Pricing Credit From The Top Down With Time-Changed Poisson Processes

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This talk is based on joint work with
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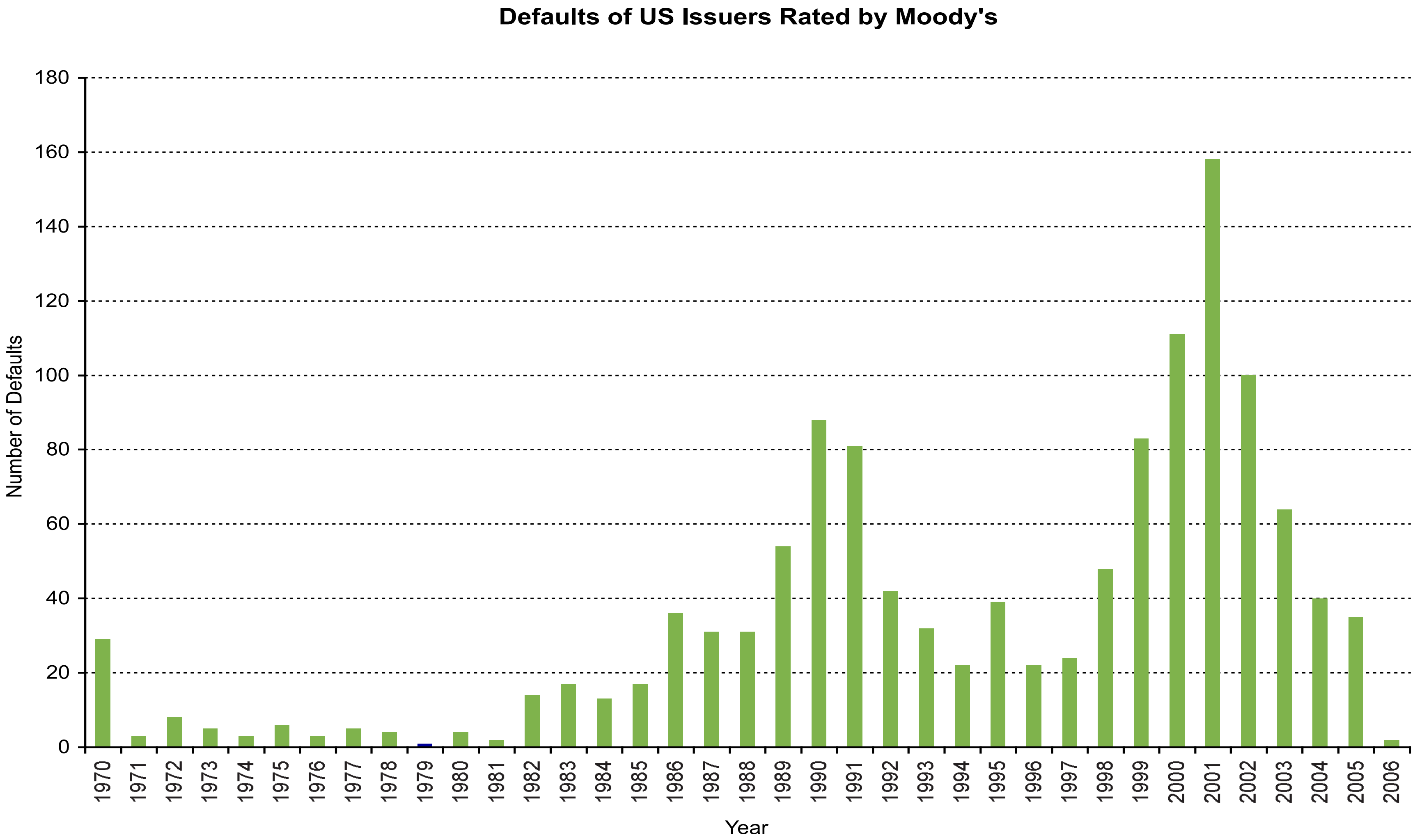
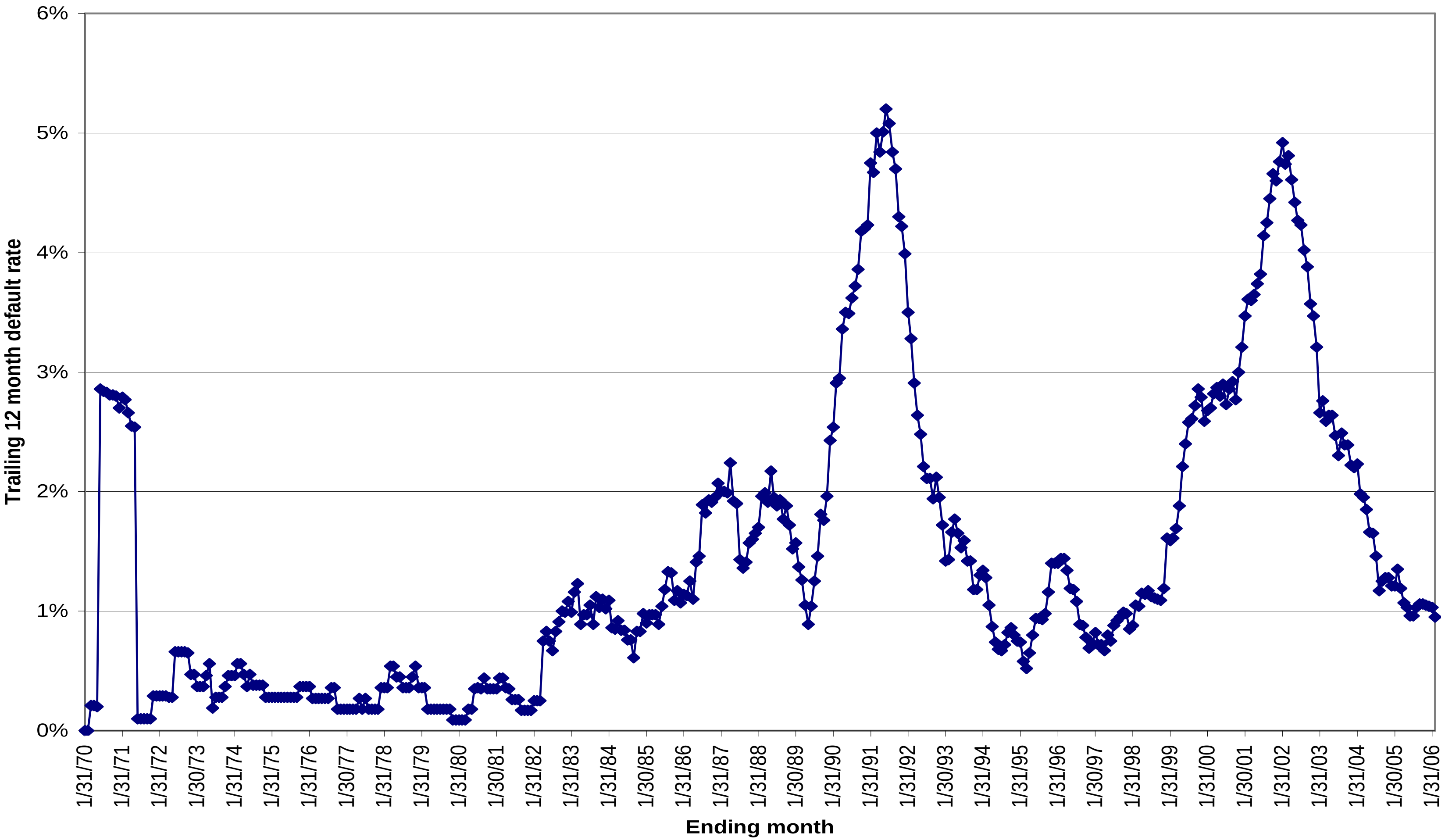
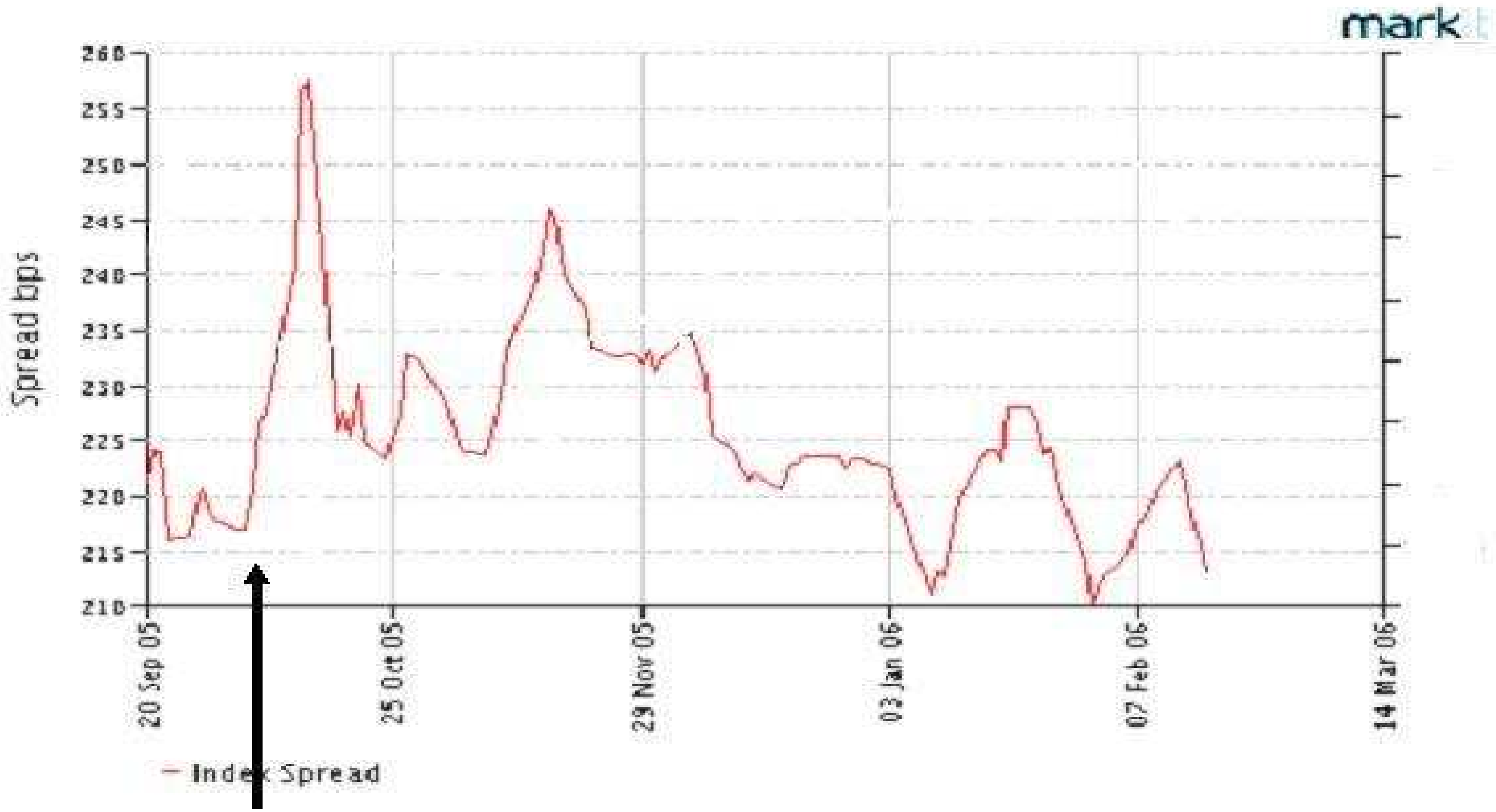
Defaults cluster (Moody's rated US issuers)



Trailing 12 month default rate (Moody's US)



CDX.NA.XO 5-yr credit swap index spread



10/11 Delphi filed Chapter 11
10/17 GM announced 1.6 bn Q3 losses

Multi-name derivatives

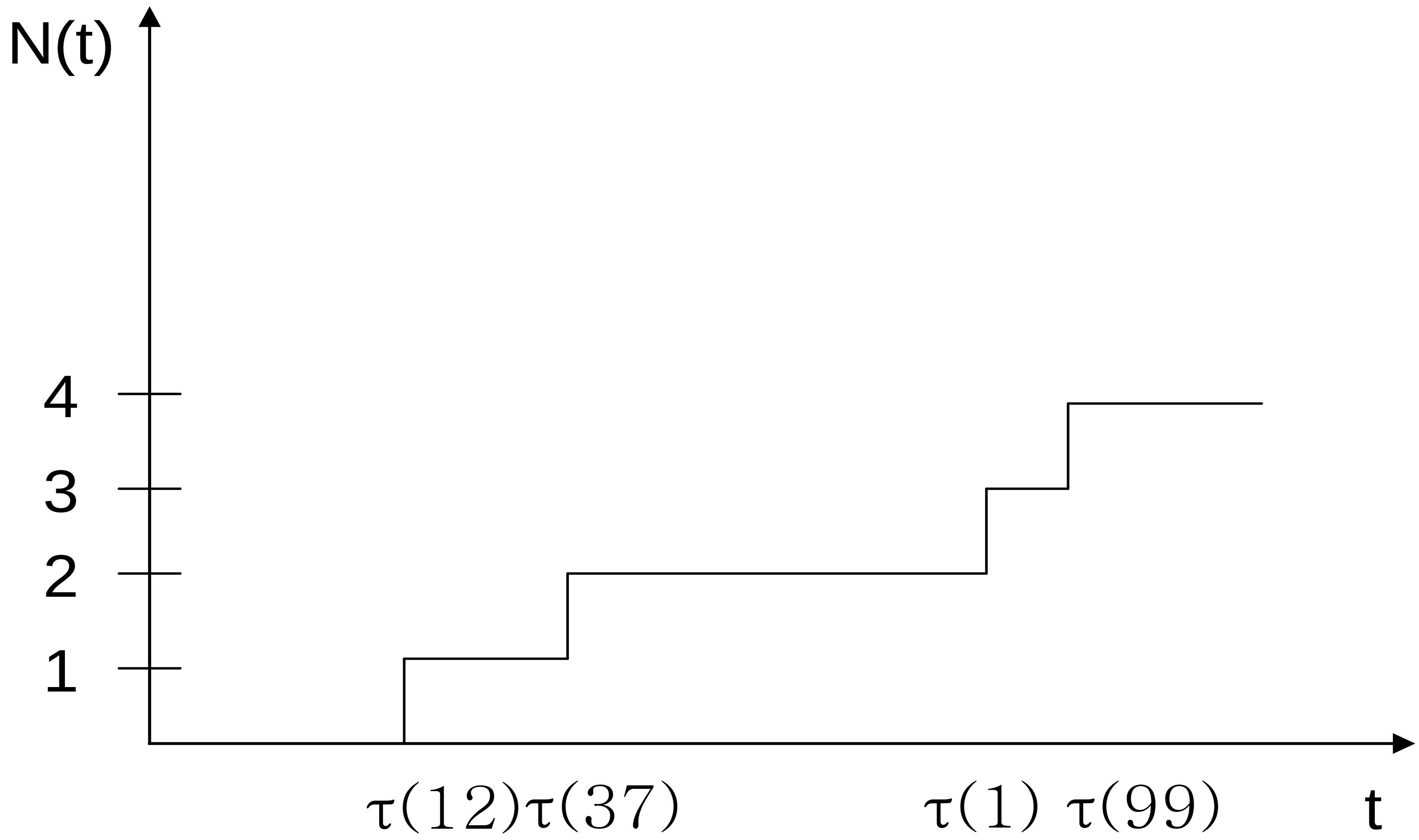
- Trading of default dependence in a portfolio of names
  - Firm sensitivity to common factors
  - Feedback of events
- Path-dependent contingent claims on portfolio defaults and losses
  - **Default process** N counts defaults:

$$N_t = \sum_i 1_{\{\tau^i \leq t\}}$$

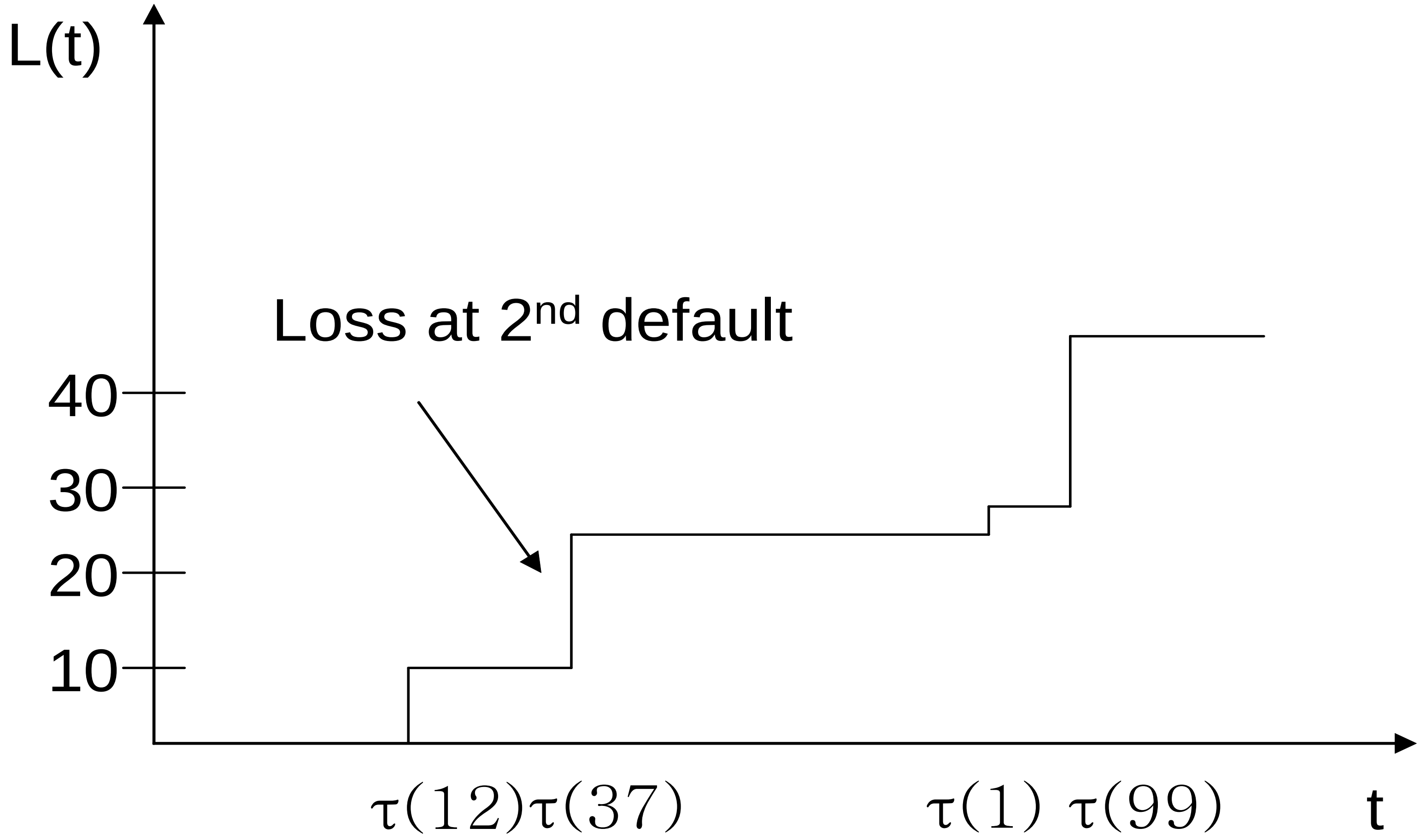
  - **Loss process** L records financial loss due to defaults:

$$L_t = \sum_i L^i 1_{\{\tau^i \leq t\}}$$

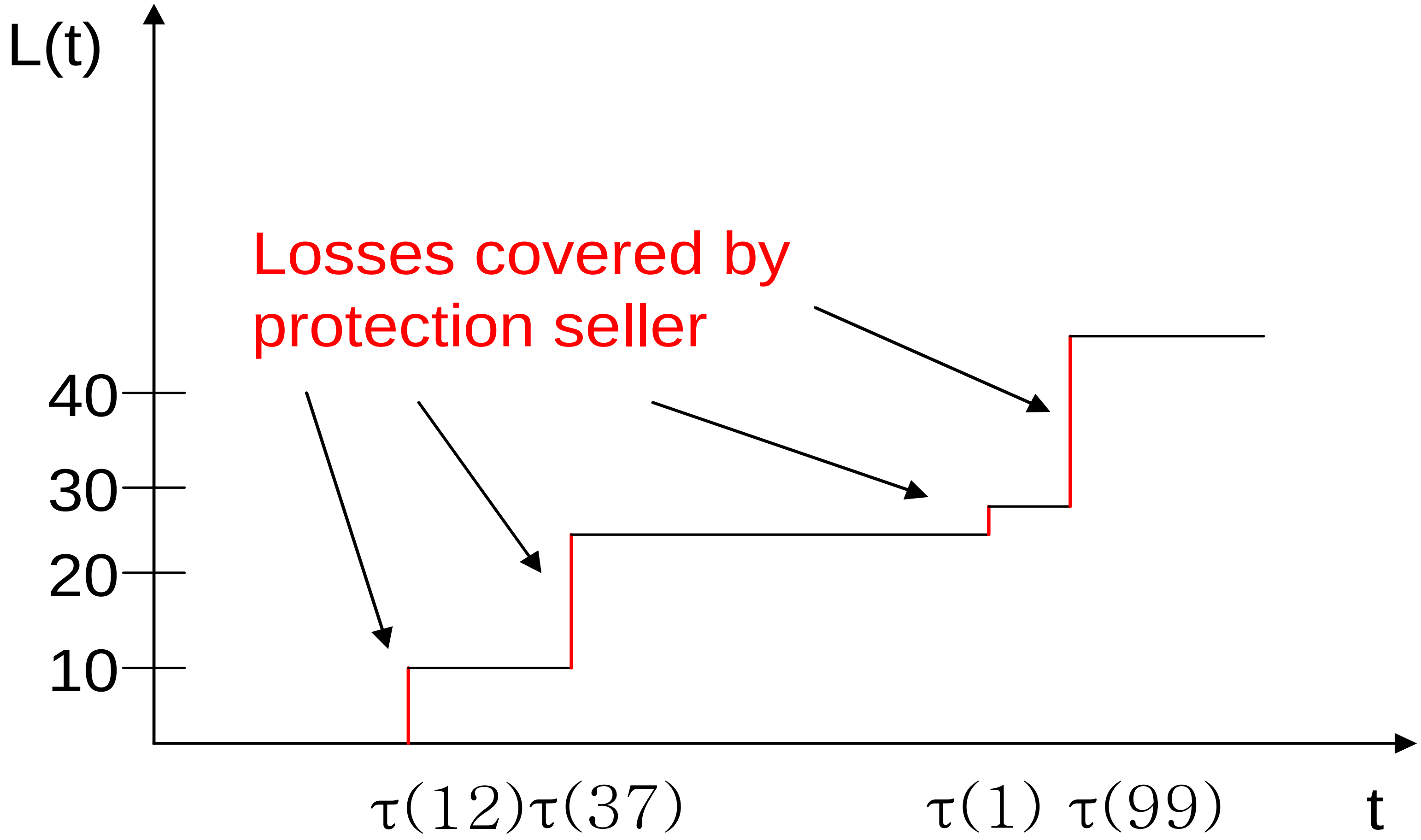
Default process



Loss process



Loss process and index swap



Index swap

- **Default leg**: stream of payments that cover the losses as they occur

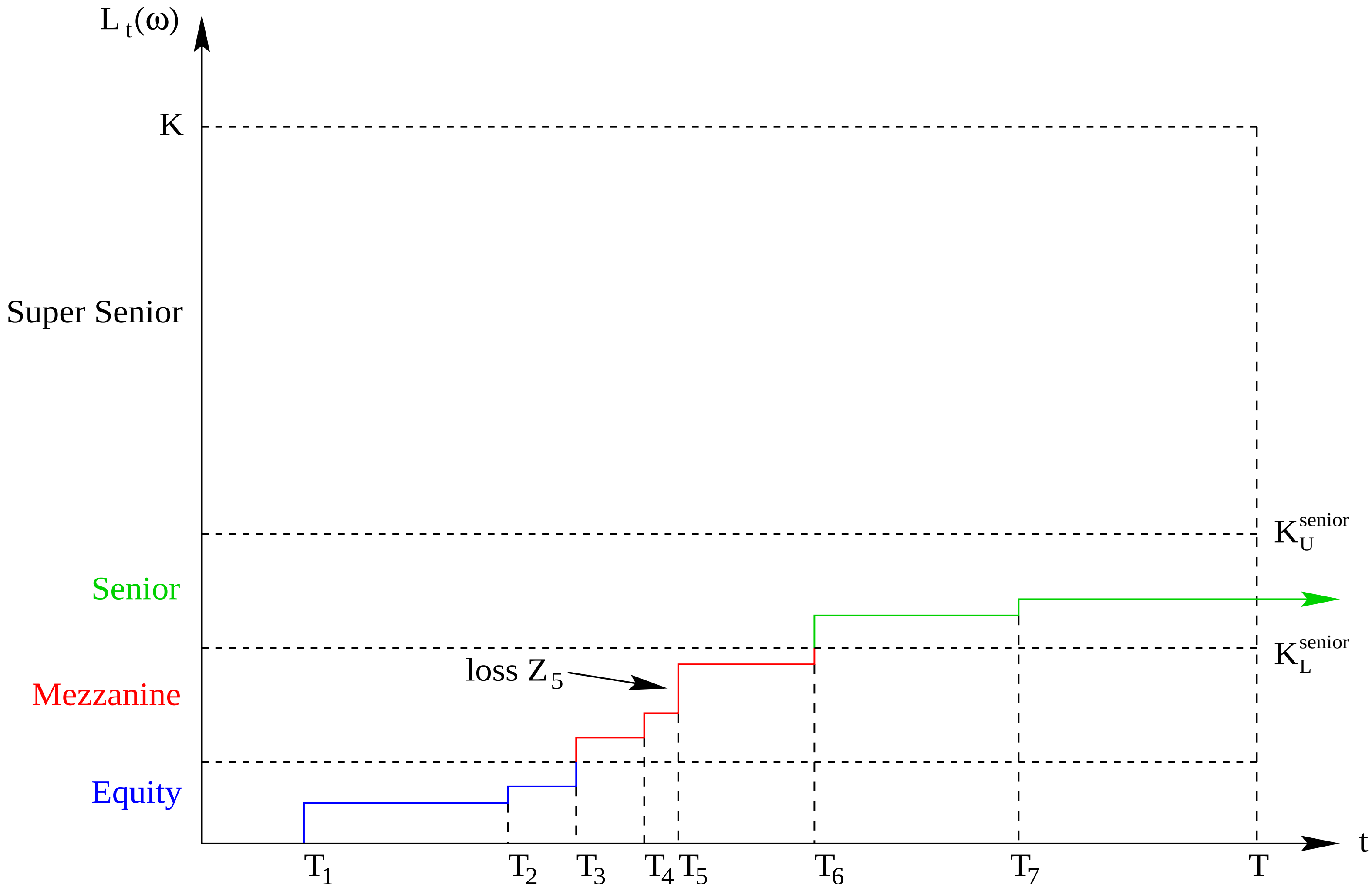
$$D_t = e^{-r(T-t)} E_t[L_T] - L_t + r \int_t^T e^{-r(s-t)} E_t[L_s] \, ds$$

- **Premium leg**: stream of payments that are proportional to the total notional on the names that have survived; with S the spread

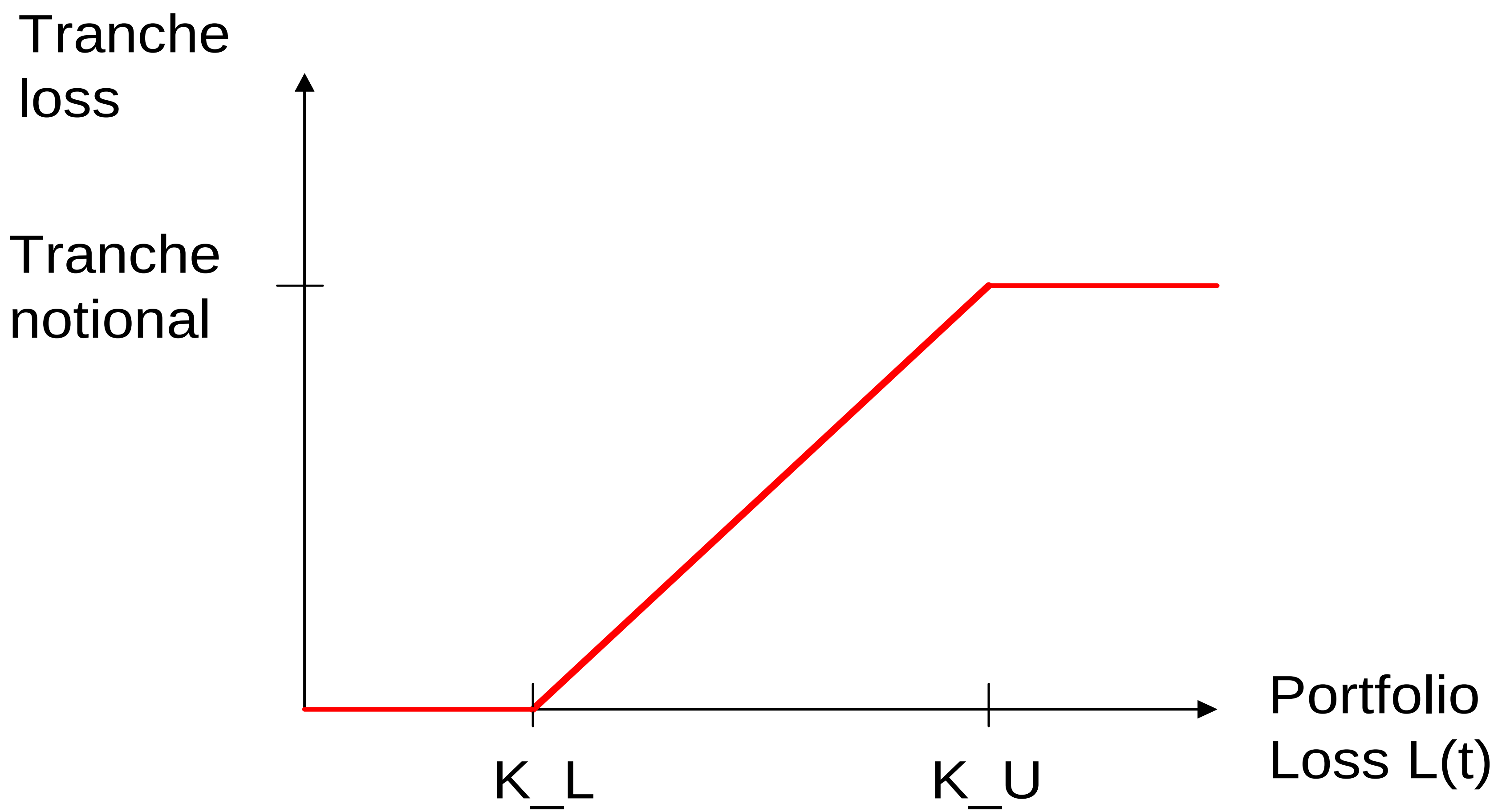
$$P_t(S) = S \sum_{t_m \geq t} e^{-r(t_m - t)} \Big(1 - \frac{1}{n} E_t[N_{t_m}] \Big)$$

- A top down estimate of the **index spread** S_t at time t is the solution $S = S_t$ to the equation $D_t = P_t(S)$
  - Depends only on expected losses and defaults at future dates

Tranching the loss



Tranche loss $U_t = (L_t - K_L)^+ - (L_t - K_U)^+$



Tranche swap

- **Default leg**: stream of payments that cover the tranche loss

$$D_t = e^{-r(T-t)} E_t[U_T] - U_t + r \int_t^T e^{-r(s-t)} E_t[U_s] \, ds$$

- **Premium leg**: stream of payments that are proportional to the tranche notional less the tranche loss; with S the spread

$$P_t(S) = S \sum_{t_m \geq t} e^{-r(t_m - t)} \big(K - E_t[U_{t_m}]\big)$$

  – Extension to include upfront payment

- A top down estimate of the **tranche spread** S_t at time t is the solution $S = S_t$ to the equation $D_t = P_t(S)$

  – Depends only on the values of call options on L

Top down approach

- Reduced form modeling of underlying process $J = (L, N)^\top$
  - Events arrive with intensity λ that responds to arrivals
  - Distribution ν governs random loss at default
- Multi-name products are contingent claims on J
  - Basic building block is the call $E_t[(J_s - c)^+]$
  - Closed form or transform based techniques
- Random thinning allocates λ to the portfolio constituents
  - Hedging of single name exposures

Related literature

- **Intensity based top down approach**: Brigo, Pallavicini and Torresetti (2006), Davis and Lo (2001), Ding, Giesecke and Tomecek (2006), Errais, Giesecke and Goldberg (2006), Giesecke and Goldberg (2005), Giesecke and Tomecek (2005), Halperin (2006), Longstaff and Rajan (2006)

- **Forward top down approach**: Schönbucher (2005), Sidenius, Piterbarg and Andersen (2005)

- **Intensity based bottom up approach**: Duffie and Garleanu (2001), Frey and Backhaus (2004), Jarrow and Yu (2001)

Basic example: Hawkes process

- Defaults are **self-affecting** with intensity

$$\lambda_t = \lambda_\infty + \int_0^t d(t-s) dL_s$$

  - $\lambda_\infty > 0$ constant first-to-default intensity
  - $d(t) = \delta e^{-\kappa t}$ deterministic response function; $\delta \geq 0$ and $\kappa \geq 0$

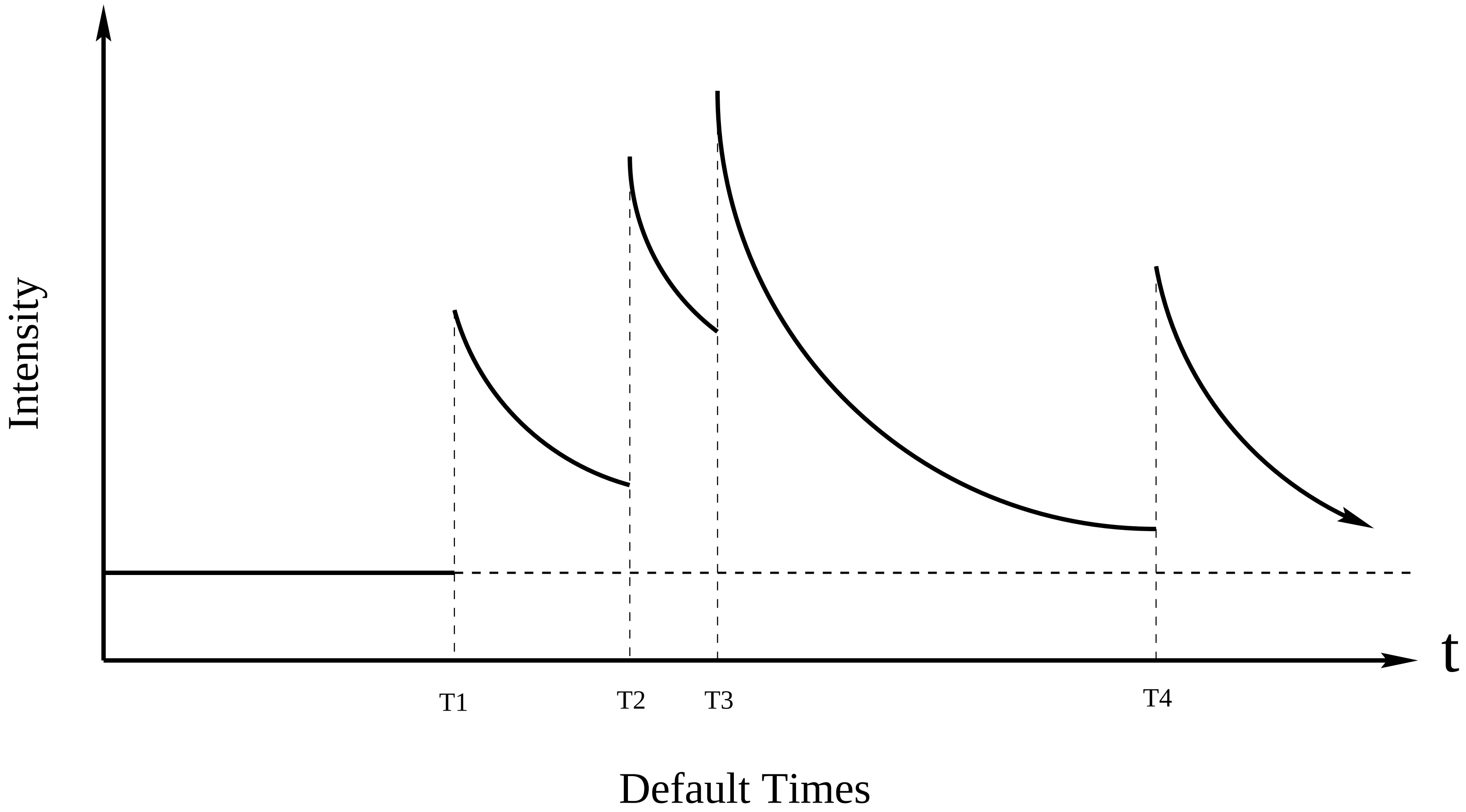
- The Hawkes intensity has dynamics

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)\, dt + \delta\, dL_t$$

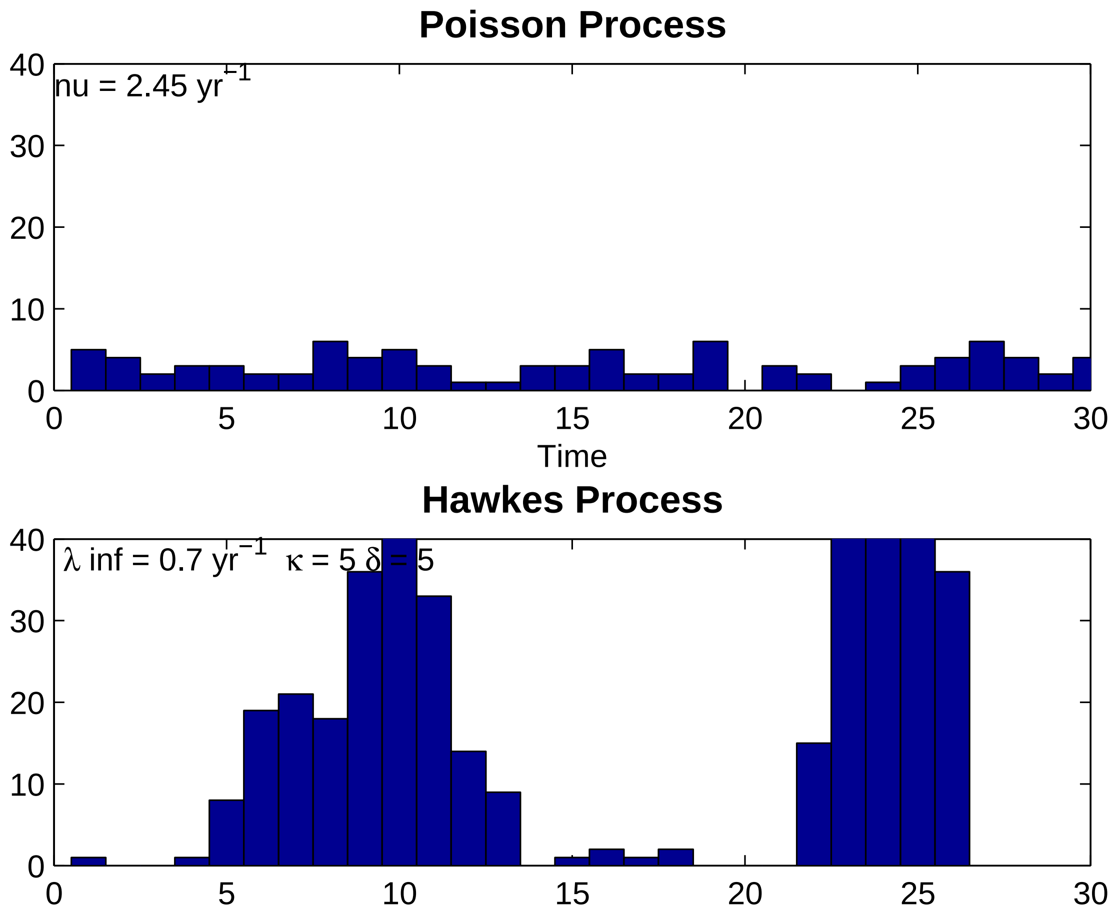
- Negative correlation between default and recovery rates

- Classical examples: Poisson process $\delta = 0$, birth process $\kappa = 0$

Sample path of Hawkes intensity



Defaults cluster



Conditional transform

- The loss at default is independent of N and has distribution ν
- The conditional transform of $J = (L, N)^\top$ is given by

$$E_t[e^{u \cdot J_s}] = e^{a(u,t,s) + b(u,t,s)\lambda_t + u \cdot J_t}$$

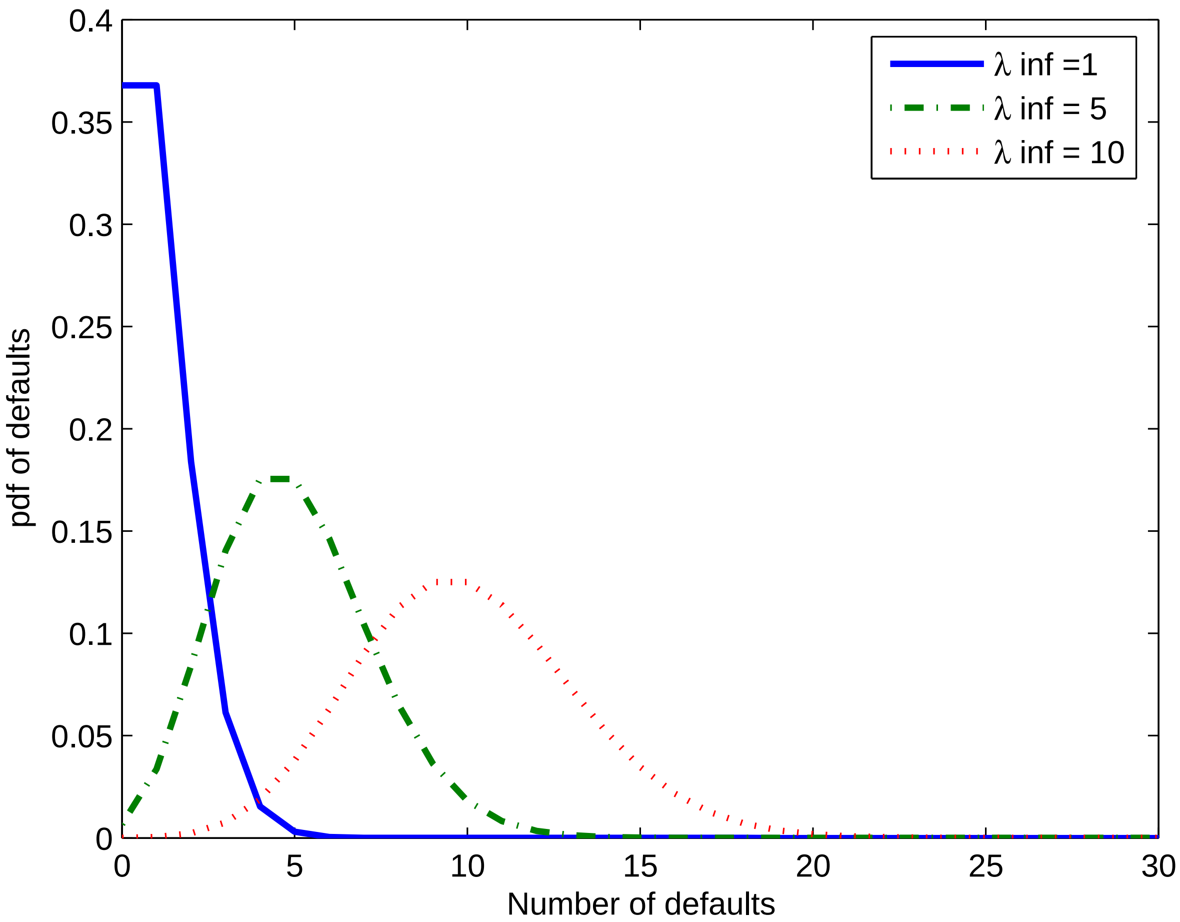
where $a(t) = a(u, t, s)$ and $b(t) = b(u, t, s)$ satisfy the ODEs

$$\partial_t b(t) = \kappa b(t) - \psi(\delta b(t), u) + 1$$
$$\partial_t a(t) = -\kappa \lambda_\infty b(t)$$

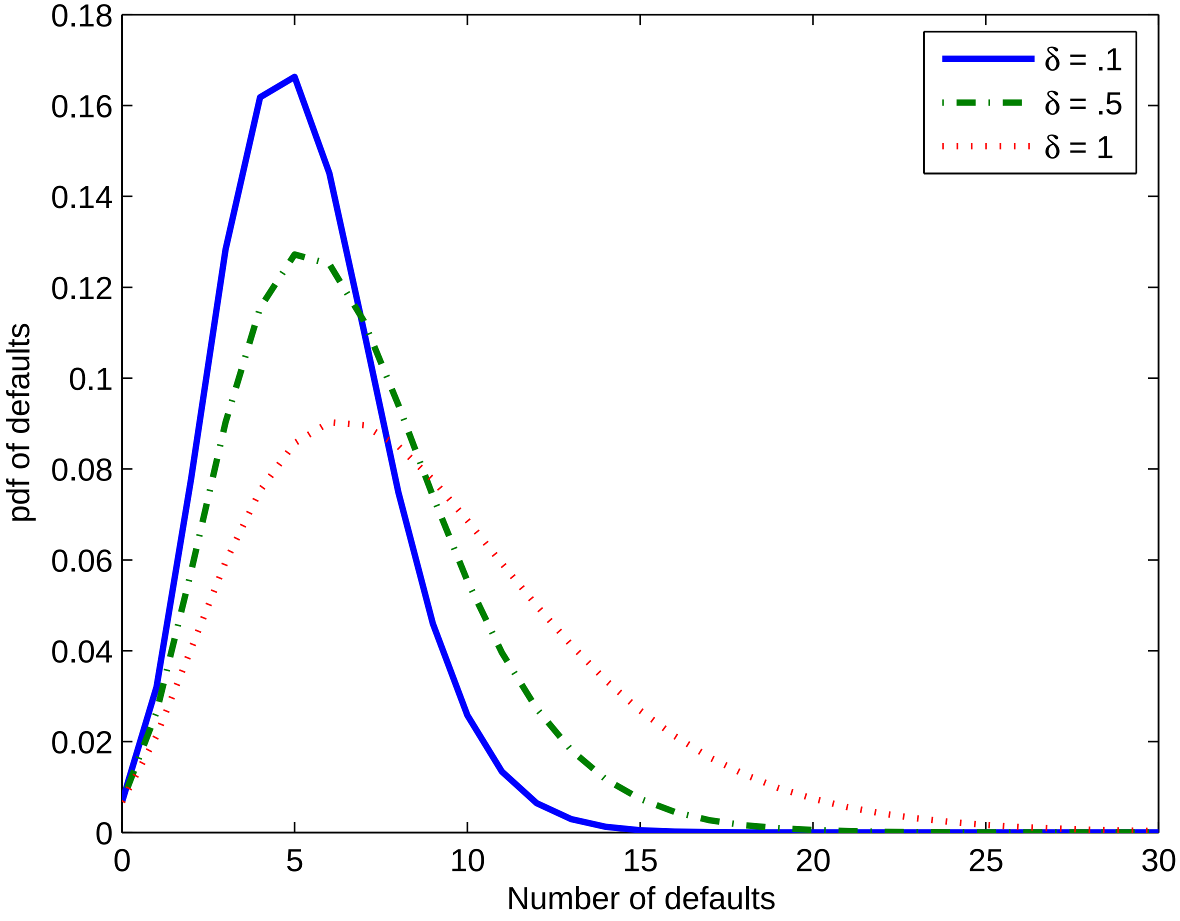
with boundary conditions $a(s) = b(s) = 0$ and

$$\psi(c, u) = e^{u \cdot (0,1)^\top} \int_{\mathbb{R}_+} e^{(c + u \cdot (1,0)^\top) z} d\nu(z)$$

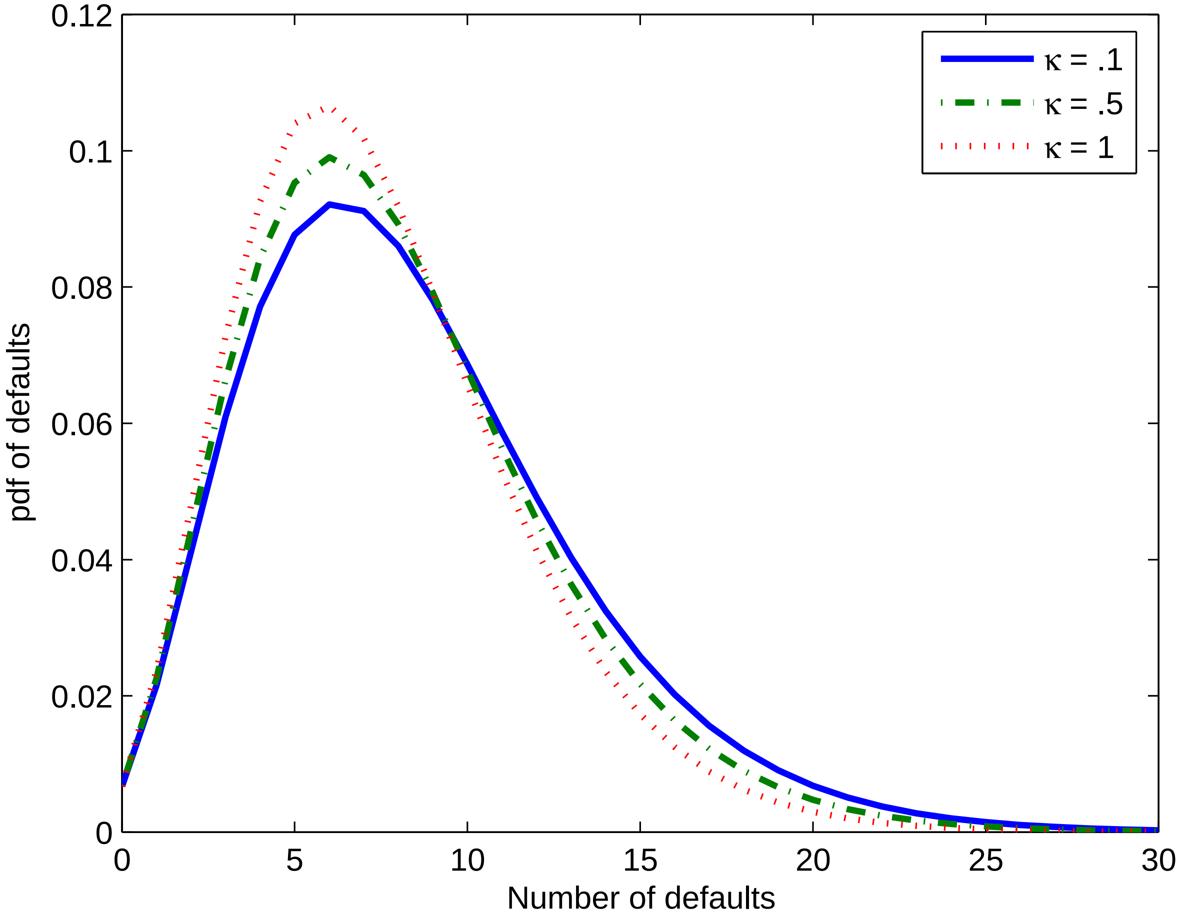
1-yr Poisson arrivals: $\kappa = \delta = 0$



1-yr birth arrivals: $\kappa = 0$, $\lambda_\infty = 5$



1-yr Hawkes arrivals: $\lambda_\infty = 5$, $\delta = 1$



Index spreads for the Hawkes process

- Need conditional expected losses and defaults at future dates
- Differentiate the transform of J with respect to u and evaluate the derivative at $u = (0,0)^\top$:

$$E_t[y \cdot J_s] = A(t,s) + B(t,s)\lambda_t + y \cdot J_t$$

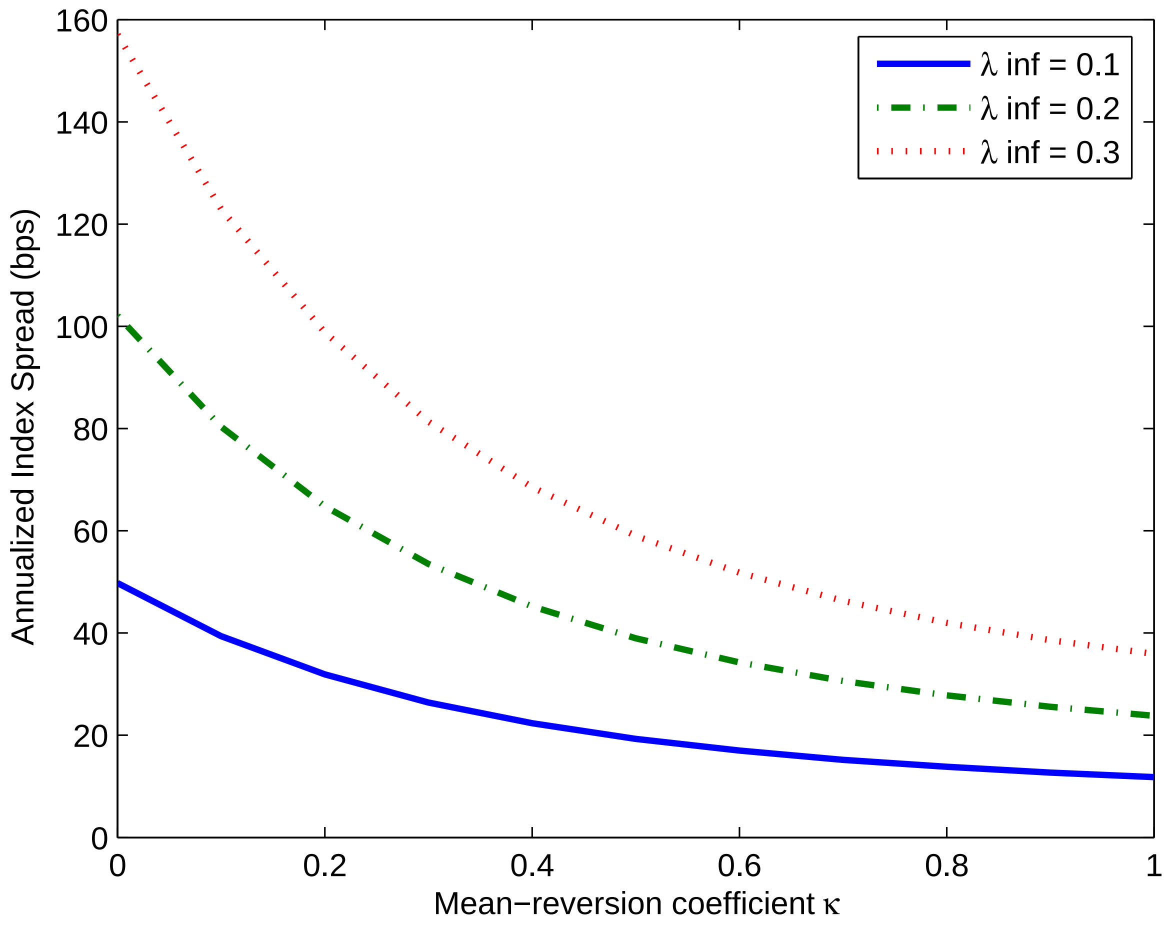
where the coefficient functions satisfy the ODEs

$$\partial_t B(t) = (\kappa - \delta\bar{\nu})B(t) - y \cdot (\bar{\nu}, 1)^\top$$
$$\partial_t A(t) = -\kappa\lambda_\infty B(t)$$

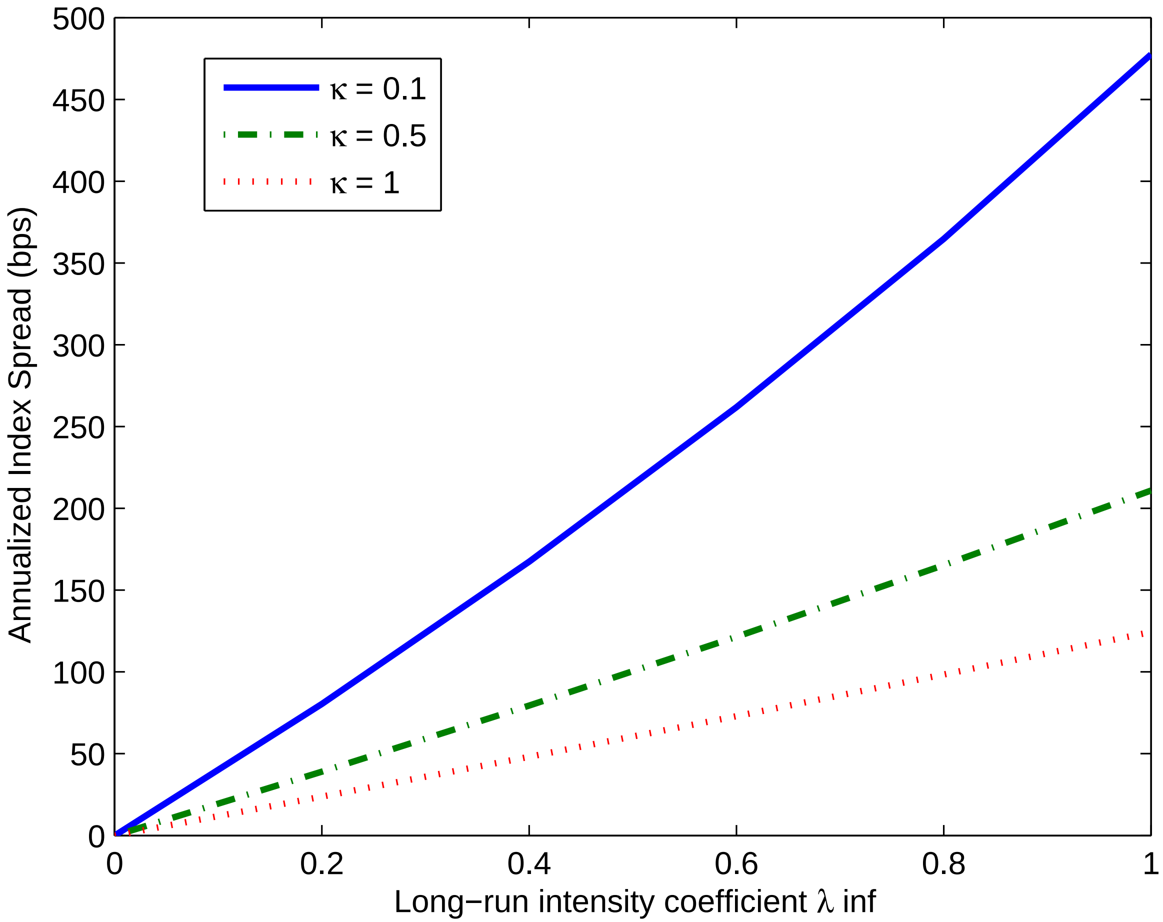
with boundary conditions $A(s) = B(s) = 0$, where $\bar{\nu} = \int z d\nu(z)$

- Closed form expressions for $E_t[N_s]$ and $E_t[L_s]$ and index spreads

5-yr index: $\delta = 1$, $\bar{\nu} = 0.7$



5-yr index: $\delta = 1$, $\bar{\nu} = 0.7$



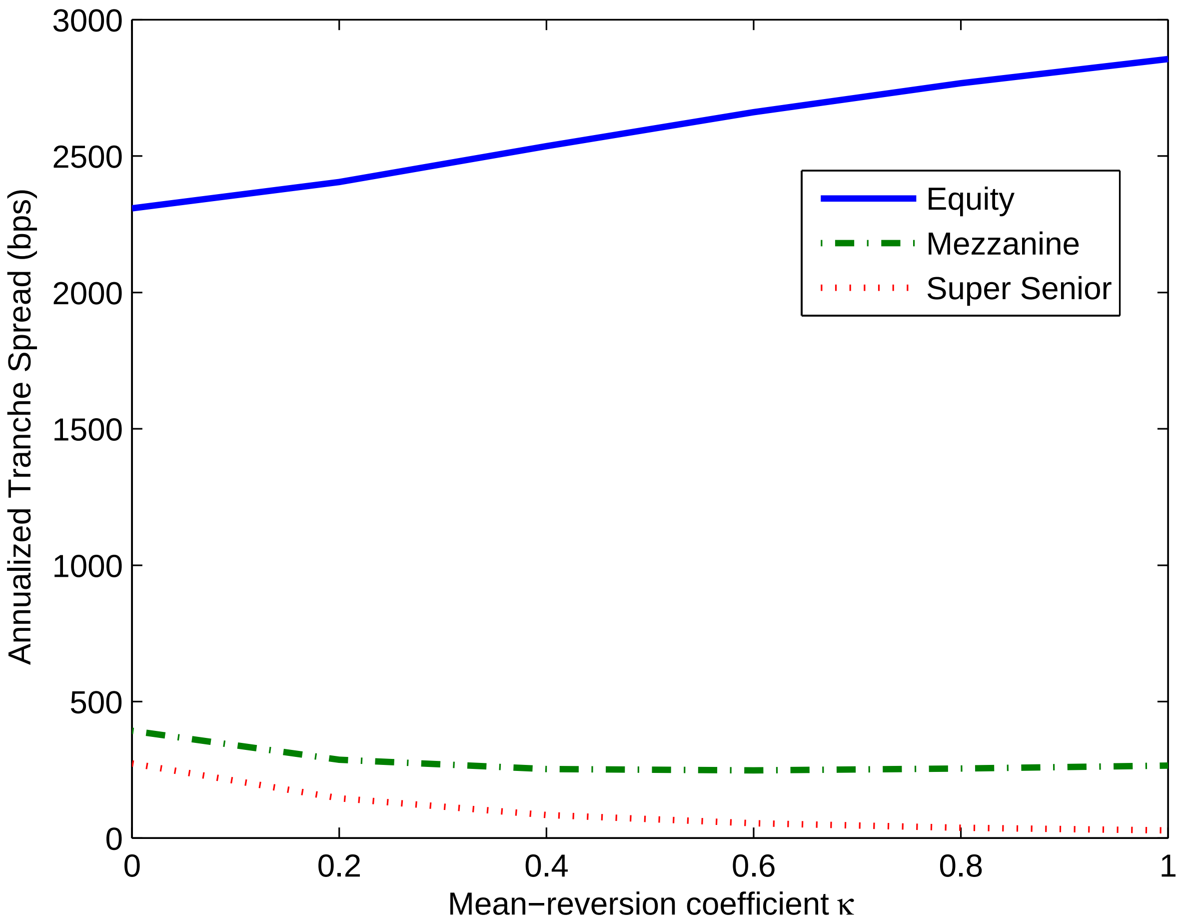
Tranche spreads for the Hawkes process

- Need values of call options on L maturing at future dates
- The conditional density $f_t(\cdot, s)$ of L_s at time t is obtained by inverting the transform and we calculate

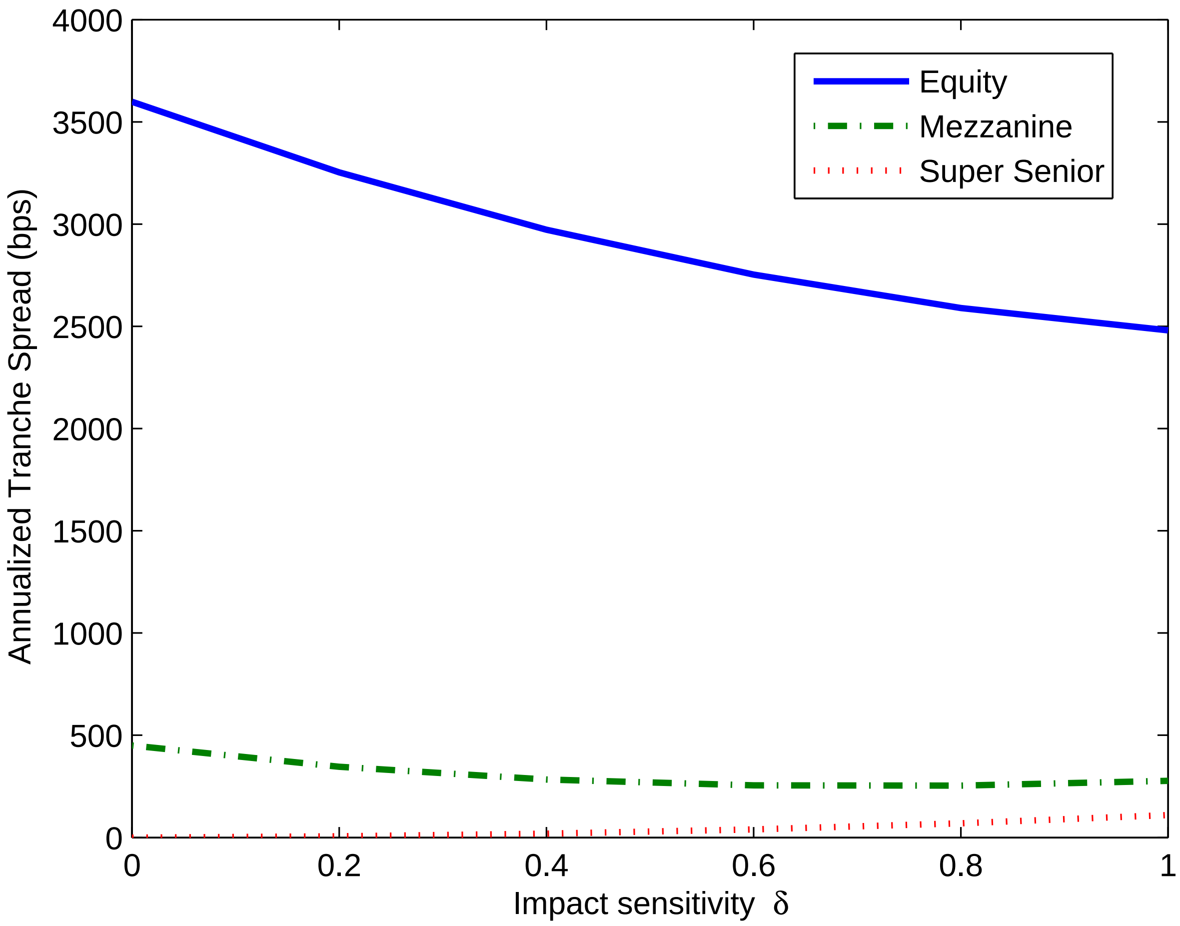
$$E_t[(L_s - c)^+] = \int_c^\infty (x - c) f_t(x, s) dx$$

- Alternatives
  - Express transform of option price in terms of transform of L_s, see Carr and Madan (1999) and Lee (2005)
  - Decompose the option payoff into $L_s 1_{\{L_s \geq c\}}$ and $c 1_{\{L_s \geq c\}}$, see Duffie, Pan and Singleton (2000)

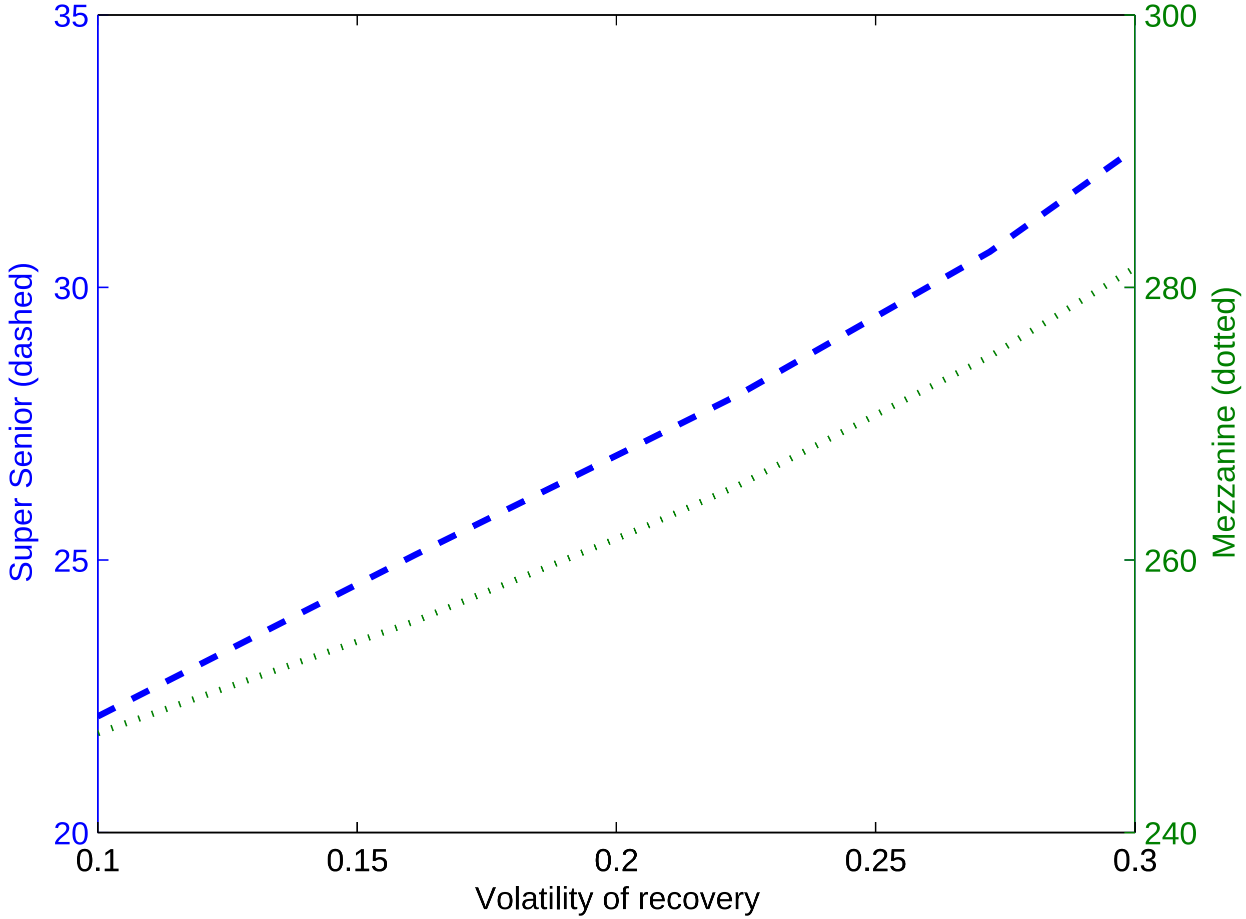
5-yr tranche: $\delta = \lambda_\infty = 1$, $\bar{\nu} = 0.7$



5-yr tranche: $\lambda_\infty = \kappa = 1$, $\bar{\nu} = 0.7$



5-yr tranche: $\lambda_\infty = \kappa = \delta = 1$, $\bar{\nu} = 0.7$



Calibrating λ_∞, κ, δ and ν to HY CDX6 5-yr index and tranche spreads (6/1/2006)

	Bid	Ask	Type	Calibration
0-10%	81.5%	82.0%	Upfront	82.1%
10-15%	45.5%	46.0%	Upfront	46.6%
15-25%	408	418	Running	406
25-35%	78	82	Running	90
Index	332	332	Running	332

Affine point process

- A vector point process J is called **affine** if its intensity λ is an affine function of a risk factor vector X that satisfies

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t + \zeta dJ_t$$

  - W is a standard Brownian motion
  - Z is a point process with intensity $h(X)$
  - μ, $\sigma\sigma^\top$ and h are affine functions
  - ζ is a diagonal matrix

- The components of J share common event times and at each event time the jump size vector is drawn from a fixed distribution

Affine point process: self-affecting examples

- Hawkes process: between events, the intensity drifts deterministically toward λ_∞:

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)\, dt + \delta\, dL_t$$

- Between events, the intensity drifts stochastically toward λ_∞:

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)\, dt + \sigma\sqrt{\lambda_t}\, dW_t + \delta\, dL_t$$

- Adding shocks governed by a point process Z:

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)\, dt + \sigma\sqrt{\lambda_t}\, dW_t + dZ_t + \delta\, dL_t$$

Affine point process: transform

- Let ρ be a non-negative process with affine dependence on X:

$$\rho(x,t) = R_0(t) + R_1(t) \cdot x$$

  for deterministic functions $R_0(t)$ and $R_1(t)$

  – Short rate (correlation between interest and default rate)

- Under technical conditions, the discounted conditional transform

$$E_t[e^{-\int_t^s \rho(X_v,v)dv} e^{u \cdot J_s}] = e^{\alpha(u,t,s)+\beta(u,t,s)\cdot X_t + u\cdot J_t}$$

  where the coefficient functions satisfy the Riccati-ODEs

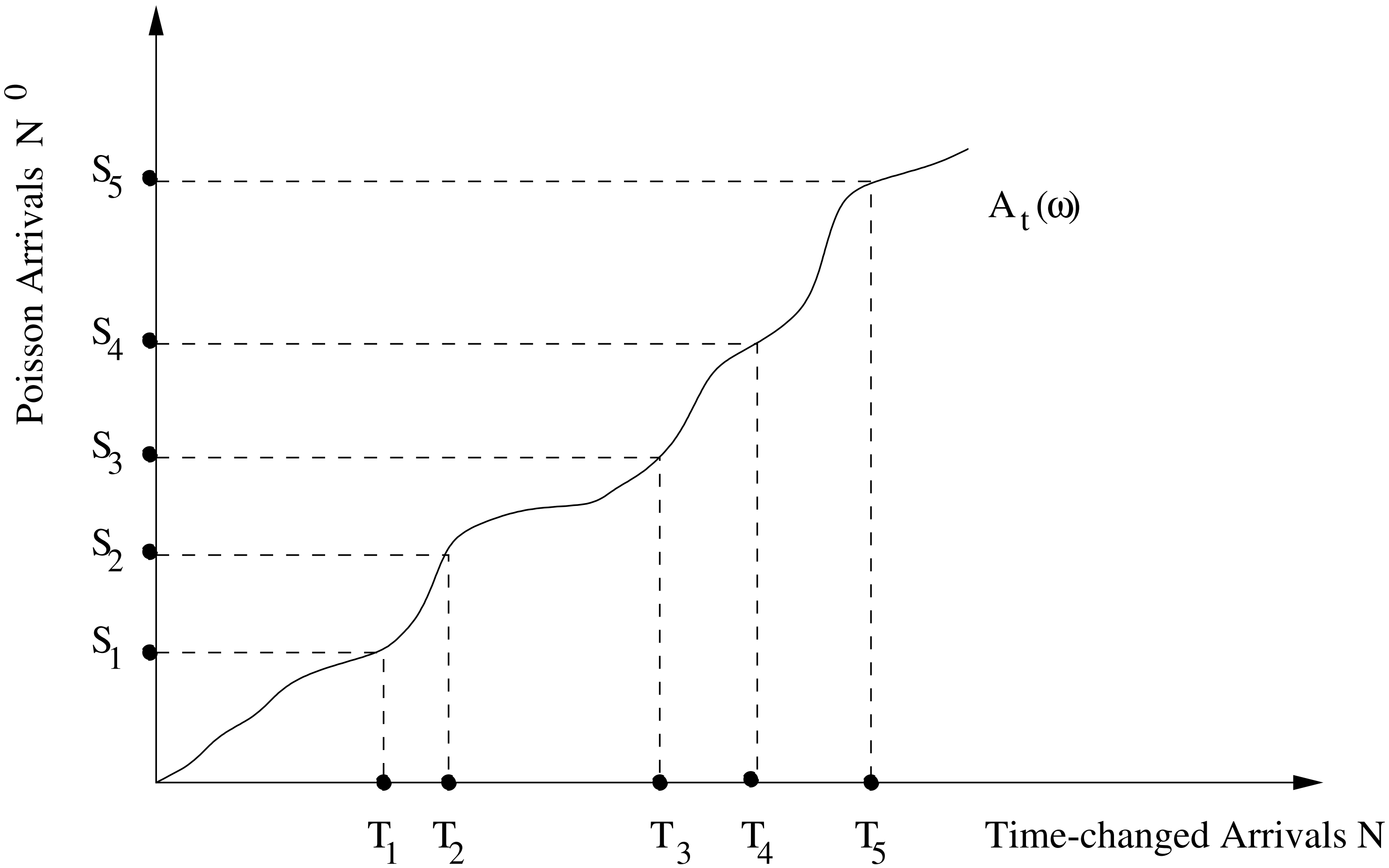
APP as time-changed Poisson process

- The fundamental model is a standard Poisson process N^0
  - Intensity is equal to 1
  - Inter-arrival times are independent and Exp(1)
- Default process is obtained by **time-changing** the Poisson process:

$$N = N^0_A \qquad \text{where} \qquad A_t = \int_0^t \lambda_s ds$$

  - Analogous to modeling the price process of an underlying by a time-changed Brownian motion in equity derivatives
  - Converse to Meyer's (1971) time change theorem
  - Worked out in Giesecke and Tomecek (2005)

Time change


Poisson Arrivals N 0
S5
S4
S3
S2
S1
A t (ω)
T1
T2
T3
T4
T5
Time-changed Arrivals N

Time change and transform

- Under mild conditions on A,

$$E_t[e^{uN_s}] = e^{uN_t}\mathcal{L}^u_{t,s}(1 - e^u)$$

where $\mathcal{L}^u_{t,s}$ is the conditional "Laplace transform" of $A_s - A_t$ under the complex valued measure defined by the density

$$e^{uN + A(1-e^u)}$$

see Carr and Wu (2004)

- Under this measure, the compensator of the default process is $e^u A$
- Modeling strategies
  - Choose time change A whose Laplace transform is convenient
  - Time change a point process whose distribution is known

Time-changed birth process

- Start with a self-affecting birth process N^0, whose intensity is $c + \delta N^0$ and whose distribution is negative binomial

- Apply an independent time change with density ν to get a self-affecting process with intensity dynamics

$$d\lambda_t = (c + \delta N_{t-}) d\nu_t + \nu_{t-} \delta dN_t$$

- For $R_t = N_t + c/\delta$, we have

$$P_t[N_s - N_t = k] = \frac{\Gamma(R_t + k)}{\Gamma(R_t) k!} \sum_{m=0}^{k} \binom{k}{m} (-1)^m \mathcal{L}_{t,s}(\delta(m + R_t))$$

  where $\mathcal{L}_{t,s}$ is the Laplace transform of the time change applied to the birth process N^0

Time-changed birth process: example

- We let the default process $N^n = N \wedge n$ with distribution $P_{t,s}(k)$
- Suppose the default losses are i.i.d. and have a gamma distribution with shape parameter m valued in the positive integers and scale parameter $\theta > 0$
- For $a = (c - L_t)/\theta$ and $\Gamma(\cdot, \cdot)$ the incomplete Gamma function

$$E_t[(L_s - c)^+] = \sum_{k=0}^{n-N_t^n} \frac{\theta P_{t,s}(k)}{(mk)!}(e^{-a}a^{mk+1} - (a - mk)\Gamma(mk+1, a))$$

- **Closed form pricing** for a broad class of processes that descend from the birth process
- Tranche options require a single numerical integration, see Ding, Giesecke and Tomecek (2006)

Hedging: random thinning

- Models for the constituent names are required to **hedge** the exposure to single name spread changes and defaults
- We need to attribute a fraction of the portfolio credit risk to a constituent name
- Given a portfolio model λ, there is a thinning process Y^i such that

$$Q_t[t < \tau^i \leq s] = \int_t^s E_t[Y_v^i \lambda_v] \, dv$$

- The value Y_t^i is the conditional probability that name i defaults next, given that a default is imminent
- Each constituent model incorporates the default dependence among all names

Hedging: random thinning

- The model default probability can be calculated explicitly for affine and time-changed birth process models
- Calibrate a parametric thinning vector $(Y^1, \dots, Y^n)$ from the swap curves of the constituent names
  - Choose the parameters of each Y^i to match the default probability functions implied by the single name swap curves
  - The fit of the portfolio intensity model λ to the multi-name market remains intact
- Estimate single name hedge ratios by bumping the input spread

Summary

- Self-affecting top down models incorporate salient market wide effects that are difficult to model from the bottom up
- The top down perspective supports closed form or semi-analytic transform based pricing, hedging and calibration of derivatives on the loss process
- Top down models are fit directly to multi-name information such as market spreads of indexes and tranches
- Top down models specify the joint evolution of aggregate and constituent losses in relation to the information revealed over time

References

Download at `www.stanford.edu/~giesecke`

- "A top down approach to multi-name credit"
  with Lisa Goldberg
- "Pricing credit from the top down with affine point processes"
  with Eymen Errais and Lisa Goldberg
- "Dependent events and changes of time"
  with Pascal Tomecek
- "Time-changed birth processes and tranche pricing"
  with Xiaowei Ding and Pascal Tomecek