Surgery and harmonic spinors

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Outline

The Dirac operator

Let M be a (fixed) compact manifold with spin structure,
$n = \dim M$.
For any metric g on M one defines

- the *spinor bundle* $\Sigma_g M$: a vector bundle with a metric, a connection and Clifford multiplication $TM \otimes \Sigma_g M \to \Sigma_g M$.
- the *Dirac operator* $D_g : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_g M)$: a self-adjoint elliptic differential operator of first order.

$\Longrightarrow \quad \dim \ker D_g$ is finite-dimensional.

The elements of $\ker D_g$ are called *harmonic spinors*.

Dirac operator and conformal change

Hitchin 1974:
If $\widetilde{g} = f^2 g$, then one can identify $\Sigma_g M$ with $\Sigma_{\widetilde{g}} M$ such that

$$D_{\widetilde{g}} = f^{-\frac{n+1}{2}} D_g f^{\frac{n-1}{2}}.$$

Hence

$$\dim \ker D_g$$

is conformally invariant.

Lichnerowicz formula

$$\int_M |D\psi|^2 = \int_M |\nabla\psi|^2 + \frac{1}{4}\int_M \mathrm{scal}\ |\psi|^2$$

Hence $\mathrm{scal}\ > 0$ implies $\ker D = \{0\}$.

Atiyah-Singer Index Theorem for $n = 4k$

Let $n = 4k$. $\Sigma_g M = \Sigma_g^+ M \oplus \Sigma_g^- M$. $D_g = \begin{pmatrix} 0 & D_g^- \\ D_g^+ & 0 \end{pmatrix}$

$$\mathrm{ind}\ D_g^+ = \dim\ker D_g^+ - \mathrm{codim}\ \mathrm{im}\ D_g^+ = \dim\ker D_g^+ - \dim\ker D_g^-$$

Theorem (Atiyah-Singer 1968)

$$\mathrm{ind}\ D_g^+ = \int_M \widehat{A}(TM)$$

Hence: $\dim\ker D_g \geq |\int \widehat{A}(TM)|$

Index Theorem for $n = 8k + 1$ and $8k + 2$

$n = 8k + 1$:

$$\alpha(M) := \dim \ker D_g \mod 2$$

$n = 8k + 2$:

$$\alpha(M) := \frac{\dim \ker D_g}{2} \mod 2$$

$\alpha(M)$ is independent of g.
However, $\alpha(M)$ depends on the choice of spin structure.

D-minimal metrics

We summarize:

$$\dim\ker D^g \geq \begin{cases} |\int \widehat{A}(TM)|, & \text{if } n = 4k; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

A metric is called *D-minimal* if we have equality.

D-minimality theorem

Theorem (D-minimality theorem, ADH 2006)

Generic metrics on connected compact spin manifolds are D-minimal.

Generic = dense in C^∞-topology and open in C^1-topology.

The investigations for this result were initiated by Hitchin (1974).
The theorem was explicitly conjectured by Bär-Dahl (2002).

History of partial solutions

In order to show that generic metrics are D-minimal, if suffices to show that one D-minimal metric exists.

- Hitchin (1974): $\dim \ker D_g$ depends on g.
- Maier (1996) proved the theorem if

$$n = \dim M \leq 4.$$

- Bär-Dahl (2002) proved the theorem when

$$n \geq 5 \text{ and } \pi_1(M) = \{e\}.$$

  They use the surgery method which has already turned out to be useful in the construction of manifolds with positive scalar curvature (Gromov-Lawson 1980, Stolz 1992).

- Our proof (ADH 2006) also uses the surgery method. It works under no restriction on n or π_1.

Large kernel conjecture

Conjecture

Let $\dim M \geq 3$*. For any* $k \in \mathbb{N}$ *there is a metric* g_k *with* $\dim \ker D \geq k$*.*

This conjecture has been proved by

- *Hitchin 1974* on $M = S^3$ for any $k \in \mathbb{N}$,
- *Hitchin 1974* in dimensions $n \equiv 0, 1, 7 \mod 8$ for $k = 1$,
- *Bär 1996* in dimensions $n \equiv 3, 7 \mod 8$ for $k = 1$,
- *Seeger 2000* on S^{2m}, $m \geq 2$, for $k = 1$,
- *Dahl 2006* on S^n, $n \geq 5$, for $k = 1$.

Many open cases!

Comparison to Kähler manifolds

Let (M, g) be Kähler.
A spin structure corresponds to a square root L of the canonical bundle.
The Dirac operator on (M, g) coincides with the Dolbeault $\bar{\partial} + \bar{\partial}^*$ acting on $(0, *)$-forms twisted by L.

Kotschick (1996) constructs complex manifolds M, on which **generic Kähler metrics** are **not** D-minimal.

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Comparison to other generalized Dirac operators: Gauss-Bonnet-Chern operator

n even

$$\Lambda^* T^* M = \Lambda^{even} T^* M \oplus \Lambda^{odd} T^* M$$

$$d + d_g^* = \begin{pmatrix} 0 & (d + d_g^*)^{odd} \\ (d + d_g^*)^{even} & 0 \end{pmatrix}$$

$$\dim \ker(d + d_g^*)^{even} = \sum_{i \text{ even}} b_i, \qquad \dim \ker(d + d_g^*)^{odd} = \sum_{i \text{ odd}} b_i,$$

$$\mathrm{ind}\,(d + d_g^*)^{even} = \sum_{i=0}^{n} (-1)^i b_i = \chi(M)$$

$$\dim \ker(d + d_g^*) = \sum_{i=0}^{n} b_i$$

If $\sum_{i=0}^{n} b_i > \chi(M)$, then no metric is "$d + d^*$-minimal".

Signature Operator

$n = 4k$

$$\Lambda^* T^* M = \Lambda^+ T^* M \oplus \Lambda^- T^* M$$

Splitting according to

$$\epsilon = i^{\frac{n}{2}+p(p-1)} * : \Gamma(\Lambda^p_{\mathbb{C}} T^* M) \to \Gamma(\Lambda^{n-p}_{\mathbb{C}} T^* M).$$

$$d + d_g^* = \begin{pmatrix} 0 & (d + d_g^*)^- \\ (d + d_g^*)^+ & 0 \end{pmatrix}$$

Let $b^+_{n/2}$ (resp. $b^-_{n/2}$) be the number of positive (resp. negative) eigenvalues of the intersection form

$$H^{n/2}(M, \mathbb{R}) \times H^{n/2}(M, \mathbb{R}) \to \mathbb{R}.$$

Then $b^+_{n/2} + b^-_{n/2} = b_{n/2}$.

Signature Operator (cont.)

$$\dim \ker(d + d_g^*)^+ = b_{n/2}^+ + \sum_{i=0}^{(n/2)-1} b_i,$$

$$\dim \ker(d + d_g^*)^- = b_{n/2}^- + \sum_{i=0}^{(n/2)-1} b_i.$$

$$\mathrm{ind}\ (d + d_g^*)^+ = b_{n/2}^+ - b_{n/2}^- = \mathrm{sign}(M)$$

$$\dim \ker(d + d_g^*) = \sum_{i=0}^{n} b_i$$

Then no metric is “minimal”, unless $b_i = 0$ for all $i \neq n/2$ and $b_{n/2}^{\pm} = 0$.

Surgery

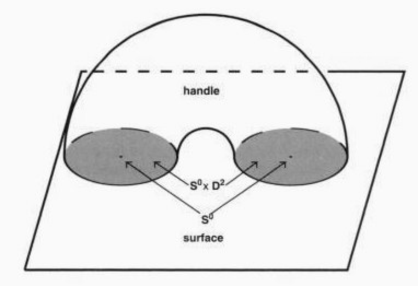
Let $f : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We define

$$M^{\#} := M \setminus f(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1})/\sim$$

where $/\sim$ means gluing the boundaries via

$$M \ni f(x,y) \sim (x,y) \in S^k \times S^{n-k-1}.$$

We say that $M^{\#}$ is obtained from M by surgery of dimension k.



Example: 0-dimensional surgery on a surface.

Scalar curvature and surgery

Theorem (Gromov-Lawson 1980)

Let $k \leq n - 3$.
If M carries a metric of positive scalar curvature, then $M^{\#}$ carries a metric of positive scalar curvature as well.

Strong consequences, in particular if $\pi_1 = \{e\}$.

Gromov-Lawson fails for $k = n - 2$ as $S^1 = S^{n-k-1}$ has scalar curvature 0.

D-minimality and surgery

Theorem (D-Surgery Theorem, ADH 2006)

Let $k \leq n - 2$.
If M carries a D-minimal metric, then $M^{\#}$ carries a D-minimal metric as well.

Bär-Dahl (2002) proved the theorem with other methods for $k \leq n - 3$.

Proof of "D-surgery Thm $\Longrightarrow$ D-minimality Thm"

We use a theorem from Stolz 1992.
The given spin manifold M is bordant to $N \cup P$, where

- P carries a metric of positive scalar curvature,
- N is a disjoint union of products of S^1, a $K3$-surface and a Bott manifold, and carries a D-minimal metric.

Perform surgery at the bordism in order to get a connected and simply connected bordism W between $N \cup P$ and M.
Decompose W into surgeries of dimensions $0, \ldots, n-2$.

Proof of the D-surgery theorem

Let g be a D-minimal metric on M and $f : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We write close to $S := f(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{round}^{n-k-1}$$

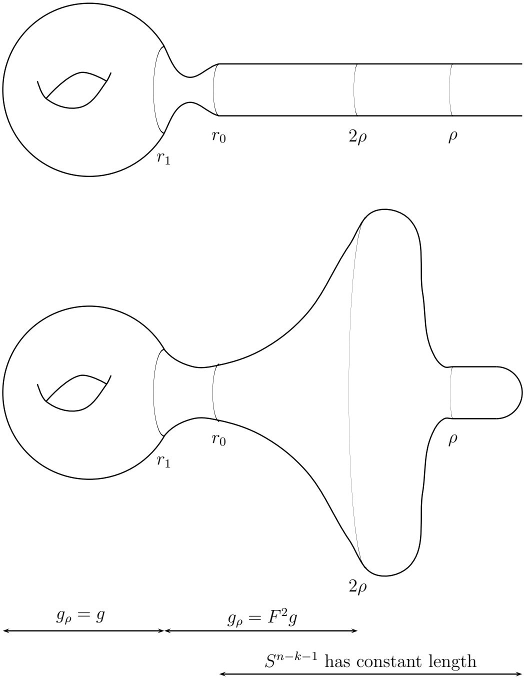
where g_{round}^{n-k-1} is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{round}^{n-k-1}$$

We define a metric

$$g_\rho^\# = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{round}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

that extends to a metric on $M^\#$.



r_1
r_0
2ρ
ρ
r_0
r_1
ρ
2ρ
$g_\rho = g$
$g_\rho = F^2 g$
S^{n-k-1} has constant length

Assume that ψ_ρ is a harmonic spinor on $(M^\#, g_\rho^\#)$ with L^2-norm 1.

The spinors ψ_ρ converge for $\rho \to 0$ in a certain weak sense to a harmonic spinor $\bar{\psi}$ on $M \setminus S$.

Show that each ψ_ρ falls off exponentially as $t \to \infty$.

The exponential fall off implies that $\bar{\psi}$ does not vanish.

It also implies regularity, harmonicity and L^2-boundedness for $\bar{\psi}$.

A removal of singularity theorem says that $\bar{\psi}$ extends to a harmonic spinor on M.

The τ-invariant

We define

$$\lambda_{\min}(M, [g_0]) := \inf_{g \in [g_0]} \lambda_1(D_g^2) \operatorname{vol}(M, g)^{2/n},$$

$$\tau(M) := \sup_{[g_0]} \lambda_{\min}(M, [g_0]).$$

One shows that $\tau(M) > 0$ iff there is a metric with $\ker D_g = \{0\}$.
Hence:

- $\tau(M) = 0$ iff there is an index theoretical reason,
- $\tau(M) > 0$ otherwise.

Monotonicity for τ

Theorem (AH2006)

Let $M^\#$ be obtained from M by 0-dimensional surgery. Then

$$\tau(M^\#) \geq \tau(M).$$

Application ($n = 2$):

$$\begin{array}{ll} \tau(M) = 0 & \text{if } \alpha(M) = 1 \\ \tau(M) = \lambda_{\min}(S^2) = 4\pi & \text{if } \alpha(M) = 0 \end{array}$$

More details in my publications:

`http://www.berndammann.de/publications`

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