The homology of Moduli Spaces of Riemann Surfaces

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Basics I

Introduction A motivating question would be the following: How can one classify all complex structures on a two dimensional manifold F? The first huge step towards a satisfactory answer, is the construction of the moduli space $\mathfrak{M}$. Its underlying points are in one-to-one correspondence with the set of equivalence classes of complex structures. The study of these moduli spaces relates topology, geometry, algebra and mathematical physics.

The moduli space $\mathfrak{M}^m_{g,n}$ Keep $g \geq 0$, $m \geq 0$ and $n \geq 1$ fixed. A surface with structure consists of

(1) a complex surface F of genus g;

(2) a set $\mathcal{P} = \{P_1, \dots, P_m\} \subset F$ of m distinct points;

(3) an ordered set $\mathcal{Q} = (Q_1, \dots, Q_n) \subset F$ of n distinct points disjoint from $\mathcal{P}$;

(4) directions $\mathcal{X} = (X_1, \dots, X_n)$ in the repsectively tangent spaces at $Q_1, \dots, Q_n$.

Two surfaces $[F, \mathcal{P}, \mathcal{Q}, \mathcal{X}]$ and $[F', \mathcal{P}', \mathcal{Q}', \mathcal{X}']$ are equivalent if and only if there is a map $\varphi\colon F \longrightarrow F'$ respecting the structure i.e.

(5) $\varphi\colon F \xrightarrow{\cong} F'$ as complex manifolds.

(6) $\varphi\colon \mathcal{P} \xrightarrow{\cong} \mathcal{P}'$ resp. $\varphi\colon \mathcal{Q} \xrightarrow{\cong} \mathcal{Q}'$ resp. $D\varphi\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}'$ as (un)ordered sets.

The set of equivalence classes embody the moduli space of Riemann surfaces $\mathfrak{M}^m_{g,n}$. The assertion $n \geq 1$ ensures that it is both a manifold of dimension $6g-6+2m+4n$ and a classifying space $B\Gamma^m_{g,n}$ for the mapping class group (because the action of $\Gamma^m_{g,n}$ on the Teichmüller space is well behaved).

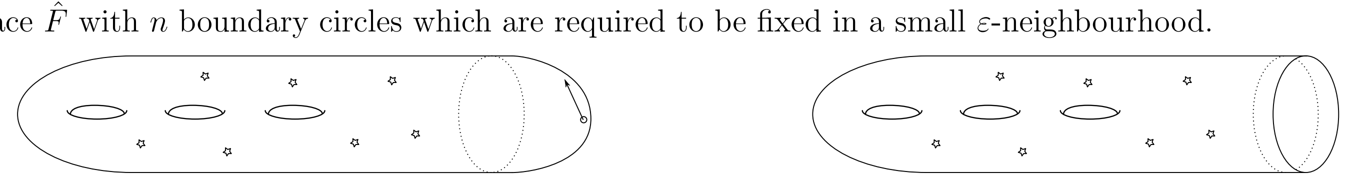
The mapping class group $\Gamma^m_{g,n}$ Consider an oriented smooth surface F of genus g with $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{X}$ as above. Let

$$\mathit{Diff}^+ = \mathit{Diff}^+(F, \mathcal{P}, \mathcal{Q}, \mathcal{X}) = \{\varphi\colon F \xrightarrow{\cong} F \mid \text{smooth, orientation preserving, respecting (6)}\}. \quad (7)$$

with the C^∞-Whitney topology and let $\mathit{Diff}^+_0 \subset \mathit{Diff}^+$ be the subspace of diffeomorphisms isotopic to the identity. The usual composition of maps turns Diff^+ into a topological group with Diff^+_0 a contractible subgroup. The mapping class group is

$$\Gamma^m_{g,n} = \mathit{Diff}^+(F, \mathcal{P}; \mathcal{Q}, \mathcal{X}) \;/\; \mathit{Diff}^+_0(F, \mathcal{P}; \mathcal{Q}, \mathcal{X}) = \pi_0 \mathit{Diff}^+(F, \mathcal{P}; \mathcal{Q}, \mathcal{X}). \quad (8)$$

Instead of fixing directions $\mathcal{X}$ at $\mathcal{Q}$, we remove an open small disc around every Q_i and obtain a compact surface $\hat{F}$ with n boundary circles which are required to be fixed in a small ε-neighbourhood.



This gives an isomorphic group

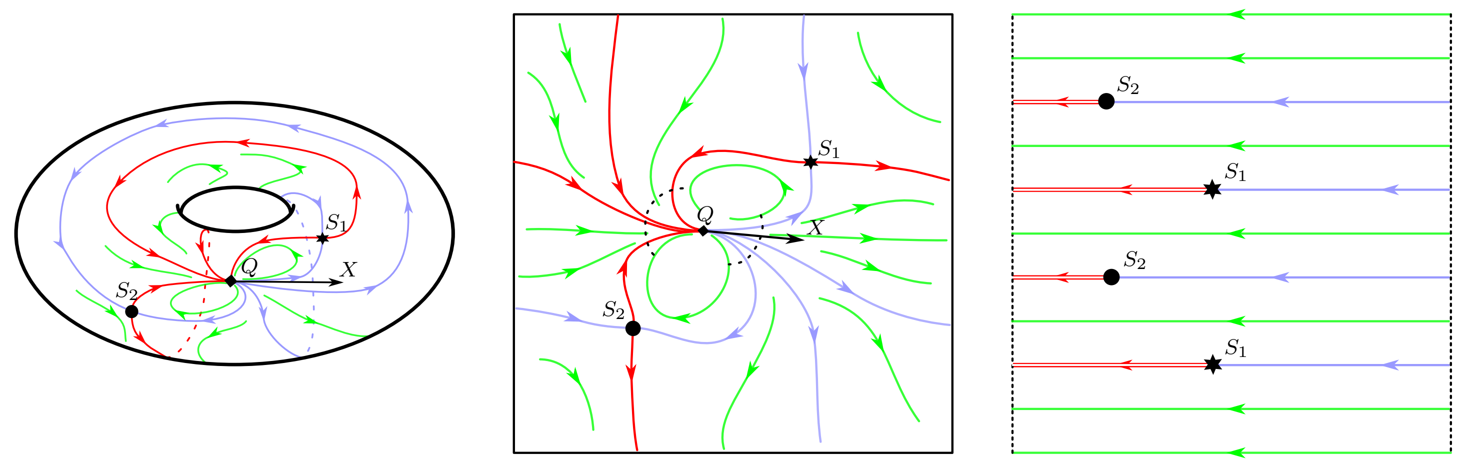
$$\Gamma^m_{g,n} = \mathit{Diff}^+(\hat{F}, \mathcal{P}; \partial\hat{F}) \;/\; \mathit{Diff}^+_0(\hat{F}, \mathcal{P}; \partial\hat{F}) = \pi_0 \mathit{Diff}^+(\hat{F}, \mathcal{P}; \partial\hat{F}). \quad (9)$$

It is finitely presented by Dehn twists.

Hilbert uniformization A method providing a nice model for $\mathfrak{M}^m_{g,n}$ is introduced in [Böd1]. In order to ease the discussion of the uniformization process, we provide a pictorial example on the next page, where $g = 1$, $m = 0$ and $n = 1$. Given a surface $[F] \in \mathfrak{M}^m_{g,n}$ we choose a map $u\colon F \longrightarrow \hat{\mathbb{R}} \subset \hat{\mathbb{C}}$ which is harmonic away from $\mathcal{P}$ and $\mathcal{Q}$. Moreover, we assert a dipole at every $Q_i \in \mathcal{Q}$ in direction X_i and with a logarithmic sink at every $P_j \in \mathcal{P}$. The flow of steepest descent has finitely many critical points $S_1, \dots, S_k$. The union of $\mathcal{Q}$, $\mathcal{P}$, all the S_l and the flow lines leaving the S_l constitute the critical graph K drawn in red.

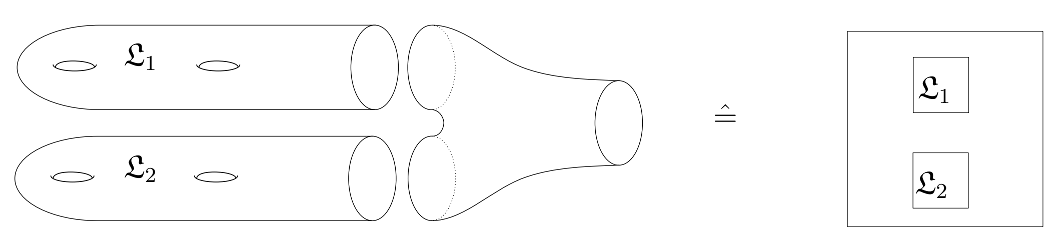
Basics II & Questions

Observe that $F - K$ consist of exactly n contractible components because every flow line starts near exactly one Q_i. The process of "straightening the remaining flow lines" defines a bihomolorphic map $u + iv$ from $F - K$ into the complex plane. The image is $\mathbb{C}$ minus a finite number of horizontal half-rays running to the left.



We denote the space of such maps $u + iv$ by $\mathcal{H}^m_{g,n}$. It is a bundle $\mathcal{H}^m_{g,n} \xrightarrow{\simeq} \mathfrak{M}^m_{g,n}$ and the choices we made constitute the fibre which is contractible. The space $\mathcal{H}^m_{g,n}$ is homeomorphic to the space of admissible slit configurations denoted by $\mathfrak{Par}^m_{g,n}$.

The E_2-space structure The data of a slit picture $\mathfrak{L} \in \mathfrak{Par}^m_{g,1}$ consists of the endpoints of the half-rays and certain glueing information. Thus, $\mathfrak{L}$ is inscribed in a square of finite area. Placing two slit pictures into disjoint squares in $\mathbb{C}$ defines an H-space structure on $\mathfrak{Par} = \coprod^m_{g,1} \mathfrak{Par}^m_{g,1}$. Observe: this operation is induced by joining the two corresponding surfaces by a pair of pants.



More generally, the little 2-cubes operad $\tilde{C}(\mathbb{C}) = \coprod_{k \geq 0} \{k \text{ disjoint, paraxial squares in } \mathbb{C}\}$ acts on $\mathfrak{Par}$. As a consequence, $H_*(\mathfrak{Par}) \cong H_*(\coprod_{g,m} \mathfrak{M}^m_{g,1})$ is not only a commutative Pontryargin ring, but a Dyer–Lashof algebra.

Questions Denote $\mathfrak{M} = \coprod_{g,m} \mathfrak{M}^m_{g,1}$.

1. What are the homology modules $H_*(\mathfrak{M}^m_{g,n})$ for given parameters g, n and m?
2. What are generators of $H_*(\mathfrak{M}^m_{g,n})$ for given parameters g, n and m?
3. How does the homology of the braid groups act on $H_*(\mathfrak{M})$?
4. How are the generators related by Browder operations, Dyer–Lashof operations and other homology operations?

Partial Answers I

The stable range and the Madsen–Weiss Theorem The Harer stabilization theorem states, that the multiplication with the generator in $H_0(\Gamma^0_{1,1})$ induces an isomorphism $H_*(\Gamma^0_{g,1}) \xrightarrow{\cong} H_*(\Gamma^0_{g+1,1})$ if $* \leq \frac{2}{3}g - 1$, compare [Wah]. Thus $\Gamma_{\infty,1} = \cup_g \Gamma_{g,1}$ is an approximation of every $\Gamma_{g,1}$ in this so called stable range. In [MW] Madsen and Weiss construct a certain spectrum $MT(d)^+$ detecting the homotopy type of a cobordism category. As a special case, a group completion theorem yields a homology isomorphism

$$\mathbb{Z} \times B\Gamma_{\infty,1} \xrightarrow{\simeq} \Omega^\infty MT(2)^+ .$$

This proves a conjecture by Mumford.

Theorem (Madsen–Weiss 2002). *The rational cohomology of $\Gamma_{\infty,1}$ is*

$$H^*(\Gamma_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots]$$

with κ_i the Mumford–Morita–Miller characteristic classes for surface bundles. In particular, $H_*(\Gamma^0_{g+1,1}; \mathbb{Q})$ is known in the stable range $* \leq \frac{2}{3}g - 1$.

Homology calculations in the unstable range The space of parallel slit domains $\mathfrak{Par}^m_{g,n}$ is a a combinatorial, relative manifold, i.e. $\mathfrak{Par}^m_{g,n} \cong \mathbb{P} - \mathbb{P}'$ with $(\mathbb{P}, \mathbb{P}')$ a pair of compact cell complexes. The homology of $\mathfrak{M}^m_{g,n}$ is therefore Poincaré dual to the cohomology of $\mathbb{P}/\mathbb{P}'$. Computations for $2g + m < 6$ were done by Ehrenfried, Mehner and Wang using this model; and Godin using another model. Bödigheimer introduces a nice filtration on $\mathbb{P}$ in [Böd2]. It descends to a certain homotopy retract of $\mathbb{P}/\mathbb{P}'$ provided by Visy. This allows explicit calculations. We state some of our results for $2g + m = 6$.

Theorem (Bödigheimer, B., Hermann 2014). *The rational betti numbers of the moduli spaces are as follows.*

	$*=0$	$*=1$	$*=2$	$*=3$	$*=4$	$*=5$	$*=6$	$*=7$	$*=8$	$*=9$
$\dim_{\mathbb{Q}} H_*(\mathfrak{M}^2_{2,1})$	1	0	1	3	0	2	2	0	0	0
$\dim_{\mathbb{Q}} H_*(\mathfrak{M}^0_{3,1})$	1	0	1	1	0	1	1	0	0	1

Bödigheimer and Mehner describe most of the generators of the known homology as embedded manifolds. For example, $H_3(\mathfrak{M}^0_{2,1}; \mathbb{Z}) = \mathbb{Z}$ is generated by the fundamental class of the sphere bundle of the universal surface bundle over the moduli space $\mathfrak{M}^0_{2,0}$. Bödigheimer and the author provide a handful of relations between generators via generatlized Browder operations.

Braid groups The moduli space $\mathfrak{M}^m_{0,1}$ is the space of m undistinguishable particles in the plane. Thus, $\pi_1(\mathfrak{M}^m_{0,1}) = \Gamma^m_{0,1}$ is the braid group on m stands. Using the theory of iterated loop spaces, Cohen provides the p-torsion of the integral homology and its description as Dyer–Lashof algebra. The classical result by Arnold and Fuks is then obtained as corollary: The homology of the braid group is a truncated subring

$$H_* = H_*(\mathfrak{M}^0_{0,1}; \mathbb{F}_2) \leq \mathbb{F}_2[\, x_1, x_2, x_3, \dots]$$

where $\deg(x_i) = 2^i - 1$ and $x = x_1^{l_1} \cdots x_k^{l_k} \in H_*$ for $\sum_i l_i 2^i \leq n$ and $x_{i+1} = Q_1(x_i)$ with Q_1 the first Dyer–Lashof operation.

Partial Answers II & References

The action of the homology of the Braid groups Forgetting the marked points defines a fibration $\mathfrak{M}^m_{g,1} \longrightarrow \mathfrak{M}^0_{g,1}$ with fibre $C^m(F_{g,1})$ the unordered configuration space on the surface without marked points. Adding a marked point near the boundary curve, defines a map α over $\mathfrak{M}^0_{g,1}$. The induced map in homology, is the multiplication with the generator in $H_0(\mathfrak{M}^1_{0,1})$.

Theorem (Bödigheimer, Tillmann 2001). *Adding a marked point $\alpha\colon \mathfrak{M}^m_{g,1} \longrightarrow \mathfrak{M}^{m+1}_{g,1}$ admits a stable retract $\Omega^\infty\Sigma^\infty\mathfrak{M}^{m+1}_{g,1} \longrightarrow \Omega^\infty\Sigma^\infty\mathfrak{M}^m_{g,1}$. In particular, the restriction of the multiplication in one argument*

$$H_0(\mathfrak{M}^1_{0,1}; \mathbb{Z}) \otimes H_*(\mathfrak{M}^m_{g,1}; \mathbb{Z}) \longrightarrow H_*(\mathfrak{M}^{m+1}_{g,1}; \mathbb{Z})$$

admits a retraction.

Using this, we obtain a family of non-trivial homology operations.

Theorem (B. 2015). *The following restriction of the multiplication is injective.*

$$H_1(\mathfrak{M}^2_{0,1}; \mathbb{Z}/2\mathbb{Z}) \otimes H_*(\mathfrak{M}^m_{g,1}; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_{*+1}(\mathfrak{M}^{m+2}_{g,1}; \mathbb{Z}/2\mathbb{Z})$$

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