Homology of Moduli Spaces of Riemann Surfaces

universitätbonn

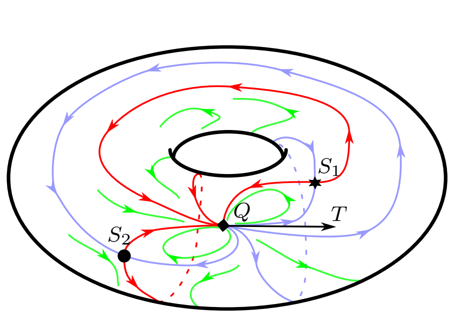
Carl-Friedrich Bödigheimer, Felix Boes, Anna Hermann

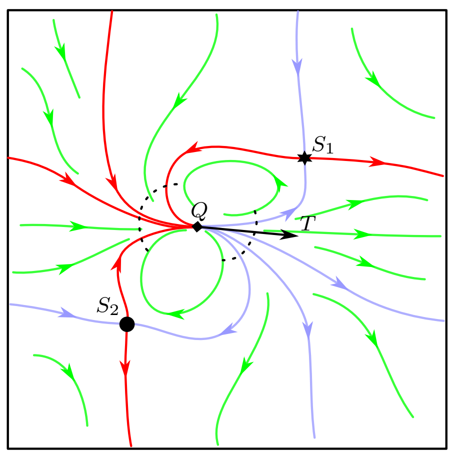
Hilbert Uniformization

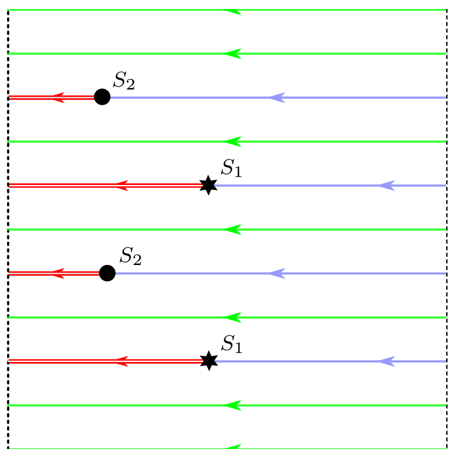
Let $\mathfrak{M}^{\mathrm{par}} = \mathfrak{M}^m_{g,n}$ resp. $\mathfrak{M}^{\mathrm{rad}} = \mathfrak{M}_g(m,n)$ be the moduli spaces of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ boundary curves and $m \geq 0$ punctures resp. with $n \geq 1$ incoming and $m \geq 1$ outgoing boundary curves. At the beginning of our studies is a uniformization method (called *Hilbert unformization*, which is useful in three aspects:

(1) it allows the definition of several operad structures and thus of homology operations,
(2) it gives a cell decomposition,
(3) which in turn leads to the computation of homology groups.

We replace (for technical reasons) in the parallel case $\mathfrak{M}^{\mathrm{par}}$ each boundary curve by a point Q with a non-zero tangent vector T. For the Hilbert uniformization we look at all harmonic functions $u\colon F \longrightarrow \mathbb{R} \cup \infty$ with the following singularities: in the case $\mathfrak{M}^{\mathrm{par}}$ we require a dipole singularity at the n points $Q_1, \ldots, Q_n$ with directions $T_1, \ldots, T_n$, and logarithmic singularities at the punctures $P_1, \ldots, P_m$; in the case $\mathfrak{M}^{\mathrm{rad}}$ we require u to be positive and constant on the incoming curves and vanishing on the outgoing curves. Such a function u is uniquely determined by choosing the obvious constants fixing the differential du; together with integration constants for u and for the harmonic conjugate dv (defined only as a differential) these constants determine a point in a vector bundle $\mathfrak{H}^{\mathrm{par}} \longrightarrow \mathfrak{M}^{\mathrm{par}}$ resp. $\mathfrak{H}^{\mathrm{rad}} \longrightarrow \mathfrak{M}^{\mathrm{rad}}$. If we cut the surface F along the critical graph (i.e. all flow lines of the gradient flow of u leaving a critical point), then F decomposes into n contractible regions (parallel case) resp. into n annular regions (radial case). We can map them biholomorphically onto n planes with parallel slits resp. onto n annuli with radial slits. In the three figures below we picture the parallel case with $g = 1$, $n = 1$, $m = 0$; the three figures to the right show the radial case with $g = 1$, $n = 1$, $m = 1$. The critical graph is drawn in red.







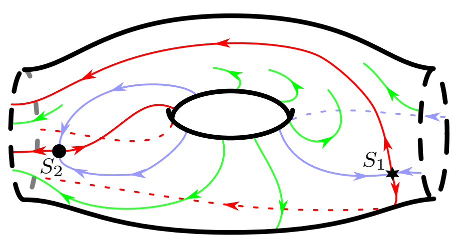
The slit pictures are determined by the combinatorics of their gluing: q permutations in the symmetric group $\mathfrak{S}_p$, where $0 \leq p \leq 2h$ is the number of critical levels of u and $0 \leq q \leq h$ is the number of critical levels of dv; here $h = 2g-2+m+2n$ in the parallel case resp. $h = 2g-2+m+n$ in the radial case. The distances of the critical levels of u resp. of dv, give two sets of barycentric coordinates. Thus we obtain a finite bi-simplicial complex $P = P^m_{g,n}$ resp. $R = R_g(m,n)$ as compactifications of $\mathfrak{H}^{\mathrm{par}}$ resp. of $\mathfrak{H}^{\mathrm{rad}}$, with subcomplexes P' resp. R' containing "degenerate" surfaces; then the Hilbert uniformization is a homoeomorphism

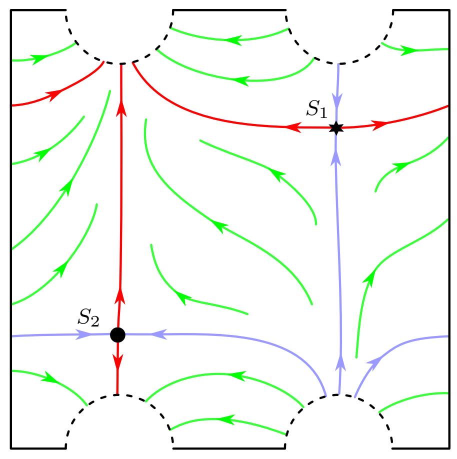
$$\mathfrak{H}^{\mathrm{par}} \xrightarrow{\cong} P - P' \qquad \text{resp.} \qquad \mathfrak{H}^{\mathrm{rad}} \xrightarrow{\cong} R - R'\,.$$

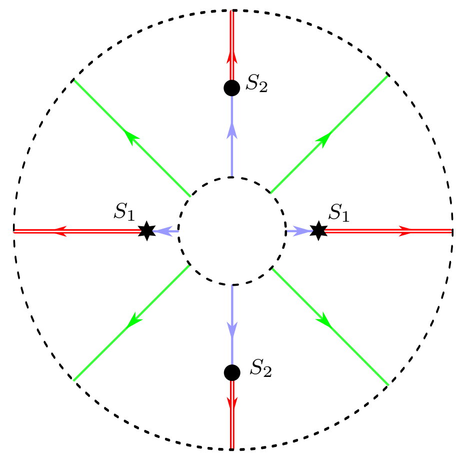
If $\mathcal{O}$ denotes the orientation system, we can compute the homology via Poincaré duality

$$H_*(\mathfrak{M}^{\mathrm{par}};\mathbb{Z}) = H_*(\mathfrak{H}^{\mathrm{par}};\mathbb{Z}) = H^{3h-*}(P,P';\mathcal{O}) \qquad \text{resp.} \qquad H_*(\mathfrak{M}^{\mathrm{rad}};\mathbb{Z}) = H_*(\mathfrak{H}^{\mathrm{rad}};\mathbb{Z}) = H^{3h+n-*}(R,R';\mathcal{O})\,.$$

Homology







Homology Operations. Staring at these slit pictures (parallel or radial) it is apparent how to define a multiplication (i.e., a connected sum operation), yet even various operads, not only the little-2-cube operad, acting on these moduli spaces. Thus there is Hopf algebra structure, Dyer-Lashof operations and many more operations on the homology of moduli spaces.

Small chain complexes. As seen above, the homology of the moduli space can be computed as the (co)homology of the associated simplicial complexes $\mathbb{P} = P/P'$ resp. $\mathbb{P} = R/R'$. The homology of this bi-complex has a remarkable property, noticed by Ehrenfried in [Ehr]: its homology is concentrated in the top degree $q = h$. The reason for this, found by Visy in [Vis], is the factoribility of the symmetric groups; this property, shared by many Coxeter groups, gives a certain normal form for their elements. This leads to a chain complex much smaller than the bar resolution computing the homology of groups. Our calculations could therefore be reduced to a manageable chain complex $\mathbb{E}$ which we call Ehrenfried complex. The following two tables demonstrate this reduction.

The number of cells of the bi-complex $\mathbb{P}$ for $\mathfrak{M}^3_{1,1}$.

$q=5$	640	12425	74610	202825	278600	189000	50400
$q=4$	800	18500	122700	357280	516880	365400	100800
$q=3$	240	7425	57375	185220	289380	217350	63000
$q=2$	10	650	6800	26600	47740	39900	12600
$q=1$	0	0	35	315	910	1050	420
	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$

The number of cells of the Ehrenfried complex $\mathbb{E}$ for $\mathfrak{M}^3_{1,1}$.

70	700	2520	4480	4270	2100	420
$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$

Cluster spectral sequence. Finally, there is a cluster filtration of $\mathbb{P}$ (and thus of $\mathbb{E}$) determined by the number of components of the critical graph. The associated spectral sequence (see [Böd-2]) collapses at page two and allows us to compute the homology up to $h \leq 6$, esp. $g = 3$.

Calculations

The cluster spectral sequence computing the rational homology of $\mathfrak{M}_1(3,1)$ is as follows.

	$E_0^{p,c}(\mathfrak{M}_1(3,1);\mathbb{Q})$				
$p \backslash c$	1	2	3	4	5
2	10				
3	96	14			
4	210	196	4		
5		610	130		
6			680	30	
7				350	
8					70

	$E_1^{p,c}(\mathfrak{M}_1(3,1);\mathbb{Q})$				
$p \backslash c$	1	2	3	4	5
2	0				
3	1	0			
4	125	0	0		
5		428	0		
6			554	0	
7				320	
8					70

	$E_2^{p,c}(\mathfrak{M}_1(3,1);\mathbb{Q})$				
$p \backslash c$	1	2	3	4	5
2	0				
3	1	0			
4	2	0	0		
5		1	0		
6			0	0	
7				1	
8					1

Using a well-known calculation for $H_1(\mathfrak{M}^0_{2,1};\mathbb{Z})$ of Powell [Pow] and one for $H_2(\mathfrak{M}^0_{2,1};\mathbb{Z})$ by Sakasai [Sak] we state here a few examples of our calculations. In the following theorem, $\blacksquare$ denotes in the last case $\mathfrak{M}^0_{3,1}$ possible p-torsion for primes $p > 23$, and in the other cases $\mathfrak{M}_2(2,1)$ possible p-torsion of the form $\mathbb{Z}/p^k\mathbb{Z}$ for p any prime and $k > k_p$ where $k_2 = 6$, $k_3 = 4$, $k_5 = 3$, $k_7 = k_{11} = k_{13} = 2$, $k_{17} = k_{19} = k_{23} = 1$ and $k_p = 0$ for $p > 23$ prime. The first example $\mathfrak{M}^0_{2,1}$ has been calculated by Ehrenfried [Ehr] and Godin [God].

Theorem (Bödigheimer, Boes, Hermann, Mehner, Wang).
The integral homology of the moduli spaces $\mathfrak{M}^0_{2,1}$ and $\mathfrak{M}_2(1,1)$ is

$$H_*(\mathfrak{M}^0_{2,1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/10 & *=1 \\ \mathbb{Z}/2 & *=2 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & *=3 \\ \mathbb{Z}/6 & *=4 \\ 0 & *\geq 5 \end{cases} \qquad \text{and} \qquad H_*(\mathfrak{M}_2(1,1);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/10 & *=1 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & *=2 \\ \mathbb{Z}^2\oplus(\mathbb{Z}/2)^2 & *=3 \\ (\mathbb{Z}/6)^2 & *=4 \\ \mathbb{Z} & *=5 \\ \mathbb{Z} & *=6 \\ 0 & *\geq 7 \end{cases}.$$

The integral homology of the moduli space $\mathfrak{M}_2(2,1)$ is

$$H_*(\mathfrak{M}_2(2,1);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & *=0 \\ (\mathbb{Z}/2)^2\oplus\mathbb{Z}/5\oplus\blacksquare & *=1 \\ \mathbb{Z}\oplus(\mathbb{Z}/2)^2\oplus\blacksquare & *=2 \\ \mathbb{Z}^3\oplus(\mathbb{Z}/2)^4\oplus\blacksquare & *=3 \\ \mathbb{Z}\oplus(\mathbb{Z}/2)^5\oplus(\mathbb{Z}/3)^3\oplus\blacksquare & *=4 \\ \mathbb{Z}^2\oplus(\mathbb{Z}/2)^4\oplus\mathbb{Z}/3\oplus\blacksquare & *=5 \\ \mathbb{Z}^2\oplus(\mathbb{Z}/2)^3\oplus\blacksquare & *=6 \\ \mathbb{Z}/2\oplus\blacksquare & *=7 \\ 0 & *\geq 8 \end{cases}.$$

The integral homology of the moduli space $\mathfrak{M}^0_{3,1}$ is

$$H_*(\mathfrak{M}^0_{3,1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & *=1 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & *=2 \\ \mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3\oplus\mathbb{Z}/4\oplus\mathbb{Z}/7\oplus\blacksquare & *=3 \\ (\mathbb{Z}/2)^2\oplus(\mathbb{Z}/3)^2\oplus\blacksquare & *=4 \\ \mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3\oplus\blacksquare & *=5 \\ \mathbb{Z}\oplus(\mathbb{Z}/2)^3\oplus\blacksquare & *=6 \\ \mathbb{Z}/2\oplus\blacksquare & *=7 \\ 0\oplus\blacksquare & *=8 \\ \mathbb{Z}\oplus\blacksquare & *=9 \\ 0 & *\geq 10 \end{cases}.$$

References

[ABE] J. Abhau, C.-F. Bödigheimer, R. Ehrenfried: *Homology of the mapping class group $\Gamma_{2,1}$ for surfaces of genus 2 with a boundary curve.* In: *The Zieschang Gedenkschrift.* Vol. 14 Geometry and Topology Monograpghs. Geometry and Topology Publications, 2008, pp.1–25.

[Böd-1] C.-F. Bödigheimer: *On the topology of moduli spaces of Riemann surfaces. Part I: Hilbert uniformization.* In: *Mathematica Gottingensis*, Hefte 7+8, 1990.

[Böd-2] C.-F. Bödigheimer: *Cluster spectral sequence.* (in preparation), 2014.

[B,H] F. Boes and A. Hermann: *Moduli Spaces of Riemann Surfaces — Homology Computations and Homology Operations.* Masters Thesis. University of Bonn, 2014.

[Ehr] R. Ehrenfried: *Die Homologie der Modulräume berandeter Riemannscher Flächen von kleinem Geschlecht.* PhD Thesis. University of Bonn, 1997.

[God] V. Godin: *The unstable integral homology of the mapping class groups of a surface with boundary.* In: *Mathematische Annalen* 337.1, 2007, pp. 15–60.

[Meh] S. Mehner: *Homologieberechnungen von Modulräumen Riemannscher Flächen durch diskrete Morsetheorie.* Diploma Thesis. University of Bonn, 2011.

[Pow] J. Powell. *Two theorems on the mapping class group of a surface.* In: *Proceedings of the American Mathematical Society* 68.3, 1978, pp. 347–350.

[Sak] T. Sakasai: *Lagrangian mapping class groups from a group homological point of view.* In: *Algebraic & Geometric Topology* 12.1, 2012, pp. 267–291.

[Vis] B. Visy: *Factorable Groups and their Homology.* PhD Thesis. University of Bonn, 2010.

[Wan] R. Wang: *Homology Computations for Mapping Class Groups, in particular for $\Gamma^0_{3,1}$.* PhD Thesis. University of Bonn, 2011.

cfb@math.uni-bonn.de
boes@math.uni-bonn.de
hermann@or.uni-bonn.de

Mathematisches Institut, Universität Bonn, Germany