Exercises in Geometry II

University of Bonn, Summer Semester 2018
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Sheet 2

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1. Curvature of curves [4 points]

a) Let $\gamma : I \to \mathbb{R}^n$ be a unit speed curve such that its curvature $\kappa_\gamma(t_0) \neq 0$ for some $t_0 \in I$. Show that there is a unique unit speed parametrized circle $c : \mathbb{R} \to \mathbb{R}^n$, called the *osculating cicle* at $\gamma(t_0)$, with the property that $c(t_0) = \gamma(t_0)$, $\dot{c}(t_0) = \dot{\gamma}(t_0)$ and $\ddot{c}(t_0) = \ddot{\gamma}(t_0)$. Further, show that $\kappa_c(t_0) = \frac{1}{R}$, where R is the radius of the osculating circle c.

b) Let M be a Riemannian manifold and suppose $\gamma : I \to M$ is a regular curve, i.e. $\dot{\gamma}(t) \neq 0$ for all $t \in I$, but not necessarily unit speed. Show that the curvature of γ at t is given by

$$\kappa(t) = \left(\frac{|D_t\dot{\gamma}(t)|^2}{|\dot{\gamma}(t)|^4} - \frac{\langle D_t\dot{\gamma}(t), \dot{\gamma}(t)\rangle^2}{|\dot{\gamma}(t)|^6} \right)^{1/2}.$$

2. Totally geodesic submanifolds [4 points]

Let $(\widetilde{M}, \tilde{g})$ be a Riemannian manifold. Show that the following are equivalent for a Riemannian submanifold $M \subset \widetilde{M}$ with induced Riemannian metric g:

1. M is totally geodesic.
2. Every g-geodesic in M is also a $\tilde{g}$-geodesic in $\widetilde{M}$.
3. The second fundamental form of M vanishes identically.

3. The Gaussian curvature of a sphere [4 points]

Show that $S^2_R \subset \mathbb{R}^3$, the round 2-dimensional sphere of radius $R > 0$ in euclidean space $\mathbb{R}^3$, has constant Gaussian curvature $\frac{1}{R^2}$.

4. Product manifolds [4 points]

Suppose $g = g_1 \oplus g_2$ is a product metric on the product manifold $M_1 \times M_2$.

a) Show that for each point $p_i \in M_i$ the submanifolds $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally geodesic.

b) Let $\Pi \subset T(M_1 \times M_2)$ be a 2-plane spanned by $X_1 \in TM_1$ and $X_2 \in TM_2$. Show that the sectional curvature $K(\Pi) = 0$.

c) Show that $S^2_R \times S^2_R$, the Riemannian product of two round 2-spheres of radius $R > 0$, has nonnegative sectional curvature.

d) Show that there is an embedding of the torus $T^2 = S^1 \times S^1$ in $S^2_R \times S^2_R$ such that the induced metric is flat.

Due on Monday, May 7.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`