Problem sheet 5 Rigid analytic geometry Winter term 2024/25

Problem 1 (7 points)**.** *Let X be a G_+-space which has a G_+-base $\mathfrak{B}$ such that $N \in \mathbb{N}$ and $\left(\Omega_i\right)_{i=1}^N \in \mathfrak{B}^N$ implies $\bigcap_{i=1}^N \Omega_i \in \mathfrak{B}$, where the empty intersection is defined as X. Show that X^* is a spectral space.*

Problem 2 (6 points)**.** *Let X be a spectral space, Z a closed subset of X and $\mathcal{G}$ a family of quasicompact open subsets of X, such that $Z \cap \bigcap_{\Omega \in \mathcal{F}} \Omega \neq \emptyset$ for all finite subsets $\mathcal{F} \subseteq \mathcal{G}$. Then $Z \cap \bigcap_{\Omega \in \mathcal{G}} \Omega \neq \emptyset$.*

Let X be a set. By a boolean algebra of subsets of X we understand a set $\mathcal{A}$ of subsets of X such that $n \in \mathbb{N}$ and $(M_i)_{i=1}^n \in \mathcal{A}^n$ implies $\bigcup_{i=1}^n M_i \in \mathcal{A}$, and such that $M \in \mathcal{A}$ implies $X \setminus M \in \mathcal{A}$. The boolean algebra generated by a set $\mathfrak{G}$ of subsets of X is the smallest boolean algebra of subsets of X containing $\mathfrak{G}$.

From now on let always X be a spectral space. The set $\mathfrak{C}_X$ of constructible subsets of X is the boolean algebra generated by the set of quasi-compact open subsets of X. It is easy to see that the elements of $\mathfrak{C}_X$ are precisely the subsets of the form $\bigcup_{i=1}^N \left(\Omega_i \cap Z_i\right)$ with $N \in \mathbb{N}$, $\Omega_i \subseteq X$ quasicompact and open and $Z_i \subseteq X$ closed such that $X \setminus Z_i$ is quasicompact. Similarly, the elements of $\mathfrak{C}_X$ are precisely the subsets of the form $\bigcap_{i=1}^N \left(\Omega_i \cup Z_i\right)$ with the same conditions of the operands.

The constructible topology on X is the topopology which has $\mathfrak{C}_X$ as a topology base. Let X_{con} denote X equipped with its constructible topology.

Problem 3 (1 point)**.** *Show that X_{con} is a Hausdorff space.*

Problem 4 (6 points)**.** *Show that X_{con} is compact.*

Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Monday November 25.