Minisymposium 6

Positive Definite Functions and Applications

Leiter des Symposiums:

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Positive definite functions play a central role in several areas of mathematical research. Among others, they appear as characteristic functions of probability distributions, as correlation functions, within the context of spectral theory and convolution algebras, or in connection with representation of groups, semigroups and hypergroups. Further on, a large variety of mathematical applications have been developed making use of positive definite functions and their properties.

The study of positive definite functions and their associated reproducing kernel Hilbert spaces lead to a better understanding of the relations between the theory of Hilbert spaces, representation theory, harmonic analysis, and special functions. On top of that, methods based on positive definite functions in applications rely on a better understanding of fundamental properties of these relations.

The aim of the symposium is to bring together scientists from different mathematical areas working on or using positive definite functions. Next to six survey lectures on central topics there will be research presentations from both, the theoretical perspective as well as the point of view of applications.

Montag, 18. September

HS II, Hauptgebäude, Regina-Pacis-Weg

15:00 – 15:50 **Heinz Langer** *(Wien)*
M.G.Krein's extension problem of positive definite functions

16:00 – 16:20 **Matthias Langer** *(Glasgow)*
Continuations of Hermitian indefinite functions and operator models for corresponding canonical systems

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16:30 – 16:50 **Klaus-Robert Müller** *(Berlin, Potsdam)*
Denoising and Dimension Reduction in Feature Space

17:00 – 17:50 **Bernhard Schölkopf** *(Tübingen)*
Applications of Kernel Methods

Dienstag, 19. September

Seminarraum 2, AVZ I, Endenicher Allee 11-13

15:00 – 15:50 **Herbert Heyer** *(Tübingen)*
Positiv- und negativ-definite Funktionen auf Hypergruppen und deren Dualräumen

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16:00 – 16:20 **Rupert Lasser** *(München)*
Positive definite sequences on polynomial hypergroups and the discrete part of their spectral measures

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16:30 – 16:50 **Martin Schlather** *(Hamburg)*
Fast and Exact Simulation of Large Gaussian Lattice Systems

17:00 – 17:50 **Tilmann Gneiting** *(Washington)*
Real-world applications of positive definite functions: A diptych

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Mittwoch, 20. September

Seminarraum 2, AVZ I, Endenicher Allee 11-13

15:00 – 15:50 **Christian Berg** *(Kopenhagen)*
Transformations of moment sequences, a fix-point-measure and its Mellin transform

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16:00 – 16:20 **Paul Ressel** *(Eichstätt-Ingolstadt)*
Positive definite functions and exchangeable random processes

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16:30 – 16:50 **Holger Wendland** *(Göttingen)*
Positive Definite Functions in Aeroelasticity: Towards Airbus's Preferred Solution

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17:00 – 17:50 **Robert Schaback** *(Göttingen)*
Kernel-based meshless methods for solving PDEs

Vortragsauszüge

Heinz Langer *(Wien)*
M.G.Krein's extension problem of positive definite functions

In the lecture a survey of the theory of extensions of positive definite functions (which M.G.Krein considered one of his most important contributions to analysis) and its applications to the inverse spectral problem and to the extrapolation problem of weakly stationary processes is given.

Matthias Langer *(Glasgow)*
Continuations of Hermitian indefinite functions and operator models for corresponding canonical systems

There is a close connection between the continuation problem of positive definite functions from a finite interval to the real axis and the inverse spectral problem for certain differential equations. In this talk this connection is discussed for a simple example of a function that has a negative square if the interval is large enough. In this case a singularity appears for the differential equation. Operator models in a Pontryagin space are presented for this situation.

Klaus-Robert Müller *(Berlin, Potsdam)*
Denoising and Dimension Reduction in Feature Space

joint work with
Mikio L. Braun, Fraunhofer FIRST.IDA, Berlin
Joachim Buhmann, Department of Computer Science, ETH Zürich

We prove that the relevant information about a classification problem in feature space is contained in a rather small number of leading kernel PCA components. This new theoretical insight means that kernels not only transform data sets such that they become ideally linearly separable in feature space. Rather kernelizing is done in a manner which makes economic use of feature space dimensions, i.e. well chosen kernels provide efficient representations of the data that are effective for classification. Thus our work provides another contribution for explaining why kernel-based learning methods work well. Practically we propose an algorithm which enables us to estimate the subspace and dimensionality relevant for good classification. Our algorithm can therefore be applied (1) to analyze the interplay of data set and kernel better, (2) to help in model selection, and to (3) de-noising in feature space in order to yield better classification results. Simulations underline these claims.

Bernhard Schölkopf *(Tübingen)*
Applications of Kernel Methods

The talk will first discuss positive definite kernels in machine learning and why they are currently rather popular. Time permitting, we will also present some applications, including problems of computer graphics (surface modeling and morphing).

Herbert Heyer *(Tübingen)*
Positiv- und negativ-definite Funktionen auf Hypergruppen und deren Dualräumen

Hypergruppen sind lokalkompakte Räume, für welche der Raum der beschränkten Masse bzgl. einer axiomatisch festgelegten Faltung zu einer Banachalgebra wird. Beispiele für Hypergruppen sind Doppelnebenklassenräume von Gelfand-Paaren. Obgleich der Ausbau der harmonischen Analyse von Hypergruppen in Anlehnung an den Spezialfall lokalkompakter Gruppen vollzogen wird, ergeben sich bereits im Falle kommutativer Hypergruppen neue Fragestellungen, die nur mit zusätzlichem analytischen Aufwand bewältigt werden können. Beispielsweise verliert der mittels der Charaktere definierbare Dualraum im allgemeinen die Eigenschaft, wieder eine Hypergruppe zu sein, womit die Pontrjagin-Eigenschaft eine Rarität wird. Trotzdem möchte man auch auf diesem Dualraum positiv-und negativ-definite Funktionen erklären und mittels der Fouriertransformation Sätze vom Bochnerschen bzw. Schoenbergschen Typ beweisen sowie die aus der Wahrscheinlichkeitstheorie bekannten Themenkreise der unendlich teilbaren Verteilungen und ihrer Einbettung in stetige Faltungshalbgruppen auf Hypergruppen übertragen. Im Vortrag soll die Problematik des aus der Kategorie der Hypergruppen herausfallenden Dualraums im Mittelpunkt stehen und gezeigt werden, welchen Einschränkungen man ausgesetzt ist, wenn man Positiv-und Negativ-Definitheit und damit einen Teil der Darstellungstheorie für möglichst große Klassen von Hypergruppen zur Wirkung bringen möchte.

Rupert Lasser *(München)*

Positive definite sequences on polynomial hypergroups and the discrete part of their spectral measures

Given a positive definite bounded sequence $(d_n)_{n\geq 0}$ on a polynomial hypergroup, Bochner's theorem yields a bounded positive Borel measure on the dual space. We derive results describing when and how the discrete part of this measure can be determined completely from the sequence $(d_n)_{n\geq 0}$ by applying certain average processes. In that way we generalize results of Ky Fan and N. Wiener.

Martin Schlather *(Hamburg)*

Fast and Exact Simulation of Large Gaussian Lattice Systems

Joint work with Tilmann Gneiting, Yindeng Jiang, Donald B. Percival and Hana Ševčíková

The circulant embedding technique allows for the fast and exact simulation of stationary and intrinsically stationary Gaussian random fields. The method uses periodic embeddings and relies on the fast Fourier transform. However, exact simulations require that

the periodic embedding is nonnegative definite, which is frequently not the case for two-dimensional simulations.
Here we consider a suggestion by Michael Stein, who proposed nonnegative definite periodic embeddings based on suitably modified, compactly supported covariance functions. Theoretical support to this proposal and software for its implementation are presented.

Tilmann Gneiting *(Washington)*
Real-world applications of positive definite functions: A diptych

In this talk, I discuss two complementary applications of positive definite functions that address important real-world problems, yet are of mathematical interest by themselves.
A major human desire is to make forecasts for an uncertain future. There are strong arguments, philosophically, scientifically and economically, that forecasts should be probabilistic in nature, taking the form of probability distributions over future events. Scoring rules assess the quality of probabilistic forecasts, by assigning a numerical score based on the forecast and on the event or value that materializes. An elegant construction originally proposed by Eaton uses positive definite functions to construct scoring rules that encourage the forecaster to make careful assessments and to be honest. We study a generalization that is based on conditionally negative definite functions and generates rich classes of proper scoring rules.
The second part of my talk considers geostatistical models for spatio-temporal data. We use classical results in harmonic analysis to construct novel classes of positive definite, nonseparable space-time covariance functions, and fit them to wind data from Ireland.

Christian Berg *(Kopenhagen)*
Transformations of moment sequences, a fix-point-measure and its Mellin transform

In recent papers A. Durán and the speaker studied some non-linear transformations from Hausdorff moment sequences (a_n) to Stieltjes moment sequences (s_n), namely

$$s_n = (a_0 a_1 \cdots a_n)^{-1}, \quad s_n = (a_0 + a_1 + \cdots + a_n)^{-1}.$$

This made it possible to unify different constructions from the theory of additive functionals of Lévy-processes. The 'sum' transformation has a fix-point (m_n) defined by the recursive equation

$$(m_0 + m_1 + \cdots + m_n)m_n = 1, \quad n \geq 0$$

i.e.

$$m_0 = 1, \quad m_1 = \frac{-1+\sqrt{5}}{2}, \quad m_2 = \frac{\sqrt{22+2\sqrt{5}}-\sqrt{5}-1}{4}, \ldots,$$

and (m_n) is the moment sequence of a probability measure μ on $[0,1]$. In a new manuscript we prove that μ has an increasing and convex density and that the Mellin transform F of μ

$$F(z) = \int_0^1 t^z \, d\mu(t),$$

can be characterized in analogy with the Bohr-Mollerup theorem about the Gamma function as the unique log-convex function $F :]-1,\infty[\to]0,\infty[$ satisfying $F(0) = 1$ and the functional equation

$$1/F(s) = 1/F(s+1) - F(s+1), \quad s > -1.$$

We also prove that F extends to a meromorphic function in the whole complex plane.

Paul Ressel *(Eichstätt-Ingolstadt)*
Positive definite functions and exchangeable random processes

There is an intimate connection betwen exchangeable random structures and harmonic analysis on semigroups, applying to classical theorems of de Finetti (and generalizations therof) as well as to much more recent results on exchangeable random orders and partitions. The short overview will be followed by some open problems in this area.

Holger Wendland *(Göttingen)*
Positive Definite Functions in Aeroelasticity: Towards Airbus's Preferred Solution

In fluid-structure interaction (FSI) the reciprocal action of a flexible structure with a flowing fluid, in which it is submersed or by which it is surrounded, is studied. Naturally, FSI has applications in many fields of engineering, such as the stability and response of aircrafts, the flow of blood through arteries, the vibration of turbine and compressor blades, and the response of bridges and tall buildings to winds.

In this talk, I will present an efficient scheme for loose coupling in fluid-structure-interaction problems as they typically appear in the context of aircraft design. This coupling scheme uses a multivariate scattered data interpolation approach, based on positive definite functions and partition of unity methods. It allows us to couple arbitrary meshes on fluid and structure side. It conserves virtual work and forces. It is designed for large scale problems and allows the coupling of entire aircraft meshes. Finally, it is currently implemented into MSC.Nastran to become Airbus's preferred solution in the field of Aeroelasticity.

Robert Schaback *(Göttingen)*
Kernel-based meshless methods for solving PDEs

This talk provides a framework to derive error bounds and convergence rates for certain unsymmetric meshless methods, including the technique started by E. Kansa in 1986 and the Meshless Local Petrov Galerkin method (MLPG) of S.N. Atluri and collaborators, dating back to 1998. It consists of four essential ingredients:

(1) continuous dependence of the solution of the analytic problem on the data,
(2) a space of *trial* functions allowing a reasonably good approximation to functions in the regularity class of the solution,
(3) a weak or strong *testing* strategy with a certain stability property with respect to the trial space,
(4) a numerical solution of an overdetermined unsymmetric linear system within a certain tolerance.

The theory is not constrained to elliptic problems. It will be shown how to apply the framework for special situations where meshless translates of *kernels* are used as trial functions. In case of weak problems, meshless translates of kernels occur also as test functions, while the test side of strong problems is handled by collocation. Thus our framework covers methods in strong and weak form. In the weak case it allows distributional data, providing error bounds in negative Sobolev norms.