Exercises in Geometry II

University of Bonn, Summer Semester 2018
Dozent: PD Dr. Fernando Galaz-Garcia
Assistant: Saskia Roos
Sheet 3

UNIVERSITÄT BONN
Rheinische Friedrich-Wilhelms-Universität Bonn

---

1. Models of the hyperbolic space [8 points]

There are three classical models of the hyperbolic space of radius $R > 0$:

- The *upper half space*, $\mathbb{H}^n_R := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ with the metric

$$g_R = R^2 \sum_{i=1}^n \frac{(dx_i)^2}{x_n^2}.$$

  This model is called the *Poincaré half-space model.*

- The *Poincaré ball*, $\mathbb{B}^n_R := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \|x\| = (\sum_{i=1}^n x_i^2)^{1/2} < R\}$ with the metric

$$h_R = 4R^4 \sum_{i=1}^n \frac{(dx_i)^2}{(R^2 - \|x\|^2)^2}.$$

- The *upper hyperboloid sheet*, $\mathbb{U}^n_R := \{(x_1, \dots, x_n, \tau) \in \mathbb{R}^{n+1} : \tau > 0,\ \tau^2 - \|x\|^2 = R^2\}$ with the metric

$$k_R = \sum_{i=1}^n (dx_i)^2 - (d\tau)^2.$$

  The metric k_R is induced by the Minkowski metric $m := \sum_{i=1}^n (dx_i)^2 - (d\tau)^2$ on $\mathbb{R}^{n+1}$, i.e. $k = \iota^* m$, where $\iota : \mathbb{U}^n_R \hookrightarrow \mathbb{R}^{n+1}$ denotes inclusion.

a) The *hyperbolic stereographic projection*

$$\pi : \mathbb{U}^n_R \to \mathbb{B}^n_R, \quad (x, \tau) \mapsto \frac{Rx}{R + \tau}$$

is a diffeomorphism with inverse

$$\pi^{-1} : \mathbb{B}^n_R \to \mathbb{U}^n_R, \quad x \mapsto \left(\frac{2R^2 x}{R^2 - \|x\|^2}, R\frac{R^2 + \|x\|^2}{R^2 - \|x\|^2} \right).$$

Show that $(\pi^{-1})^* k_R = h_R$, i.e. that π and π^{-1} is an isometry.

b) In the following we let $(x_1, \dots, x_n) = (y, x_n)$. The *generalized Cayley transform*

$$\sigma : \mathbb{B}^n_R \to \mathbb{H}^n_R, \quad (y, x_n) \mapsto \left(\frac{2R^2 y}{\|y\|^2 + (x_n - R)^2}, R\frac{R^2 - \|y\|^2 - x_n^2}{\|y\|^2 + (x_n - R)^2} \right)$$

is a diffeomorphism with inverse

$$\begin{aligned}\sigma^{-1} &: \mathbb{H}^n_R \to \mathbb{B}^n_R,\\ (y, x_n) &\mapsto \left(\frac{2R^2 y}{\|x\|^2 + (y+R)^2}, R\frac{\|y\|^2 + x_n^2 - R^2}{\|y\|^2 + (x_n + R)^2} \right).\end{aligned}$$

Show that $\sigma^* g_R = h_R$, which shows that σ and σ^{-1} are isometries.

Hint: First look at the 2-dimensional case to obtain a "feeling" for these models.

2. Transitive action on the hyperboloid [4 points]

The group $\mathrm{O}_+(n,1)$ is the group of all real matrices A such that

$$A^t \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & -1 \end{pmatrix}$$

with $\det(A) = 1$. This is the group of linear maps from $\mathbb{R}^{n+1}$ to itself that preserve the Minkowski metric.

Show that $\mathrm{O}_+(n,1)$ acts transitively on the set of orthonormal bases on $\mathbb{U}^n_R$, i.e. $\mathbb{U}^n_R$ is homogeneous and isotropic.

Hint: Show that for any $p \in \mathbb{U}^n_R$ and any orthonormal basis $(e_1, \ldots, e_n)$ of $T_p\mathbb{U}^n_R$ there is an orthogonal map that maps the point $N = (0, \ldots, 0, R)$ to p and the standard basis $(\partial_1, \ldots, \partial_n)$ to $(e_1, \ldots, e_n)$.

3. Sectional curvature of the hyperbolic space [4 points]

Show that the sectional curvature of the hyperbolic space of radius R has everywhere constant sectional curvature $-\frac{1}{R^2}$.

Possible strategy: By Exercise 2, it suffices to calculate the sectional curvature only at one point for one orthonormal basis (Why?). Then one can reduce the problem to the 2-dimensional case.

Due on Monday, May 14.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`