It follows from the results of the first four problems from the previous sheet that the category of spectral spaces has arbitrary limits over small categories and that the functors from spectral spaces to topological spaces or to sets commute with such limits. This is so since limits $\lim_C F$ of a functor $C \xrightarrow{F} \mathfrak{S}$ from a small catgory C to a category $\mathfrak{S}$ with products and equalizers can be expressed as the equalizer of the two morphisms

$$\prod_{\alpha \in \mathrm{Ob} C} F(\alpha) \xrightarrow{a, b} \prod_{\phi \in \mathrm{Mor} C} F(\mathrm{targ} \phi)$$

given in the special case of the target categories of sets, groups, or (toplogical, spectral, Priestley) spaces by

$$\begin{aligned} \Big(a\big((f_\alpha)_{\alpha \in \mathrm{Ob} C}\big)\Big)_\phi &= f_{\mathrm{targ}\phi} \\ \Big(b\big((f_\alpha)_{\alpha \in \mathrm{Ob} C}\big)\Big)_\phi &= F(\phi)\big(f_{\mathrm{src}\phi}\big). \end{aligned}$$

Our final general result about spectral spaces is the fact that every such space can be written as a limit of finite T_0 spaces. For this, let X be a spectral space and $\mathfrak{F}_X$ the partially ordered (by $\subseteq$) set of finite subsets of $\mathfrak{Qc}(X)$. For $F \in \mathfrak{F}_X$, we have a map

$$X \xrightarrow{\pi_F} 2^F$$

to the set of subsets of F, given by

$$\pi_F(x) = \big\{\Omega \in F \bigm| x \in \Omega\big\}.$$

Let X_F be the image of π_F equipped with the quotient topology.

Problem 1 (2 points)**.** *Show that X_F is a finite T_0 space.*

Problem 2 (7 points)**.** *Define a functor from $\mathfrak{F}_X$ to spectral spaces sending the object $F \in \mathfrak{F}_X$ to X_F, together with a homeomorphism*

$$X \to \lim_{F \in \mathfrak{F}_X} X_F,$$

demonstrating that this is indeed a homeomorphism!

For my taste this seems easiest to do when the category of Priestley spaces is used instead of using spectral spaces directly. With the previous two problems the exposition of the general theory of spectral spaces given in the exercise sessions is finished.

In the following four problems, let always R be a nat ring whose topology is given by a ring norm $\|\cdot\|$.

Problem 3 (2 points)**.** *Show that*

$$\big\{r \in R \bigm| \|r\| < 1\big\} \subseteq R^{oo}$$

with equality in the case where $\|\cdot\|$ is power multiplicative!

Problem 4 (3 points). *For every positive real number t, show that*

$$K_t = \{ r \in R \mid \|r\| \le t \}$$

is a bounded subset of R! In the case where the ring norm is multiplicative and the topology not discrete, show that every bounded subset of R is contained in K_t for some positive real number t!

Problem 5 (2 points). *Show that*

$$\{ r \in R \mid \|r\| \le 1 \} \subseteq R^o$$

with equality in the case where $\|\cdot\|$ is multiplicative and the topology not discrete!

Problem 6 (6 points). *Show that the Tate norm*

$$\left\| \sum_{k=0}^{\infty} f_k T^k \;\middle|\; R\langle T\rangle \right\| = \max_{0\le k<\infty} \|f_k\|$$

is multiplicative (resp. power multiplicative) when $\|\cdot\|$ has this property!

For the remainder of that sheet let always K be a field equipped with an ultrametric absolute value $|\cdot|$.

Problem 7 (2 points). *If K is of positive characteristic and perfect, show that its completion is also perfect!*

Problem 8 (3 points). *If $(\alpha_i)_{i=1}^d \in K^d$ and $\prod_{i=1}^d (T-\alpha_i) = \sum_{j=0}^d p_j T^j$, show that*

$$\max_{1\le i\le d} |\alpha_i| = \max_{0\le j<d} \sqrt[d-j]{p_j}!$$

Problem 9 (Krasner's Lemma, 4 points). *Assume K to be complete, and let L be a Galois extension of K. The unique extension of $|\cdot|$ to L will also be denoted $|\cdot|$. Let α and β be elements of L such that $|\alpha-\beta| < |\alpha - \sigma(\alpha)|$ for all $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\alpha) \ne \alpha$.*

Problem 10 (6 points). *If K is separably closed, show that its completion $\hat{K}$ is also separably closed!*

The idea is to consider a finite Galois extension $L/\hat{K}$ with primitive element α and to approximate $\mathrm{Min}_{\alpha/\hat{K}}$ by an element of $K[T]$ so well that Krasner's lemma can be applied.

Seventeen of the 37 points from this sheet are bonus points. Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Monday January 6.