $(\varphi^n)^{-1} \circ X^n(e^{it}) = \gamma(e^{i\sigma^n(t)})$ $\forall\, t \in [0, 2\pi]$. By (**??**) we conclude that

$$\begin{aligned} \max_{t\in[0,2\pi]} \mid \gamma(e^{i\sigma^n(t)}) - X^n(e^{it}) \mid = \max_{t\in[0,2\pi]} \mid \gamma(e^{i\sigma^n(t)}) - \varphi^n(\gamma(e^{i\sigma^n(t)})) \mid \\ = \max_{P\in\Gamma} \mid P - \varphi^n(P) \mid \longrightarrow 0 \qquad for\ n \to \infty. \end{aligned} \tag{5.27}$$

Furthermore Helley's selection principle (see [?], p. 248) yields a subsequence $\{\sigma^{n_k}\}$ and a non-decreasing function σ on $[0,2\pi]$ such that

$$\sigma^{n_k}(t) \longrightarrow \sigma(t) \qquad \forall\, t \in [0,2\pi], \qquad \textit{for } k \to \infty, \tag{5.28}$$

thus also $\gamma(e^{i\sigma^{n_k}(t)}) \longrightarrow \gamma(e^{i\sigma(t)})$ $\forall\, t \in [0,2\pi]$. Hence together with (??) we arrive at

$$X^{n_k}(e^{it}) \longrightarrow \gamma(e^{i\sigma(t)}) \qquad \forall\, t \in [0,2\pi], \qquad \textit{for } k \to \infty, \tag{5.29}$$

which especially implies $\gamma(e^{i\sigma(\psi_k)}) = P_k$, $k = 0,1,2$, due to the required three-point-condition imposed on the $X^n\mid_{\partial B}$. Hence, since $P_j \neq P_i$ for $i \neq j$ we see that

$$\sigma(\psi_i) \neq \sigma(\psi_j) \qquad \mod\ 2\pi, \qquad \textit{for } i \neq j. \tag{5.30}$$

Now an extension of Helley's selection principle (see [?], p. 63 and p. 226) provides the uniform convergence of the σ^{n_k} if σ is known to be continuous, what we are going to prove now. We assume σ not to be continuous. As σ is weakly monotonic there exist the one-sided limits $\sigma(t+0)$ and $\sigma(t-0)$, $\forall\, t \in [0,2\pi]$, where we mean $\sigma(0-0) := \sigma(2\pi-0) - 2\pi$ and $\sigma(2\pi+0) := \sigma(0+0) + 2\pi$. The points of discontinuity of σ coincide with those points t^* in which we have $0 < \sigma(t^*+0) - \sigma(t^*-0)$. Moreover there holds $\sigma(t^*+0) - \sigma(t^*-0) < 2\pi$, otherwise on account of the monotonicity of σ and $\sigma(2\pi) = \sigma(0) + 2\pi$ we would have $\sigma(t) \equiv \sigma(t^*-0)$ on $[0,t^*)$ and $\sigma(t) \equiv \sigma(t^*+0)$ on $(t^*,2\pi]$, which contradicts (??). Hence, we conclude that $\sigma(t^*+0) \neq \sigma(t^*-0) \mod 2\pi$ and therefore by the injectivity of γ

$$\gamma(e^{i\sigma(t^*+0)}) \neq \gamma(e^{i\sigma(t^*-0)}) \tag{5.31}$$

in all discontinuity points t^* of σ. Now we fix such a point t^* which we suppose to lie in $(0,2\pi)$ without loss of generality. By (??) we have $\mid \gamma(e^{i\sigma(t^*+0)}) - \gamma(e^{i\sigma(t^*-0)}) \mid = \epsilon > 0$ for some $\epsilon > 0$. Moreover by the existence of the one-sided limits $\sigma(t+0)$, $\sigma(t-0)$ and by the continuity of γ there is some sufficienty small $\alpha > 0$ such that $[t^*-\alpha, t^*+\alpha] \subset (0,2\pi)$ and

$$\begin{aligned} &\mid \gamma(e^{i\sigma(t)}) - \gamma(e^{i\sigma(t^*-0)}) \mid < \frac{\epsilon}{3} \qquad \forall\, t \in (t^*-\alpha, t^*) \\ \text{and} \qquad &\mid \gamma(e^{i\sigma(t)}) - \gamma(e^{i\sigma(t^*+0)}) \mid < \frac{\epsilon}{3} \qquad \forall\, t \in (t^*, t^*+\alpha), \end{aligned}$$

which implies together with (??):

$$\lim_{k\to\infty} \mid X^{n_k}(e^{it'}) - X^{n_k}(e^{it''}) \mid = \mid \gamma(e^{i\sigma(t')}) - \gamma(e^{i\sigma(t'')}) \mid > \frac{\epsilon}{3} \tag{5.32}$$

$\forall\, t' \in (t^*-\alpha, t^*)$ and $\forall\, t'' \in (t^*, t^*+\alpha)$. Now we only consider pairs t', t'' such that $0 < t'' - t^* = t^* - t' < \alpha$. For $r := 2\sin\left(\frac{t^*-t'}{2}\right)$ we have $\partial B_r(e^{it^*}) \cap \partial B = \{e^{it'}, e^{it''}\}$. We introduce the notation $\{w_1(\rho), w_2(\rho)\} := \partial B_\rho(e^{it^*}) \cap \partial B$, for $\rho < 2\sin\left(\frac{\alpha}{2}\right)$. Now making use of the requirement $\mathcal{D}(X^n) \leq \text{const.} =: M \quad \forall\, n \in \mathbb{N}$ and of the Hölder inequality one easily infers from Fatou's lemma that $\liminf_{k\to\infty} \mid X^{n_k}(w_1(\rho)) - X^{n_k}(w_2(\rho)) \mid^2 \frac{1}{\rho} \in L^1([\delta, \sqrt{\delta}])$ for $\delta < 4\sin^2\left(\frac{\alpha}{2}\right)$ and that there holds (see [?], p. 207):

$$\frac{1}{2\pi}\int_\delta^{\sqrt{\delta}} \liminf_{k\to\infty} \mid X^{n_k}(w_1(\rho)) - X^{n_k}(w_2(\rho)) \mid^2 \frac{1}{\rho}\, d\rho \leq M.$$

Combining this with (??) we achieve:

$$M > \frac{\epsilon^2}{18\pi}\int_\delta^{\sqrt{\delta}} \frac{1}{\rho}\, d\rho = \frac{\epsilon^2}{36\pi}\log\left(\frac{1}{\delta}\right) \qquad \forall\, \delta < 4\sin^2\left(\frac{\alpha}{2}\right),$$

which yields a contradiction letting $\delta \searrow 0$. Hence, σ must be continuous on $[0, 2\pi]$ and therefore the convergence in (??) even uniform:

$$\sigma^{n_k} \longrightarrow \sigma \qquad in\ C^0([0,2\pi]).$$

As γ is uniformly continuous on $\mathbb{S}^1$ this yields

$$\gamma(e^{i\sigma^{n_k}(\,\cdot\,)}) \longrightarrow \gamma(e^{i\sigma(\,\cdot\,)}) \qquad in\ C^0([0,2\pi],\mathbb{R}^3),$$

and together with (??) we finally arrive at

$$X^{n_k}(e^{i\,(\,\cdot\,)}) \longrightarrow \gamma(e^{i\sigma(\,\cdot\,)}) \qquad in\ C^0([0,2\pi],\mathbb{R}^3). \tag{5.33}$$

Hence, defining $\beta : \mathbb{S}^1 \longrightarrow \Gamma$ via $\beta(e^{i\,(\,\cdot\,)}) := \gamma(e^{i\sigma(\,\cdot\,)})$ we see that β has in fact the asserted properties due to the continuity and weak monotonicity of σ and since γ is a homeomorphism. Finally $\beta(e^{i\psi_k}) = P_k,\ k = 0,1,2,$ follows immediately from (??).

◇

5.1.2 Limit Superior of continua

In this subsection we are concerned with the following objects (see Section 6.1 in [?]):

Definition 5.3 *Let (Y,d) be some metric space. For any sequence of subsets $\{M^n\}_{n\in\mathbb{N}}$ of Y we define its limit inferior by*

$$\liminf_{n\in\mathbb{N}} M^n := \{y \in Y \mid \exists \text{points } m_n \in M^n \text{ such that } d(m_n, y) \longrightarrow 0 \text{ for } n \to \infty\}$$

and its limit superior by

$$\limsup_{n\in\mathbb{N}} M^n := \{y \in Y \mid \exists \text{some subseq. } \{M^{n_j}\} \text{ of } \{M^n\} \text{ and points } m_j \in M^{n_j} \text{ such that } d(m_j, y) \longrightarrow 0 \text{ for } j \to \infty\}.$$

Furthermore we will make use of the identity

$$\limsup_{n\in\mathbb{N}} M^n = \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\geq k} M^n}, \tag{5.34}$$

which is proved in [?], p. 86. The result of this subsection is (see also [?], p. 388)

Proposition 5.4 *Let $\{M^n\}_{n\in\mathbb{N}}$ be some sequence of compact and connected subsets (continua) of a metric space (Y, d) such that $\overline{\bigcup_{n\in\mathbb{N}} M^n}$ is compact and $\liminf_{n\in\mathbb{N}} M^n \neq \emptyset$. Then $\limsup_{n\in\mathbb{N}} M^n$ is compact and connected, i.e. a continuum again.*

Proof: Using (**??**) we see that

$$M := \limsup_{n\in\mathbb{N}} M^n \subset \overline{\bigcup_{n\in\mathbb{N}} M^n}, \tag{5.35}$$

thus that M is a closed subset of a compact set, by hypothesis, hence compact itself. Now we assume that M is not connected, i.e. there are open subsets O', O'' of Y such that $M' := M \cap O'$ and $M'' := M \cap O''$ satisfy

$$M' \neq \emptyset, \qquad M'' \neq \emptyset, \qquad M' \cup M'' = M, \qquad M' \cap M'' = \emptyset. \tag{5.36}$$

One easily verifies that M' and M'' are closed in M and therefore also compact. Thus together with (**??**) we conclude that $\delta := dist(M', M'') > 0$. Now we set $\epsilon := \frac{\delta}{4}$ and consider the disjoint, open ϵ-neighborhoods M'_ϵ and M''_ϵ of M' and M'' in Y. We choose a point $y \in \liminf_{n\in\mathbb{N}} M^n \subset M$, for which by Definition **??** there exists some sequence $y_n \in M^n$ with $d(y, y_n) \to 0$. Without loss of generality we assume that $y \in M'$, thus there exists some $N(\epsilon) \in \mathbb{N}$ such that

$$M^n \cap M'_\epsilon \neq \emptyset \qquad \forall\, n > N(\epsilon). \tag{5.37}$$

Furthermore by $M'' \neq \emptyset$ and Definition **??** there has to exist some subsequence $\{M^{n_j}\}$ with $M^{n_j} \cap M''_\epsilon \neq \emptyset \;\; \forall j \in \mathbb{N}$. Hence, assuming that $n_j > N(\epsilon) \;\; \forall j \in \mathbb{N}$ we obtain together with (**??**):

$$M^{n_j} \cap M'_\epsilon \neq \emptyset \qquad \text{and} \qquad M^{n_j} \cap M''_\epsilon \neq \emptyset \qquad \forall\, j \in \mathbb{N}.$$

Now, since the sets M^{n_j} are compact and connected we infer from Satz 4.14 in [**?**], p. 46, that there exists for every pair $x_1, x_2 \in M^{n_j}$ and every $\rho > 0$ a finite sequence $\{z_1, \ldots, z_m\} \subset M^{n_j}$ with $z_1 = x_1$, $z_m = x_2$ and $d(z_i, z_{i-1}) < \rho$ for $i = 2, \ldots, m$, where j is fixed now. Hence choosing $x_1 \in M^{n_j} \cap M'_\epsilon$, $x_2 \in M^{n_j} \cap M''_\epsilon$ and $\rho := \epsilon$ we obtain by $\text{dist}(M'_\epsilon, M''_\epsilon) > \delta - 2\epsilon = 2\epsilon$ the existence of some point $z^j \in M^{n_j}$ with $z^j \notin M'_\epsilon \cup M''_\epsilon$, i.e. with

$$dist(z^j, M) \geq \epsilon \qquad \text{for each} \quad j \in \mathbb{N}, \tag{5.38}$$

if we recall (**??**). Now using the required compactness of $\overline{\bigcup_{n\in\mathbb{N}} M^n}$ we obtain the existence of some convergent subsequence $z^{j_k} \longrightarrow z^*$, where the limit point z^* has to lie in M by Definition **??**, which contradicts (**??**).

◇

5.1.3 Mountain pass situation and instability

For the convenience of the reader we firstly recall the definition of the "mountain pass situation" of a pair of surfaces in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ and a pair of points in the configuration space $T \subset (0, 2\pi)^N$ assigned to a simple closed polygon with $N+3$ vertices (see Def. 7.4 and 7.7 in [?]).

Definition 5.4 *(i) Let X_1, X_2 be a pair of surfaces in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$, then we define:*

$$\mathcal{P}_{(X_1,X_2)} := \{\Sigma \subset \mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3) \mid \Sigma \text{ is compact and connected and } \Sigma \supset \{X_1, X_2\}\}.$$

(ii) For a fixed polygon Γ with $N+3$ vertices, $N \geq 1$, and any pair $\tau_1, \tau_2 \in T \subset (0, 2\pi)^N$ we also consider

$$\wp_{(\tau_1,\tau_2)} := \{P \subset T \mid P \text{ is compact and connected, } P \supset \{\tau_1, \tau_2\}\}.$$

Definition 5.5 *a) Two different surfaces $X_1, X_2 \in (\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ are in a "mountain pass situation" with respect to the evaluations by $\mathcal{K} := \mathcal{J}, \mathcal{I}$ if*

$$\sup_{\Sigma} \mathcal{K} > \max\{\mathcal{K}(X_1), \mathcal{K}(X_2)\} \qquad \forall\, \Sigma \in \mathcal{P}_{(X_1,X_2)}.$$

b) Let Γ be a fixed polygon with $N+3$ vertices, $N \geq 1$. Then a pair of different points $\tau_1, \tau_2 \in T \subset (0, 2\pi)^N$ is in a "mountain pass situation" with respect to the evaluation by $f^\Gamma = \mathcal{I} \circ \psi^\Gamma$ (see Def. 6.3 in [?]) if

$$\max_{P} f^\Gamma > \max\{f^\Gamma(\tau_1), f^\Gamma(\tau_2)\} \quad \forall\, P \in \wp_{(\tau_1,\tau_2)}.$$

c) A set $P^* \in \wp_{(\tau_1,\tau_2)}$ with the property

$$\max_{P^*} f^\Gamma = \inf_{P \in \wp_{(\tau_1,\tau_2)}} \max_{P} f^\Gamma =: \beta(\tau_1, \tau_2)$$

is called a minimizing connected set (with respect to (τ_1, τ_2)) and we set

$$P^*_\beta := \{\tau \in P^* \mid f^\Gamma(\tau) = \beta(\tau_1, \tau_2)\}$$

Now analogously to the proof of Proposition 7.8 in [?] we derive

Proposition 5.5 *If there exist two different conformally parametrized surfaces $X_1 \neq X_2$ in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ that are in a mountain pass situation with respect to $\mathcal{J}$, then the unique $\mathcal{I}$-surfaces X_l^* in the boundary value classes $H^{1,2}_{X_l|_{\partial B}}(B, \mathbb{R}^3)$, $l = 1, 2$, are in a mountain pass situation with respect to $\mathcal{I}$.*

Proof: Since X_1 and X_2 are assumed to be conformally parametrized and $\mathcal{I} \geq \mathcal{J}$ we obtain by hypothesis

$$\sup_{\Sigma} \mathcal{I} \geq \sup_{\Sigma} \mathcal{J} > \max\{\mathcal{J}(X_1), \mathcal{J}(X_2)\} = \max\{\mathcal{I}(X_1), \mathcal{I}(X_2)\} \tag{5.39}$$

$\forall\, \Sigma \in \mathcal{P}_{(X_1,X_2)}$; thus the pair (X_1, X_2) is in a mountain pass situation with respect to $\mathcal{I}$, as well. By Lemma 2.2 and Theorem 4.3 in [?] there exist unique $\mathcal{I}$-surfaces X_l^* in the boundary value classes $H^{1,2}_{X_l|\partial B}(B, \mathbb{R}^3)$, $l = 1, 2$, and Corollary 4.5 in [?] guarantees that the functions $\mathcal{I}(H_l(\,\cdot\,)) : [0, 1] \longrightarrow \mathbb{R}$ are non-decreasing, where $H_l(t) := X_l^* + t\,(X_l - X_l^*)$ for $t \in [0, 1]$, $l = 1, 2$. Combining this with (**??**) we obtain

$$\mathcal{I}(H_l(t)) < \sup_{\Sigma} \mathcal{I} \qquad \forall\, \Sigma \in \mathcal{P}_{(X_1,X_2)} \text{ and } \forall\, t \in [0, 1],\ l = 1, 2. \tag{5.40}$$

Suppose now that X_1^* and X_2^* could be connected by some $\Pi \in \mathcal{P}_{(X_1^*,X_2^*)}$ satisfying

$$\sup_{\Pi} \mathcal{I} < \sup_{\Sigma} \mathcal{I} \qquad \forall\, \Sigma \in \mathcal{P}_{(X_1,X_2)}. \tag{5.41}$$

Since $\text{image}(H_l) \subset \mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ is compact and connected and $\text{image}(H_l) \cap \Pi = \{X_l^*\}$, for $l = 1, 2$, the union $\tilde{\Pi} := \text{image}(H_1) \cup \Pi \cup \text{image}(H_2)$ is a compact connected subset of $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ that contains X_1 and X_2, hence $\tilde{\Pi} \in \mathcal{P}_{(X_1,X_2)}$. On the other hand (**??**) and (**??**) imply $\sup_{\tilde{\Pi}} \mathcal{I} < \sup_{\Sigma} \mathcal{I}\ \forall\, \Sigma \in \mathcal{P}_{(X_1,X_2)}$, in contradiction to $\tilde{\Pi} \in \mathcal{P}_{(X_1,X_2)}$. Thus together with (**??**) we obtain that for every $\Pi \in \mathcal{P}_{(X_1^*,X_2^*)}$ there is some $\Sigma^* \in \mathcal{P}_{(X_1,X_2)}$ with the property

$$\sup_{\Pi} \mathcal{I} \geq \sup_{\Sigma^*} \mathcal{I} > \max\{\mathcal{I}(X_1), \mathcal{I}(X_2)\} \geq \max\{\mathcal{I}(X_1^*), \mathcal{I}(X_2^*)\},$$

hence, the pair (X_1^*, X_2^*) is in a mountain pass situation with respect to $\mathcal{I}$.

◇

Finally we recall the notion of "instability" of $\mathcal{J}$-extremal surfaces.

Definition 5.6 *We call a $\mathcal{J}$-extremal surface $X^* \in (\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ $\mathcal{K}$-unstable, for $\mathcal{K} = \mathcal{I}, \mathcal{J}$, if in every ϵ-ball $B_\epsilon(X^*) \cap \mathcal{C}^*(\Gamma)$ around X^* there is some surface $\tilde{X}$ such that*

$$\mathcal{K}(\tilde{X}) < \mathcal{K}(X^*).$$

5.2 Proof of the main result

Firstly by Prop. **??** we obtain the existence of two $\mathcal{I}$-surfaces $X_l^* \in H^{1,2}_{X_l|\partial B}(B, \mathbb{R}^3)$, $l = 1, 2$, that satisfy

$$\sup_{\Sigma} \mathcal{I} > \max_{l=1,2}\{\mathcal{I}(X_l^*)\} \qquad \forall\, \Sigma \in \mathcal{P}_{(X_1^*,X_2^*)}. \tag{5.42}$$

Now let $\{\Gamma^n\}$ be a fixed sequence of polygonal approximations as in Prop. **??** whose vertices are given in (**??**) and $Z_l^n := \varphi^n(X_l^* \mid_{\partial B})$, for $l = 1, 2$, $n \in \mathbb{N}$. As explained in (**??**) and (**??**) we gain two sequences of tuples $\tau_l^n \in T^n \subset (0, 2\pi)^{N_n}$ with

$$Z_l^n(e^{i(\tau_l^n)_j}) = A_j^n, \qquad l = 1, 2, \quad j = 1, \ldots, N_n, \quad \forall\, n \in \mathbb{N},$$

that yield the unique minimizers $X(\tau_l^n)$ of $\mathcal{I}$ in $\mathcal{U}(\Gamma^n, \tau_l^n)$ which satisfy by Prop. **??**:

$$X(\tau_l^n) \longrightarrow X_l^* \qquad in\ C^0(\bar{B}, \mathbb{R}^3), \quad l = 1, 2, \tag{5.43}$$

$$\mathcal{I}(X(\tau_l^n)) \longrightarrow \mathcal{I}(X_l^*) \qquad for\ n \to \infty, \quad l = 1, 2. \tag{5.44}$$

Furthermore by Prop. 7.6 in [?] there exists a minimizing connected set $P^n \in \wp_{(\tau_1^n, \tau_2^n)}$ w. r. to the pair $\{\tau_l^n\}$ for every $n \in \mathbb{N}$, and we firstly prove that

$$\beta^n := \max_{P^n} f^{\Gamma^n} \le \max\{\mathcal{I}(X(\tau_1^n)), \mathcal{I}(X(\tau_2^n)), C\,\mathcal{L}(\Gamma^n)^2\} \qquad \forall\, n \in \mathbb{N}, \tag{5.45}$$

with $C := \left(1 + \frac{k}{m_1}\right)\frac{m_2}{4}$. For, if we assume that $\beta^n > \max\{\mathcal{I}(X(\tau_1^n)), \mathcal{I}(X(\tau_2^n))\} = \max\{f^{\Gamma^n}(\tau_1^n), f^{\Gamma^n}(\tau_2^n)\}$ for some $n \in \mathbb{N}$, then the pair $\{\tau_l^n\}$ is in a mountain pass situation w. r. to f^{Γ^n}, and the "finite dimensional" mountain pass lemma, Lemma 7.10 in [?], yields the existence of a critical point $\bar{\tau}^n \in P_{\beta^n}^n$ of f^{Γ^n}. Then by Theorem 6.17 in [?] the surface $X(\bar{\tau}^n) = \psi(\bar{\tau}^n)$ is a (a.e.) conformally parametrized $\mathcal{I}$-surface. Hence, in combination with $f^{\Gamma^n} = \mathcal{I} \circ \psi^{\Gamma^n}$ and the isoperimetric inequality for $\mathcal{J}$, Corollary **??**, we gain:

$$\beta^n = \max_{P^n} f^{\Gamma^n} = f^{\Gamma^n}(\bar{\tau}^n) = \mathcal{I}(X(\bar{\tau}^n)) = \mathcal{J}(X(\bar{\tau}^n)) \le C\,\mathcal{L}(\Gamma^n)^2,$$

with $C := \left(1 + \frac{k}{m_1}\right)\frac{m_2}{4}$, which proves (**??**). Combining (**??**) with (**??**) and (**??**) we obtain a convergent subsequence

$$\beta^{n_k} \longrightarrow d \qquad \text{for some} \quad d \le \max\{\mathcal{I}(X_1^*), \mathcal{I}(X_2^*), C\,\mathcal{L}(\Gamma)^2\}. \tag{5.46}$$

We rename $\{n_k\}$ into $\{n\}$ again and work with this subsequence henceforth. Now we consider the images $\Pi^n := \psi^{\Gamma^n}(P^n)$ which are compact and connected subsets of $(\mathcal{C}^*(\Gamma^n) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ on account of the continuity of ψ^{Γ^n} with respect to this topology on the target space, in particular, by Theorem 6.6 (i) in [?]. Now we are going to prove the relative compactness of the union $\bigcup_{n\in\mathbb{N}} \Pi^n$ (w. r. to $\|\cdot\|_{C^0(\bar{B})}$). To this end we firstly consider an arbitrary sequence $\{Y^k\} \subset \bigcup_{n\in\mathbb{N}} \Pi^n$. If $\{Y^k\}$ is contained in only finitely many Π^n then we can certainly select a convergent subsequence of $\{Y^k\}$ due to the compactness of the Π^n. Hence, we shall suppose the contrary, which means that we can select a subsequence $\{Y^{k_j}\}$ satisfying $Y^{k_j} \in \Pi^{n_j}$ $\forall\, j \in \mathbb{N}$, where $\{n_j\}$ is a monotonically increasing sequence in $\mathbb{N}$. In particular we have $Y^{k_j} \in \mathcal{C}^*(\Gamma^{n_j}) \cap C^0(\bar{B}, \mathbb{R}^3)$, thus $Y^{k_j}(e^{i\psi_k}) = P_k$ by (**??**), $\forall\, j \in \mathbb{N}$. Furthermore as (**??**) implies $\mathcal{I}(Y) \le \beta^n \le$ const. $\forall\, Y \in \Pi^n$ and $\forall\, n \in \mathbb{N}$, we obtain especially

$$\mathcal{D}(Y) \le const. \qquad \forall\, Y \in \bigcup_{n\in\mathbb{N}} \Pi^n. \tag{5.47}$$

Therefore we may apply Prop. **??** yielding a further subsequence $\{Y^{k_l}\}$ with equicontinuous and uniformly bounded boundary values. Hence, due to (**??**) and since the sets $\Pi^n = \psi^{\Gamma^n}(P^n)$ consist of $\mathcal{I}$-surfaces we see that the Y^{k_l} meet all requirements of Theorem **??** which just guarantees the existence of a further convergent subsequence of $\{Y^{k_l}\}$ w. r. to $\| \cdot \|_{C^0(\bar{B})}$. Now together with a standard argument one also shows that every sequence $\{Y^k\} \subset \overline{\bigcup_{n\in\mathbb{N}} \Pi^n} \setminus \bigcup_{n\in\mathbb{N}} \Pi^n$ possesses a convergent subsequence, aswell, which yields the asserted compactness of $\overline{\bigcup_{n\in\mathbb{N}} \Pi^n}$. Moreover by $X(\tau_l^n) = \psi(\tau_l^n) \in \Pi^n$, for $l = 1, 2$, and recalling (**??**) we infer that

$$\{X_l^*\} \subset \liminf_{n\in\mathbb{N}} \Pi^n. \tag{5.48}$$

Hence, we see that the sequence $\{\Pi^n\}$ satisfies all requirements of Proposition **??** implying that $\Pi := \limsup_{n\in\mathbb{N}} \Pi^n$ is compact and connected, i.e. a continuum again. Now we examine Π. By the definition of Π for any $X \in \Pi$ there exists a subsequence $\{\Pi^{n_k}\}$ and $\mathcal{I}$-surfaces $X^k \in \Pi^{n_k} \subset \mathcal{C}^*(\Gamma^{n_k}) \cap C^0(\bar{B}, \mathbb{R}^3)$ that satisfy

$$X^k \longrightarrow X \qquad in\ C^0(\bar{B}, \mathbb{R}^3). \tag{5.49}$$

Now recalling (**??**) Theorem **??** yields that X has to be an $\mathcal{I}$-surface again which lies in $\mathcal{C}^*(\Gamma)$ on account of Proposition **??** (see again (**??**)). Hence, Π is a continuum consisting of $\mathcal{I}$-surfaces in $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ and containing the pair $\{X_l^*\}$ due to (**??**), which implies $\Pi \in \mathcal{P}_{(X_1^*, X_2^*)}$ in particular and therefore

$$\sup_{\Pi} \mathcal{I} > \max_{l=1,2}\{\mathcal{I}(X_l^*)\} \tag{5.50}$$

on account of (**??**). Next we prove that

$$\beta := \sup_{\Pi} \mathcal{I} \leq d. \tag{5.51}$$

If this would be wrong then there would have to exist some surface $X \in \Pi$ with $\mathcal{I}(X) > d$. By the definition of Π we infer the existence of some sequence $\{X^k\}$ as in (**??**) which implies together with (**??**) $\| X^k \|_{H^{1,2}(B)} \leq$ const.. Hence, we obtain some subsequence $X^j \in \Pi^{n_j}$ with

$$X^j \rightharpoonup X \qquad in\ H^{1,2}(B, \mathbb{R}^3),$$

which yields by the weak lower semicontinuity of $\mathcal{I}$ and (**??**):

$$d < \mathcal{I}(X) \leq \liminf_{j\to\infty} \mathcal{I}(X^j) \leq \liminf_{j\to\infty} \beta^{n_j} = \lim_{n\to\infty} \beta^n = d,$$

which is a contradiction. Hence, combining (**??**) with (**??**), (**??**), (**??**) and $f^{\Gamma^n} = \mathcal{I} \circ \psi^{\Gamma^n}$ we conclude that there exists some $n_0 \in \mathbb{N}$ such that

$$\beta^n > \max_{l=1,2}\{\mathcal{I}(X(\tau_l^n))\} = \max_{l=1,2}\{f^{\Gamma^n}(\tau_l^n)\} \qquad \forall\, n > n_0. \tag{5.52}$$

As below (**??**) this yields by Lemma 7.10 in [**?**] a critical point $\bar{\tau}^n \in P^n_{\beta^n}$ of f^{Γ^n} and by Theorem 6.17 in [**?**] a conformally parametrized $\mathcal{I}$-surface $X(\bar{\tau}^n) \in \Pi^n$ satisfying

$$\beta^n = \mathcal{I}(X(\bar{\tau}^n)) \qquad \forall\, n > n_0. \tag{5.53}$$

Now as below (**??**) we firstly infer by (**??**) (and (**??**)) that we may apply Prop. **??** yielding a subsequence $\{X(\bar{\tau}^{n_k})\}$ with converging boundary values in $C^0(\partial B, \mathbb{R}^3)$, which enables us to apply Theorem **??** to the $\mathcal{I}$-surfaces $X(\bar{\tau}^{n_k})$ guaranteeing the existence of a further convergent subsequence:

$$X(\bar{\tau}^{n_j}) \longrightarrow \bar{X} \qquad \textit{in } C^0(\bar{B}, \mathbb{R}^3). \tag{5.54}$$

Hence, since $X(\bar{\tau}^{n_j}) \in \Pi^{n_j}$ we obtain $\bar{X} \in \Pi$ by the definition of Π, which implies in particular that $\bar{X}$ has to be again an $\mathcal{I}$-surface lying in $\mathcal{C}^*(\Gamma)$. Since we additionally know that the $\mathcal{I}$-surfaces $X(\bar{\tau}^{n_j})$ are conformally parametrized and that

$$\mathcal{L}(X(\bar{\tau}^{n_j})\,|_{\partial B}) = \mathcal{L}(\Gamma^{n_j}) \longrightarrow \mathcal{L}(\Gamma) = \mathcal{L}(\bar{X}\,|_{\partial B}) \qquad \textit{for } j \to \infty$$

on account of the weak monotonicity of the boundary values and (**??**), we infer from Corollary **??** that

$$\mathcal{I}(X(\bar{\tau}^{n_j})) \longrightarrow \mathcal{I}(\bar{X}) \qquad \textit{for } j \to \infty \tag{5.55}$$

and that $\bar{X}$ is also conformally parametrized on B, hence in particular a $\mathcal{J}$-extremal surface by Lemma 3.6 in [**?**]. Now combining (**??**), (**??**), (**??**) and (**??**) with the fact that $\bar{X} \in \Pi$ we arrive at:

$$\beta \leq d \longleftarrow \beta^{n_j} = \mathcal{I}(X(\bar{\tau}^{n_j})) \longrightarrow \mathcal{I}(\bar{X}) \leq \sup_{\Pi} \mathcal{I} = \beta \qquad \textit{for } j \to \infty, \tag{5.56}$$

which implies at once:

$$\mathcal{I}(\bar{X}) = d = \beta, \tag{5.57}$$

i.e. $\bar{X}$ "sits on the top of Π". This gives rise to consider the following set of $\mathcal{J}$-extremal surfaces:

$$\Pi^* := \{X \in \Pi \mid \mathcal{I}(X) = \beta,\ X \text{ is conform. param. on } B\}\ (\neq \emptyset). \tag{5.58}$$

Furthermore (**??**) guarantees that $\Pi \setminus \Pi^* \neq \emptyset$. Now we prove that Π^* is closed. To this end we consider a convergent sequence $\{Y^j\} \subset (\Pi^*, \|\cdot\|_{C^0(\bar{B})})$, i.e.

$$Y^j \longrightarrow Y \qquad \textit{in } C^0(\bar{B}, \mathbb{R}^3).$$

First of all we see that $Y \in \Pi$, as Π is closed. As all Y^j are conformally parametrized $\mathcal{I}$-surfaces in $\mathcal{C}^*(\Gamma)$, satisfying $\mathcal{L}(Y^j\,|_{\partial B}) \equiv \mathcal{L}(\Gamma)$ and $\mathcal{D}(Y^j) \leq \frac{\beta}{k}\ \forall\, j \in \mathbb{N}$ by (**??**) we see due to Corollary **??** that firstly $\beta \equiv \mathcal{I}(Y^j) \longrightarrow \mathcal{I}(Y)$, thus $\mathcal{I}(Y) = \beta$, and secondly that Y is conformally parametrized on B again. Hence, in fact we confirm that $Y \in \Pi^*$. Now combining this with the facts that both Π^* and $\Pi \setminus \Pi^*$ are non-empty and Π connected we can conclude that the boundary $\partial\Pi^*$ of Π^* in Π is also non-empty, i.e. there exist points $X^* \in \Pi^*$ which satisfy $B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*) \neq \emptyset \quad \forall \epsilon > 0$. We choose such a boundary point X^* and show firstly that X^* is $\mathcal{I}$-unstable. To this end we consider the (non-empty) intersection $B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$ for an arbitrarily fixed $\epsilon > 0$. If there were a surface $\tilde{X}$ in $B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$ with $\mathcal{I}(\tilde{X}) < \beta = \mathcal{I}(X^*)$, then we were done. Hence, we have to consider the case in which $\mathcal{I}(Y) \geq \beta \quad \forall Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$, but then we have

$$\beta \leq \mathcal{I}(Y) \leq \sup_{\Pi} \mathcal{I} = \beta, \qquad \text{i.e.} \quad \mathcal{I}(Y) = \beta \quad \forall Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*). \tag{5.59}$$

Now we fix some $Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$ and choose another ball $B_\delta(Y) \subset B_\epsilon(X^*)$ around Y for a sufficiently small $\delta > 0$. Again we only have to consider the case in which

$$\mathcal{I}(Z) \geq \beta = \mathcal{I}(Y) \qquad \forall Z \in B_\delta(Y) \cap \mathcal{C}^*(\Gamma), \tag{5.60}$$

otherwise we were done. Now we choose an arbitrary family $\phi_\epsilon : \bar{B} \xrightarrow{\cong} \bar{B}$ of inner variations of "medium type", i.e. of the class $\mathcal{V}$, as defined in Def. 6.7 in [?], which do not affect the three points $\{e^{i\psi_k}\}$ of the three-point-condition. Then the inner variations $Y \circ \phi_\epsilon$ still satisfy $Y \circ \phi_\epsilon \in B_\delta(Y) \cap \mathcal{C}^*(\Gamma)$, for $| \epsilon | \leq \epsilon_0$ sufficiently small. Hence, we infer by (??):

$$\mathcal{F}(Y) + k\,\mathcal{D}(Y) = \mathcal{I}(Y) \leq \mathcal{I}(Y \circ \phi_\epsilon) = \mathcal{F}(Y \circ \phi_\epsilon) + k\,\mathcal{D}(Y \circ \phi_\epsilon) \qquad \forall \ | \epsilon | \leq \epsilon_0.$$

Together with the invariance of the parametric functional $\mathcal{F}$ w. r. to orientation preserving reparametrizations of $\bar{B}$ we arrive at

$$\mathcal{D}(Y) \leq \mathcal{D}(Y \circ \phi_\epsilon) \qquad \forall \ | \epsilon | \leq \epsilon_0,$$

yielding

$$\partial\mathcal{D}(Y, \lambda) = \frac{d}{d\epsilon}\mathcal{D}(Y \circ \phi_\epsilon) \mid_{\epsilon=0} = 0, \tag{5.61}$$

with $\lambda := \frac{\partial}{\partial \epsilon}\phi_\epsilon \mid_{\epsilon=0}$ (see Prop. 6.10 in [?]). Moreover an arbitrary family $\{\phi_\epsilon\} \in \mathcal{V}$ can be "renormed" by a uniquely determined family of Moebius transformations $\{K_\epsilon\} \subset \text{Aut}(B)$, which means that $\tilde{\phi}_\epsilon := \phi_\epsilon \circ K_\epsilon$ satisfies $\tilde{\phi}_\epsilon(e^{i\psi_k}) \equiv e^{i\psi_k}$ and again $\{\tilde{\phi}_\epsilon\} \in \mathcal{V}$ (see Remark 6.11 in [?] and p. 71 in [?]). Since $\mathcal{D}$ is invariant with respect to conformal reparametrizations of $\bar{B}$ we infer together with (??) for an arbitrary family $\{\phi_\epsilon\} \in \mathcal{V}$:

$$\partial\mathcal{D}(Y, \lambda) = \frac{d}{d\epsilon}\mathcal{D}(Y \circ \phi_\epsilon) \mid_{\epsilon=0} = \frac{d}{d\epsilon}\mathcal{D}(Y \circ \tilde{\phi}_\epsilon) \mid_{\epsilon=0} = \partial\mathcal{D}(Y, \tilde{\lambda}) = 0,$$

with $\lambda := \frac{\partial}{\partial \epsilon}\phi_\epsilon \mid_{\epsilon=0}$ and $\tilde{\lambda} := \frac{\partial}{\partial \epsilon}\tilde{\phi}_\epsilon \mid_{\epsilon=0}$. Now by Lemma 6.18 and Prop. 6.19 in [?] we conclude from this that Y is conformally parametrized on B. Thus together with (??) we conclude $Y \in \Pi^*$, in contradiction to our choice $Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$. Thus in fact there has to be a surface $\tilde{X} \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*) \subset B_\epsilon(X^*) \cap \mathcal{C}^*(\Gamma)$ with $\mathcal{I}(\tilde{X}) < \mathcal{I}(X^*)$. Now using $\mathcal{J} \leq \mathcal{I}$ and that X^* is conformally parametrized we conclude from this:

$$\mathcal{J}(\tilde{X}) \leq \mathcal{I}(\tilde{X}) < \mathcal{I}(X^*) = \mathcal{J}(X^*),$$

which proves the $\mathcal{J}$-instability of the $\mathcal{J}$-extremal surface $X^* \in \mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$.

$\diamond$

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