On computations of the homology of moduli spaces of Riemann surfaces

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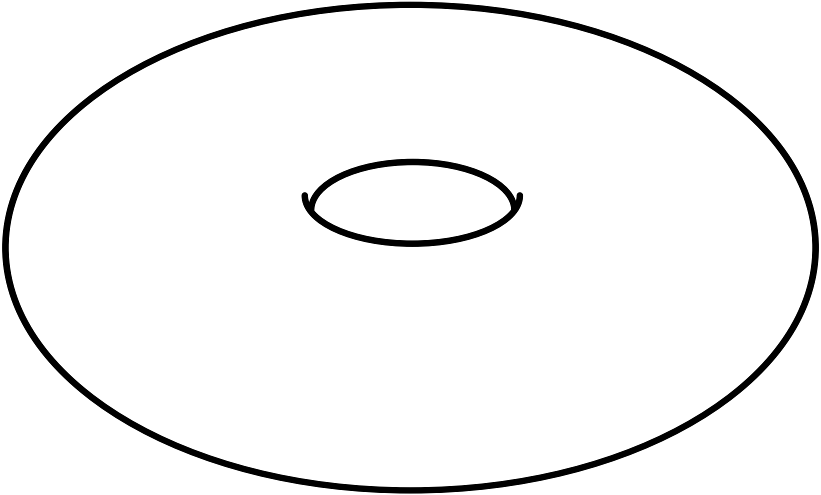
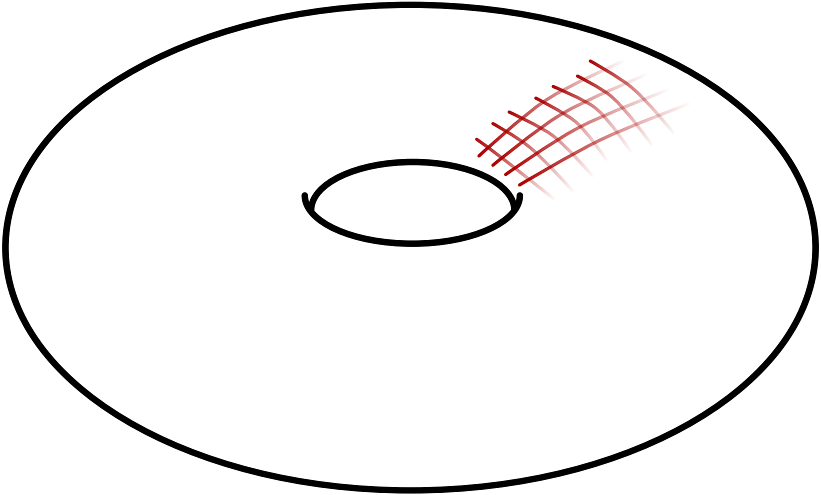
The Question





Definition

Fix a topological surface S.

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Question

What is the homology of this space?

Fact

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Fact

There are several constructions for arbitrary surface types.

The Model

We consider surfaces (in our setting often called cobordisms)

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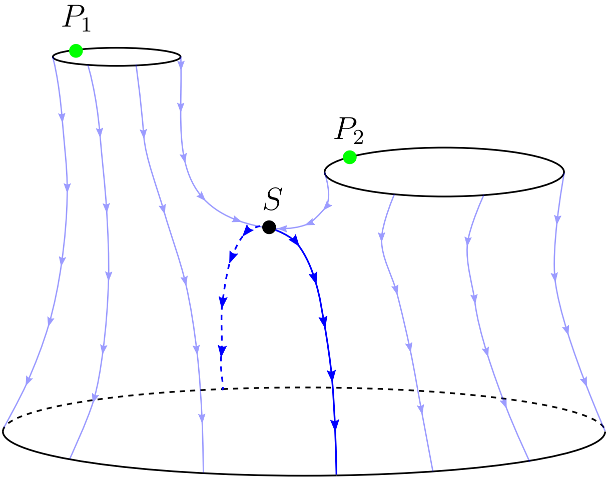
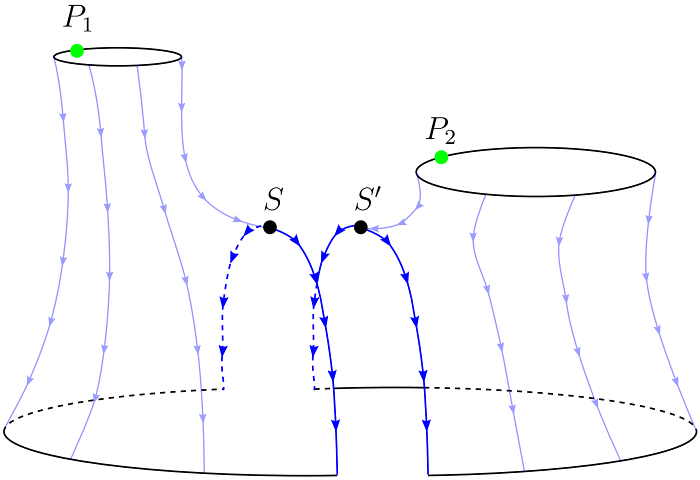
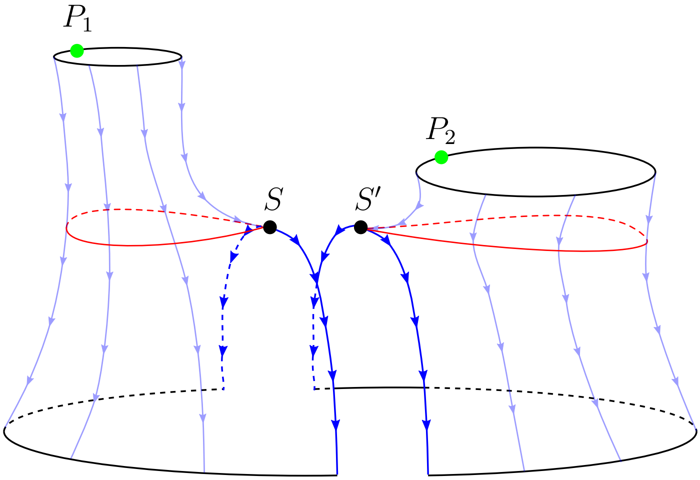
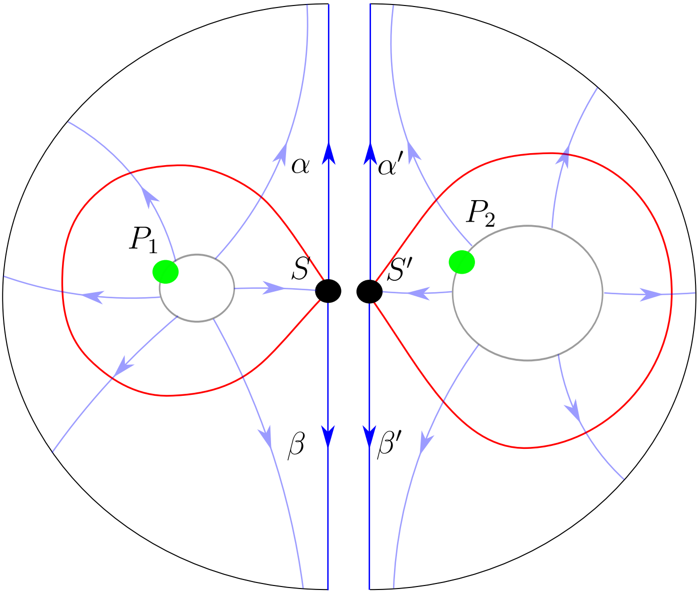
We use the following shorthand $\mathfrak{M}^m_{g,n}$.

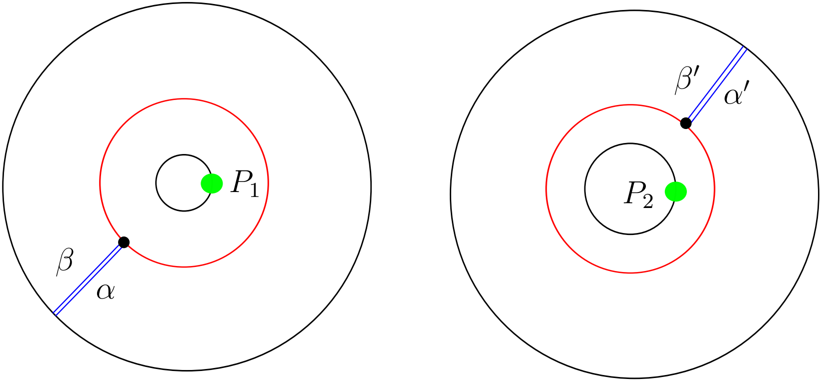


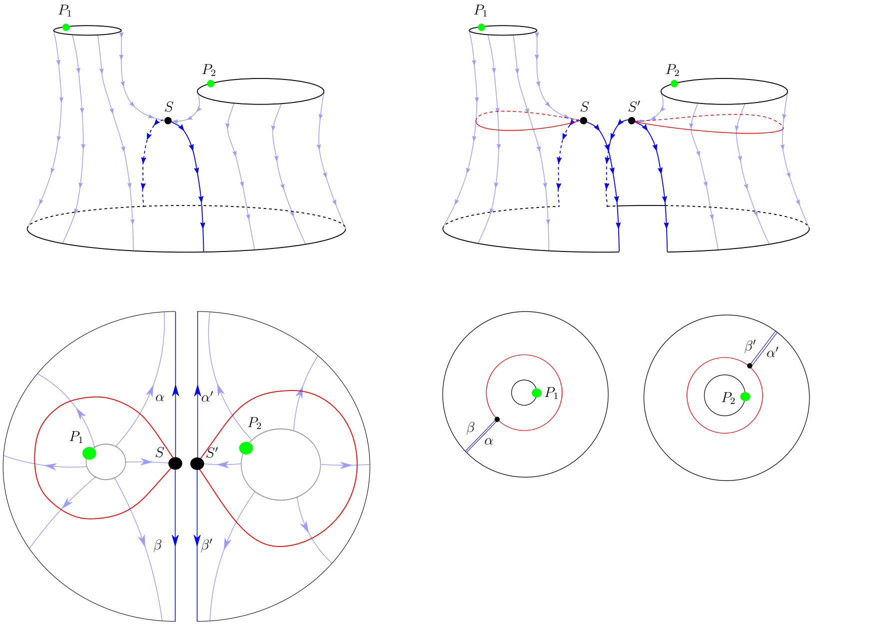
P_1
P_2
S


P_1
P_2
S
S'


P_1
P_2
S
S'


α
α'
P_1
P_2
S
S'
β
β'


P_1
β
α
P_2
β'
α'


P_1
P_2
S
S'
α
β
α'
β'

Theorem (Bödigheimer 1990)

The moduli space $\mathfrak{M} = \mathfrak{M}^m_{g,n}$ is s finite cell complex. In particular, its homology is computable in terms of a finite chain complex $K = K(\mathfrak{M}^m_{g,n})$.

Reductions

Fact

The number of cells of every chain module grows factorially $\mathcal{O}(h!)$ *for* $h = 2g + m$.

Corollary (Ehrenfried 1998, Vicy 2010)

There is a discrete Morse flow on K.

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The number of cells of the bi-complex $K(\mathfrak{M}^3_{1,1})$:

$q=5$	640	12425	74610	202825	278600	189000	50400
$q=4$	800	18500	122700	357280	516880	365400	100800
$q=3$	240	7425	57375	185220	289380	217350	63000
$q=2$	10	650	6800	26600	47740	39900	12600
$q=1$	0	0	35	315	910	1050	420
	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$

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The number of cells of the Morse complex $Morse(\mathfrak{M}^3_{1,1})$:

70	700	2520	4480	4270	2100	420
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Corollary (Bödigheimer 2014)

There is a filtration of K which descends to the Morse complex.

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The number of cells of the 0^{th} page:

	$p = 4$	$p = 5$	$p = 6$	$p = 7$	$p = 8$	$p = 9$	$p = 10$
$c = 1$	70	640	1470				
$c = 2$		60	1035	3850			
$c = 3$			15	630	4130		
$c = 4$					140	2100	
$c = 5$							420

Computational Results

Theorem (Wang 2011, B., Hermann 2014)

$$H_*(\mathfrak{M}^4_{1,1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}\oplus C_2 \oplus \boxed{\cdots} & *=1 \\ C_2^3 \oplus \boxed{\cdots} & *=2 \\ \mathbb{Z}^2 \oplus C_2^3 \oplus \boxed{\cdots} & *=3 \\ \mathbb{Z}^3 \oplus C_2^2 \oplus \boxed{\cdots} & *=4 \\ \mathbb{Z}^2 \oplus C_2 \oplus \boxed{\cdots} & *=5 \\ \mathbb{Z} \oplus \boxed{\cdots} & *=6 \\ 0 & *\geq 7 \end{cases}$$

Theorem (Wang 2011, B., Hermann 2014)

$$H_*(\mathfrak{M}^2_{2,1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ C_2^2 \oplus C_5 \oplus \boxed{\cdots} & * = 1 \\ \mathbb{Z} \oplus C_2^2 \oplus \boxed{\cdots} & * = 2 \\ \mathbb{Z}^3 \oplus C_2^4 \oplus \boxed{\cdots} & * = 3 \\ \mathbb{Z} \oplus C_2^5 \oplus C_3^3 \oplus \boxed{\cdots} & * = 4 \\ \mathbb{Z}^2 \oplus C_2^4 \oplus C_3 \oplus \boxed{\cdots} & * = 5 \\ \mathbb{Z}^2 \oplus C_2^3 \oplus \boxed{\cdots} & * = 6 \\ \mathbb{C}_2 \oplus \boxed{\cdots} & * = 7 \\ 0 & * \geq 8 \end{cases}$$

Theorem (Wang 2011, B., Hermann 2014)

$$H_*(\mathfrak{M}^0_{3,1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & *=1 \\ \mathbb{Z}\oplus C_2 & *=2 \\ \mathbb{Z}\oplus C_2\oplus C_3\oplus C_4\oplus C_7\oplus \boxed{\cdots} & *=3 \\ C_2^2\oplus C_3^2\oplus \boxed{\cdots} & *=4 \\ \mathbb{Z}\oplus C_2\oplus C_3\oplus \boxed{\cdots} & *=5 \\ \mathbb{Z}\oplus C_2^3\oplus \boxed{\cdots} & *=6 \\ C_2\oplus \boxed{\cdots} & *=7 \\ 0\oplus \boxed{\cdots} & *=8 \\ \mathbb{Z}\oplus \boxed{\cdots} & *=9 \\ 0 & *\geq 10 \end{cases}$$

Theoretical Results

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- via embedded manifolds;
- via operations applied to already known classes;
- ...

We proceed as follows.

- guess a representation;
- let the computer verify;
- try again;

Fact (Arnold 1969, Fuks 1970)

The $\mathbb{F}_2$ homology of the inifite braid group is a graded polynomial ring

$$H_*(Br_\infty;\mathbb{F}_2) \cong \mathbb{F}_2[b_1, b_2, \ldots] \qquad \textit{with} \qquad |b_i| = 2^i - 1\,.$$

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Fact (Bödigheimer 1990)

Using a similar model, the homology

$$\bigoplus_{g,m} H_*(\mathfrak{M}^m_{g,1};\mathbb{F}_2)$$

is a module over $\mathbb{F}_2[b_1, b_2, \ldots]$.

Theorem (B. 2015)

The homology

$$\bigoplus_{g,m} H_*(\mathfrak{M}^m_{g,1};\mathbb{F}_2)$$

is torsion free over $\mathbb{F}_2[b_1]$.

Familiar Models and Spaces

There is a so called harmonic compactification $\overline{\mathfrak{M}}$ of $\mathfrak{M}$.

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The cells are given by Sullivan diagrams.

Theorem (B., Egas Santander 2015)

The homology of $\overline{\mathfrak{M}^m_{g,1}}$ *is*

$g = 0$

m \ $*$	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0	0	0
3	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	0
4	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	0
5	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	0	0
6	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$
7	$\mathbb{Z}$	0	0	0	0	0	0	$\mathbb{Z}$	0

$g = 1$

m \ $*$	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9
1	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	0	0
2	$\mathbb{Z}$	C_2	0	$\mathbb{Z}$	0	0	0	0	0	0
3	$\mathbb{Z}$	0	0	C_3	0	$\mathbb{Z}^2$	$\mathbb{Z}$	0	0	0
4	$\mathbb{Z}$	0	0	C_2	0	$\mathbb{Z} \oplus C_2$	C_2	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
5	$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^3$	C_2

$g = 2$

m \ $*$	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}
1	$\mathbb{Z}$	0	$\mathbb{Z}$	C_5	0	$\mathbb{Z}^2$	C_3	0	0	0	0
2	$\mathbb{Z}$	C_2	0	C_2	0	$\mathbb{Z} \oplus C_2$	$\mathbb{Z} \oplus C_2$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus C_2$	C_2	0
3	$\mathbb{Z}$	0	0	C_3	C_2	0	$\mathbb{Z}^4$	$\mathbb{Z}^9 \oplus C_2$	$\mathbb{Z}^4 \oplus C_{18}$	$\mathbb{Z} \oplus C_2$	$\mathbb{Z}$

$g = 3$

m \ $*$	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}
1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	C_{35}	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z} \oplus C_{12}$	0	C_2	C_2

Theorem (B., Egas Santander, Lutz 2015)

The harmonic compactification $\overline{\mathfrak{M}^m_{g,1}}$ is $(m-2)$ connected.

Theorem (B., Egas Santander 2015)

The stabilization map $\overline{\mathfrak{M}^m_{g,1}} \longrightarrow \overline{\mathfrak{M}^m_{g+1,1}}$ *is a* π_**-isomorphism for* $* \leq m + g - 3$.

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Theorem (B., Egas Santander 2015)

Considering parametrized outgoing boundaries, the stabilization map $\overline{\mathfrak{M}^m_{g,1}} \longrightarrow \overline{\mathfrak{M}^m_{g+1,1}}$ *is a* H_**-isomorphism for* $* \leq g - 1$.

Thank You