Generalized Torelli Groups

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Abstract

Let F_n be the free group on $n \geq 2$ elements and $\mathrm{Aut}(F_n)$ its group of automorphisms. A well-known representation of $\mathrm{Aut}(F_n)$ is given by

$$\rho_1 : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/F_n') \cong \mathrm{GL}(n, \mathbb{Z}),$$

where F_n' is the commutator subgroup of F_n. The kernel of ρ_1 is called the classical Torelli group. In [5] Grunewald and Lubotzky construct more representations of finite index subgroups of $\mathrm{Aut}(F_n)$. By choosing a finite group G and a presentation $\pi : F_n \to G$ they obtain an integral linear representation $\rho_{G,\pi} : \Gamma(G, \pi) \to \mathcal{G}_{G,\pi}(\mathbb{Z})$, where $\Gamma(G, \pi)$ is a finite index subgroup of $\mathrm{Aut}(F_n)$.

In this thesis I study the special case $G = C_2$ of this construction. The map $\rho_{C_2,\pi}$ leads to the integral linear representation

$$\sigma_{-1} : \Gamma^+(C_2, \pi) \to \mathrm{GL}(n-1, \mathbb{Z}).$$

Let K_n denote the kernel of σ_{-1}, which fits into the following exact sequence

$$1 \to K_n \to \Gamma^+(C_2, \pi) \to \mathrm{GL}(n-1, \mathbb{Z}) \to 1. \tag{0.1}$$

We call the kernel K_n a generalized Torelli group. The first main theorem of this thesis states that K_n is finitely generated as a group. In the proof we give a set of generators explicitly. Note that this theorem corresponds to the famous theorem of Nielsen and Magnus, which states that the classical Torelli group is finitely generated.

Further we study the abelianized group K_n^{ab}, which becomes by the exaxt sequence (0.1) a $\mathrm{GL}(n-1, \mathbb{Z})$-module. Finally we consider higher quotients of the lower central series

$$K_n = \gamma_0(K_n) \geq \gamma_1(K_n) \geq \gamma_2(K_n) \geq \gamma_3(K_n) \geq \dots.$$

Our second main theorem states the surprising fact that for $i \geq 1$ the quotients $\gamma_i(K_n)/\gamma_{i+1}(K_n)$ are finite abelian groups of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{n,i}}$ with some $b_{n,i} \in \mathbb{N}_0$.

Contents

Introduction

Let F_n be the free group on $n \geq 2$ elements and $\mathrm{Aut}(F_n)$ its group of automorphisms. A theorem of Nielsen says that $\mathrm{Aut}(F_n)$ is a finitely presented group. A well-known representation of $\mathrm{Aut}(F_n)$ is given by

$$\rho_1 : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/F_n') \cong \mathrm{GL}(n, \mathbb{Z}),$$

where F_n' is the commutator subgroup of F_n and $\rho_1(\varphi)$ is the automorphism of the abelian group F_n/F_n' induced by $\varphi \in \mathrm{Aut}(F_n)$. The kernel of ρ_1 is called the *classical Torelli group* and is denoted by $\mathrm{IA}(F_n)$.

A theorem of Nielsen and Magnus ([13], [11]) says that the classical Torelli group is finitely generated. Taking a free basis $x_1, \ldots, x_n$ of F_n they prove:

Theorem: *The group $\mathrm{IA}(F_n)$ is generated by the following automorphisms*

$$K_{ij} : \{x_i \mapsto x_j x_i x_j^{-1}\} \quad \text{and} \quad K_{ijk} : \{x_i \mapsto x_i x_j x_k x_j^{-1} x_k^{-1}\}$$

(*values not given are identical to the argument*).

By the exactness of the sequence

$$1 \to \mathrm{IA}(F_n) \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z}) \to 1$$

the abelianized group $\mathrm{IA}(F_n)^{\mathrm{ab}}$ becomes a $\mathrm{GL}(n, \mathbb{Z})$-module. It is a well-known theorem of Formanek (see [6]) that

$$\mathrm{IA}(F_n)^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n \oplus V_n$$

as a $\mathrm{GL}(n, \mathbb{C})$-module, where V_n is a certain irreducible $\mathrm{GL}(n, \mathbb{C})$-module of dimension $\dim_{\mathbb{C}}(V_n) = n(n+1)(n-2)/2$.

In [5] Grunewald and Lubotzky construct more representations of finite index subgroups of $\mathrm{Aut}(F_n)$. Let G be a finite group and $\pi : F_n \to$

G a surjective homomorphism with kernel R. Define the finite index subgroup $\Gamma(G,\pi)$ of $\mathrm{Aut}(F_n)$ by

$$\Gamma(G,\pi) := \{\varphi \in \mathrm{Aut}(F_n) \mid \varphi(R) = R,\ \varphi \text{ induces the identity on } F_n/R\}.$$

Define further $\bar{R} := R/R' = R^{\mathrm{ab}}$ to be the abelianization of R. Let t denote the $\mathbb{Z}$-rank of this finitely generated free abelian group. The group G acts on $\bar{R}$ by conjugation. Every automorphism $\varphi \in \Gamma(G,\pi)$ induces a linear automorphism $\bar{\varphi}$ of $\bar{R}$ which is G-equivariant. Let

$$\mathcal{G}_{G,\pi} := \mathrm{Aut}_G(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}) \leq \mathrm{GL}(t,\mathbb{C}).$$

The group $\mathcal{G}_{G,\pi}$ is the centralizer of the group G acting on $\mathbb{C}\otimes_{\mathbb{Z}}\bar{R}$ through matrices with rational entries. Define

$$\mathcal{G}_{G,\pi}(\mathbb{Z}) := \{\Phi \in \mathcal{G}_{G,\pi} \mid \Phi(\bar{R}) = \bar{R}\}.$$

Choosing a $\mathbb{Z}$-basis of $\bar{R}$, we obtain an integral linear representation

$$\begin{array}{rcl} \rho_{G,\pi} : \Gamma(G,\pi) & \rightarrow & \mathcal{G}_{G,\pi}(\mathbb{Z}) \\ \varphi & \mapsto & \bar{\varphi}. \end{array}$$

In the special case $G = \{1\}$ this construction leads to the classical representation $\rho_1 : \mathrm{Aut}(F_n) \rightarrow \mathrm{GL}(n,\mathbb{Z})$. Thus the kernel of $\rho_{G,\pi}$ can be considered as a natural generalization of $\mathrm{IA}(F_n)$. Therefore it is called a *generalized Torelli group.*

In my work I study another special case of the construction by Grunewald and Lubotzky. Let F_n $(n \geq 2)$ be the free group generated by $x, y_1, \ldots, y_{n-1}$ and C_2 the cyclic group of order two generated by g. Moreover let $\pi : F_n \rightarrow C_2$ be the surjective homomorphism defined by

$$\pi(x) := g,\ \pi(y_1) := 1,\ \ldots,\ \pi(y_{n-1}) := 1.$$

The kernel R of this map is, by the formula of Reidemeister and Schreier, a free group of rank $2n-1$, which means that $t = 2n-1$. By the construction above we obtain a homomorphism

$$\rho_{C_2,\pi} : \Gamma(C_2,\pi) \rightarrow \mathrm{GL}(\bar{R}) \cong \mathrm{GL}(2n-1,\mathbb{Z}).$$

We set

$$\Gamma^+(C_2,\pi) := \{\varphi \in \Gamma(C_2,\pi) \mid \det(\rho_1(\varphi)) = 1\}.$$

This is a subgroup of index two in $\Gamma(C_2, \pi)$. An important feature is that we are able to present a finite set of generators of $\Gamma^+(C_2, \pi)$ (see Chapter 4.2). The restriction of $\rho_{C_2,\pi}$ leads to the representation

$$\rho_{C_2,\pi} : \Gamma^+(C_2, \pi) \to \mathrm{GL}(\bar{R}) \cong \mathrm{GL}(2n-1, \mathbb{Z}).$$

The $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$ decomposes as $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} = V_1 \oplus V_{-1}$, where V_1, V_{-1} are the ± 1 eigenspaces of g, respectively. Set $\bar{R}_1 := \bar{R} \cap V_1$ and $\bar{R}_{-1} := \bar{R} \cap V_{-1}$. It turns out that the $\mathbb{Z}$-rank of $\bar{R}_1$ equals n and the $\mathbb{Z}$-rank of $\bar{R}_{-1}$ equals $n-1$. Since $\Gamma^+(C_2, \pi)$ leaves $\bar{R}_1$ and $\bar{R}_{-1}$ invariant, we obtain representations

$$\sigma_1 : \Gamma^+(C_2, \pi) \to \mathrm{GL}(n, \mathbb{Z}), \quad \sigma_{-1} : \Gamma^+(C_2, \pi) \to \mathrm{GL}(n-1, \mathbb{Z}).$$

The map σ_1 is equivalent to ρ_1 restricted to $\Gamma^+(C_2, \pi)$. In contrast the representation σ_{-1} is somewhat less expected and is studied in this work. In Chapter 4.2 it is shown, that the map σ_{-1} is surjective by analysing the images of the generators of $\Gamma^+(C_2, \pi)$. Let K_n denote the kernel of σ_{-1}, which fits into the following exact sequence

$$1 \to K_n \to \Gamma^+(C_2, \pi) \to \mathrm{GL}(n-1, \mathbb{Z}) \to 1.$$

By the exactness of this sequence, the index of K_n in $\Gamma^+(C_2, \pi)$ is infinite for $n \geq 3$ and two for $n = 2$. The first main theorem of this thesis states that K_n is finitely generated as a group. The proof, in which the generators are given explicitly, is provided in Chapter 4.3. As a corollary we obtain the following theorem.

Theorem: *Let $n \geq 2$. The group K_n is generated by the following automorphisms:*

$$\varepsilon_i : \{x \mapsto x y_i\}, \qquad \psi_i^2 : \{y_i \mapsto y_i x^2\},$$

$$\alpha_i : \left\{ \begin{array}{ccc} x & \mapsto & x^{-1} \\ y_i & \mapsto & x y_i^{-1} x^{-1} \end{array} \right\}$$

for $1 \leq i \leq n-1$ (values not given are identical to the argument). In particular K_n is finitely generated as a group.

Note that this theorem corresponds to the theorem of Nielsen and Magnus. The idea of the proof is the following. Starting with a finite presentation of $\mathrm{GL}(n-1,\mathbb{Z})$ and the generator set of $\Gamma^+(C_2,\pi)$ we are able to construct a finite number of elements in K_n whose normal closure coincides with K_n. Then we show that the group generated by these elements is already a normal subgroup of $\Gamma^+(C_2,\pi)$. This means that K_n is finitely generated as a group.

As above K_n^{ab} becomes a $\mathrm{GL}(n-1,\mathbb{Z})$-module. In Chapter 6 we study the structure of this module.

Proposition: *Let $n \geq 2$. Then the group K_n^{ab} is generated by $[\varepsilon_i]$, $[\alpha_i]$ and $[\psi_i^2]$ for $i = 1, \ldots, n-1$. The order of $[\alpha_i]$ is either one or two. For $n \geq 3$ the order of $[\psi_i^2]$ is also either one or two.*

In Chapter 6.2 we construct for $n \geq 3$ a surjective $\mathrm{GL}(n-1,\mathbb{Z})$-equivariant homomorphism

$$\Phi_n : V_{n-1} \oplus M_{n-1} \twoheadrightarrow K_n^{\mathrm{ab}},$$

where $V_{n-1} \oplus M_{n-1}$ is a certain $\mathrm{GL}(n-1,\mathbb{Z})$-module with underlying abelian group $(\mathbb{Z}^{n-1} \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}) \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}$. For the precise structure of $V_{n-1} \oplus M_{n-1}$ see Chapter 6.1. It is difficult to compute the kernel of Φ_n, but I conjecture that the kernel is trivial:

Conjecture: *Let $n \geq 3$. Then the $\mathrm{GL}(n-1,\mathbb{Z})$-equivariant epimorphism*

$$\Phi_n : V_{n-1} \oplus M_{n-1} \twoheadrightarrow K_n^{\mathrm{ab}}$$

is an isomorphism.

Chapter 6.4 is concerned with higher quotients of the lower central series

$$K_n = \gamma_0(K_n) \geq \gamma_1(K_n) \geq \gamma_2(K_n) \geq \gamma_3(K_n) \geq \ldots .$$

The second main theorem states the surprising fact that the quotients $\gamma_i(K_n)/\gamma_{i+1}(K_n)$ are finite for $i \geq 1$.

Theorem: *Let $n \geq 2$ and $i \geq 1$. Then the group $\gamma_i(K_n)/\gamma_{i+1}(K_n)$*

is a finite abelian group of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{n,i}}$ *with*

$$0 \leq b_{n,i} \leq (3n-3)^{i-1} \cdot \left(3n^2 - 7n + 4\right).$$

In the special case $n = 2$ we give a finite presentation of K_2 and obtain that the group K_2^{ab} is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. Furthermore, in this case it is possible to compute the exponents $b_{2,i}$. Here is the result for $b_{2,1}, \ldots, b_{2,9}$:

$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$b_{2,5}$	$b_{2,6}$	$b_{2,7}$	$b_{2,8}$	$b_{2,9}$	$\ldots$
2	4	6	10	14	22	32	48	70	

More information on the $b_{2,i}$ is contained in Chapter 6.3.

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Notation

In this thesis we generally apply functions on the *left*, i.e. the image of x under a function φ is written as $\varphi(x)$. If $\varphi : X \to Y$ and $\psi : Y \to Z$ are two functions, we write $\psi \circ \varphi : X \to Z$ for the product of φ and ψ, i.e. $(\psi \circ \varphi)(x) = \psi(\varphi(x))$.

$G, H, \dots$	Groups
$\alpha, \beta, \gamma, \dots$	Homomorphisms
$x, y, z, \dots$	Elements of a group
$[x, y]$	$xyx^{-1}y^{-1}$
$H \cong G$	H is isomorphic to G
$H \leq G$, $H < G$	H is a subgroup, a proper subgroup of G
$\langle G_i \mid i \in I \rangle$	subgroup generated by subsets G_i of a group
$\langle X \mid R \rangle$	Group presented by generators X and relators R
F_n	Free group on n generators
$\mathrm{Aut}(G)$	Automorphism group of a group G
$\mathrm{Hom}_\Omega(G, H)$	Set of Ω-homomorphisms from G to H
$G' = \gamma_1(G) = [G, G]$	Derived subgroup of a group G
G^{ab}	G/G'
$\gamma_i(G)$	i-th term of the lower central series of G
$\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Sets of natural numbers, natural numbers with 0, integers, rational numbers, real numbers, complex numbers
C_n	$\mathbb{Z}/n\mathbb{Z}$
$\mathrm{GL}(V)$	Group of nonsingular linear transformations of a vector space V
$\mathrm{GL}(n, \mathbb{Z}), \mathrm{SL}(n, \mathbb{Z})$	General linear and special linear groups
I_n	$(n \times n)$-identity matrix

Let us introduce some elementary matrices in $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{GL}(n,\mathbb{Z})$ for $n \geq 2$ and $1 \leq i, j \leq n$ (our convention is that entries not given are identical to zero):

- Let $E_{ij} \in \mathrm{SL}(n,\mathbb{Z})$ be the $(n \times n)$-identity matrix with an additional 1 in the (i,j)-th position $(i \neq j)$

$$E_{ij} := \left(\begin{array}{ccccccccccc}
1 & & & | & & & & | & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & | & & & & | & & & \\
- & - & - & 1 & - & - & - & 1 & - & - & - \\
 & & & | & 1 & & & | & & & \\
 & & & & & \ddots & & & & & \\
 & & & | & & & 1 & | & & & \\
- & - & - & 0 & - & - & - & 1 & - & - & - \\
 & & & | & & & & | & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & | & & & & | & & & 1
\end{array}\right)
\begin{array}{l}
\\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \\
\end{array}$$

- Let $E_{ij}(a) \in \mathrm{SL}(n,\mathbb{Z})$ be the $(n \times n)$-identity matrix with an additional $a \in \mathbb{Z}$ in the (i,j)-th position $(i \neq j)$

$$E_{ij}(a) := \left(\begin{array}{ccccccccccc}
1 & & & | & & & & | & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & | & & & & | & & & \\
- & - & - & 1 & - & - & - & a & - & - & - \\
 & & & | & 1 & & & | & & & \\
 & & & & & \ddots & & & & & \\
 & & & | & & & 1 & | & & & \\
- & - & - & 0 & - & - & - & 1 & - & - & - \\
 & & & | & & & & | & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & | & & & & | & & & 1
\end{array}\right)
\begin{array}{l}
\\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \\
\end{array}$$

- Let $P_{ij} \in \mathrm{SL}(n,\mathbb{Z})$ be the following permutation matrix $(i \neq j)$

$$P_{ij} := \begin{array}{cl} \begin{pmatrix} 1 & & & | & & & & | & & & \\ & \ddots & & & & & & & & & \\ & & 1 & | & & & & | & & & \\ - & - & - & 0 & - & - & - & 1 & - & - & - \\ & & & | & 1 & & & | & & & \\ & & & & & \ddots & & & & & \\ & & & | & & & 1 & | & & & \\ - & - & - & 1 & - & - & - & 0 & - & - & - \\ & & & | & & & & | & 1 & & \\ & & & & & & & & & \ddots & \\ & & & | & & & & | & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \\ \end{array} \end{array}$$

- Let $O_i \in \mathrm{GL}(n,\mathbb{Z})$ be the following diagonal matrix with a -1 in the (i,i)-th position

$$O_i := \begin{array}{cl} \begin{pmatrix} 1 & & & | & & & \\ & \ddots & & & & & \\ & & 1 & | & & & \\ - & - & - & (-1) & - & - & - \\ & & & | & 1 & & \\ & & & & & \ddots & \\ & & & | & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \end{array} \end{array}$$

- Let $O_{1i} \in \mathrm{SL}(n,\mathbb{Z})$ be the following diagonal matrix with a -1 in the $(1,1)$-th and in the (i,i)-th position $(i \neq 1)$

$$O_{1i} := \begin{array}{cl} \begin{pmatrix} -1 & & & & | & & & \\ & 1 & & & & & & \\ & & \ddots & & | & & & \\ & & & 1 & & & & \\ - & - & - & - & (-1) & - & - & - \\ & & & & | & 1 & & \\ & & & & & & \ddots & \\ & & & & | & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \end{array} \end{array}$$

Chapter 1

Presentation of Groups

In this thesis, we often work with finite presentations of groups. The aim of this chapter is to give an introduction to this subject. In Section 1.1 the definition and notations of presentations are given. Section 1.2 is devoted to finite presentations of $\mathrm{SL}(n, \mathbb{Z})$ and $\mathrm{GL}(n, \mathbb{Z})$. These fundamental presentations are used consistently in the whole thesis. The last Section 1.3 is concerned with the following problem. Given a surjective homomorphism $\varphi : G \twoheadrightarrow H$ of a finitely generated group G onto a finitely presented group H. What can we say about the kernel of φ, i.e. is the kernel finitely generated? If the answer is positive, we give a method to find a set of generators. These ideas will be very useful in the proof of the main theorem in Chapter 4.

1.1 Basic concepts

A well-known theorem in the theory of free groups states that every group G is a homomorphic image of some free group. This means that for every group G, there exists a surjective homomorphism $\pi : F \twoheadrightarrow G$ of a free group F onto G. This homomorphism π is called a *presentation of the group G*.

Let $R := \ker(\pi)$ be the kernel of π. Then R is a normal subgroup of F and $F/R \cong G$. The elements in R are called the *relators* of the presentation.

Now choose a set of free generators for F, say Y, and a subset $S \subseteq R$ with the property that the normal closure of S equals R. Then the image $X := \pi(Y)$ is clearly a set of generators of the group G.

The presentation π in combination with the choice of Y and S, determines a *set of generators and defining relators* for G. We denote this in the following way

$$G = \langle Y \mid S \rangle. \tag{1.1}$$

In practice it is often more convenient to list the generators of G and the *defining relations* $s(X) = 1$ for $s \in S$:

$$G = \langle X \mid s(X) = 1, s \in S \rangle. \tag{1.2}$$

Here $s(X)$ denotes the word obtained from s by replacing formally the generators Y by the generators X of the group G. We refer to (1.1) or (1.2) as a presentation of the group G.

A group G is said to be *finitely generated* if there is a presentation $G = \langle Y \mid S \rangle$ such that the set Y is finite. Moreover, it is said to be *finitely presented* if there exists a presentation such that both Y and S are finite sets. This definition is independent of the particular presentation chosen in the sense of the following proposition.

Proposition 1.1 (B. H. Neumann)**.** *Let G be a finitely presented group and let $g_1, \dots, g_n$ be generators of G. Then there are defining relations $r_1 = 1, \dots, r_m = 1$ such that*

$$G = \langle g_1, \dots, g_n \mid r_1 = 1, \dots, r_m = 1 \rangle.$$

Proof. See for example [15]. □

Examples of finitely presented groups, which are interesting for us, are

- cyclic groups of finite order m:

$$C_m = \langle g \mid g^m = 1 \rangle,$$

- free groups of finite rank n:

$$F_n = \langle g_1, \dots, g_n \mid \text{no relations } \rangle = \langle g_1, \dots, g_n \rangle,$$

- all finite groups,
- the special linear group $\mathrm{SL}(n, \mathbb{Z})$ with $n \in \mathbb{N}$ and
- the general linear group $\mathrm{GL}(n, \mathbb{Z})$ with $n \in \mathbb{N}$.

1.2 Presentations of $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{GL}(n,\mathbb{Z})$

Let $E_{ij} \in \mathrm{SL}(n,\mathbb{Z})$ be the $(n \times n)$-identity matrix with an additional 1 in the (i,j)-th position $(i \neq j)$ and $O_i := \mathrm{diag}(1,\ldots,1,-1,1,\ldots,1)$ be the diagonal matrix with a -1 at the (i,i)-th position (see Notation).

The aim of this section is to give finite presentations of $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{GL}(n,\mathbb{Z})$ with the matrices E_{ij} and O_i as generators. Such a presentation of $\mathrm{SL}(n,\mathbb{Z})$ can be found in the literature (see [12]) and is listed below. However, a finite presentation of this kind of $\mathrm{GL}(n,\mathbb{Z})$ seems not to be published.

Proposition 1.2 (Presentation of $\mathrm{SL}(n,\mathbb{Z})$)**.**

(a) $\mathrm{SL}(1,\mathbb{Z}) = 1$.

(b) $\mathrm{SL}(2,\mathbb{Z})$ *has a finite presentation with the two generators* E_{12} *and* E_{21} *subject to the following relations*

$$\begin{aligned}&E_{12}E_{21}^{-1}E_{12}E_{21}E_{12}^{-1}E_{21} = 1,\\&(E_{12}E_{21}^{-1}E_{12})^4 = 1.\end{aligned}$$

(c) *For* $n \geq 3$*, the group* $\mathrm{SL}(n,\mathbb{Z})$ *has a finite presentation with* $n(n-1)$ *generators* E_{ij} $(i \neq j)$ *subject to the following relations*

$$\begin{aligned}&[E_{ij},E_{kl}] = 1 \text{ if } j \neq k,\ i \neq l,\\&[E_{ij},E_{jk}] = E_{ik} \text{ if } i,j,k \text{ are pairwise distinct,}\\&(E_{12}E_{21}^{-1}E_{12})^4 = 1.\end{aligned}$$

Proof. (a) is clear and (c) can be found in [12]. Let us now prove (b). We know from [16] Chapter 1.5, that

$$\mathrm{SL}(2,\mathbb{Z}) = \langle X, Y \mid X^4 = 1, X^2 = Y^3\rangle,$$

where $X = \left(\begin{smallmatrix}0 & 1\\-1 & 0\end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix}0 & 1\\-1 & 1\end{smallmatrix}\right)$. We now apply Tietze transformations (a good reference for the notion of Tietze transformations is [16]). First notice that

$$Y^{-1}X = \begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = E_{12},$$

$$YX^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = E_{21}.$$

Thus, we see that

$$\begin{aligned} & \langle X, Y \mid X^4 = 1, X^2 = Y^3 \rangle \\ = \; & \langle X, Y, E_{12}, E_{21} \mid X^4 = 1,\ X^2 = Y^3,\ E_{12} = Y^{-1}X,\ E_{21} = YX^{-1} \rangle \\ = \; & \langle X, Y, E_{12}, E_{21} \mid X = YE_{12},\ X^4 = 1,\ Y^{-1}X^2 = Y^2,\ E_{21} = YX^{-1} \rangle \\ = \; & \langle Y, E_{12}, E_{21} | (YE_{12})^4 = 1, E_{12}YE_{12} = Y^2, E_{21} = YE_{12}^{-1}Y^{-1}E_{12}^{-1}E_{12} \rangle \\ = \; & \langle Y, E_{12}, E_{21} \mid (YE_{12})^4 = 1,\ E_{12}YE_{12} = Y^2,\ E_{21} = YY^{-2}E_{12} \rangle \\ = \; & \langle Y, E_{12}, E_{21} \mid (YE_{12})^4 = 1,\ E_{12}YE_{12} = Y^2,\ Y = E_{12}E_{21}^{-1} \rangle \\ = \; & \langle E_{12}, E_{21} \mid (E_{12}E_{21}^{-1}E_{12})^4 = 1,\ E_{12}E_{21}^{-1}E_{12}E_{21}E_{12}^{-1}E_{21} = 1 \rangle. \end{aligned}$$

□

Our next aim is to find a finite presentation of $\mathrm{GL}(n, \mathbb{Z})$ in terms of the matrices E_{ij} and O_i. To do this, we use the following exact sequence

$$1 \to \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}) \overset{\det}{\to} \{-1, 1\} \to 1.$$

More general, let G be an extension of H by N, say

$$1 \to N \overset{i}{\to} G \overset{\pi}{\to} H \to 1.$$

Assume further that N has the following finite presentation

$$N = \langle n_1, \ldots, n_r \mid R_1(n_1, \ldots, n_r), \ldots, R_k(n_1, \ldots, n_r) \rangle$$

and that H has the finite presentation

$$H = \langle h_1, \ldots, h_s \mid W_1(h_1, \ldots, h_s), \ldots, W_l(h_1, \ldots, h_s) \rangle.$$

We wish to find a finite presentation of G.

Since π is surjective, there are $g_1, \ldots, g_s \in G$ with $\pi(g_i) = h_i$ for $1 \leq i \leq s$. By identifying N with the kernel of π in G, it is easy to see that G is generated by $g_1, \ldots, g_s$ and $n_1, \ldots, n_r$. Thus, we have found generators for G. We start collecting relations in terms of $g_1, \ldots, g_s$ and $n_1, \ldots, n_r$:

- The relations $R_1(n_1, \ldots, n_r), \ldots, R_k(n_1, \ldots, n_r)$ in N are, of course, also relations in G.

- Let $W_i(g_1, \ldots, g_s)$ be the word obtained from $W_i(h_1, \ldots, h_s)$ by replacing each h_j by g_j. We have
$$\pi(W_i(g_1, \ldots, g_s)) = W_i(\pi(g_1), \ldots, \pi(g_s)) = W_i(h_1, \ldots, h_s).$$
Hence $W_i(g_1, \ldots, g_s) \in \ker(\pi)$, i.e. $W_i(g_1, \ldots, g_s) \in N$. This means that we can write $W_i(g_1, \ldots, g_s)$ as a product of the n_i, say
$$W_i(g_1, \ldots, g_s) = \widetilde{W}_i(n_1, \ldots, n_r).$$
This gives us more relations in G.

- Finally, we mention that, since N is a normal subgroup in G, each conjugate $g_i n_j g_i^{-1}$ and $g_i^{-1} n_j g_i$ is in N. Thus, we get relations
$$g_i n_j g_i^{-1} = V_{ij}(n_1, \ldots, n_r) \qquad g_i^{-1} n_j g_i = U_{ij}(n_1, \ldots, n_r).$$

The next proposition tells us that the above relations are sufficient for a presentation of G.

Proposition 1.3 (P. Hall)**.** *Let G be an extension of H by N*
$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} H \to 1.$$
If N has the finite presentation
$$N = \langle n_1, \ldots, n_r \mid R_1(n_1, \ldots, n_r), \ldots, R_k(n_1, \ldots, n_r)\rangle$$
and H has the finite presentation
$$H = \langle h_1, \ldots, h_s \mid W_1(h_1, \ldots, h_s), \ldots, W_l(h_1, \ldots, h_s)\rangle,$$
then G has the following finite presentation
$$\begin{aligned} G = \langle n_1, \ldots, n_r, g_1, \ldots, g_s \quad \mid \quad & R_1(n_1, \ldots, n_r), \ldots, R_k(n_1, \ldots, n_r), \\ & g_i n_j g_i^{-1} = V_{ij}(n_1, \ldots, n_r), \\ & g_i^{-1} n_j g_i = U_{ij}(n_1, \ldots, n_r), \\ & W_i(g_1, \ldots, g_s) = \widetilde{W}_i(n_1, \ldots, n_r)\rangle, \end{aligned}$$
where g_i, $V_{ij}(n_1, \ldots, n_r)$, $U_{ij}(n_1, \ldots, n_r)$ and $\widetilde{W}_i(n_1, \ldots, n_r)$ are as above.

Proof. For a proof see [10] Chapter 13. But be careful, in the theorem and in the proof the relations $g_i^{-1} n_j g_i = U_{ij}(n_1, \dots, n_r)$ are missing. $\square$

Now we can use Proposition 1.3 and the presentation of $\mathrm{SL}(n, \mathbb{Z})$ given in Proposition 1.2 to compute a finite presentation of $\mathrm{GL}(n, \mathbb{Z})$. In order to do this define $O_1 := \mathrm{diag}(-1, 1, \dots, 1)$ to be the diagonal matrix with an entry -1 at the first position (see Notation).

Proposition 1.4 (First Presentation of $\mathrm{GL}(n, \mathbb{Z})$)**.**

(a) $\mathrm{GL}(1, \mathbb{Z}) = \langle O_1 \mid O_1^2 = 1 \rangle = C_2$.

(b) $\mathrm{GL}(2, \mathbb{Z})$ *has a finite presentation with the three generators* E_{12}, E_{21} *and* O_1 *subject to the following relations*

$$\begin{aligned}
&E_{12} E_{21}^{-1} E_{12} E_{21} E_{12}^{-1} E_{21} = 1, \\
&(E_{12} E_{21}^{-1} E_{12})^4 = 1, \\
&(O_1 E_{12})^2 = 1, \\
&(O_1 E_{21})^2 = 1, \\
&O_1^2 = 1.
\end{aligned}$$

(c) *For* $n \geq 3$, *the group* $\mathrm{GL}(n, \mathbb{Z})$ *has a finite presentation with* $n(n-1)+1$ *generators* E_{ij} *and* O_1 *subject to the following relations*

$$\begin{aligned}
&[E_{ij}, E_{kl}] = 1 \ \text{if } j \neq k,\ i \neq l, \\
&[E_{ij}, E_{jk}] = E_{ik} \ \text{if } i, j, k \text{ are pairwise distinct}, \\
&(E_{12} E_{21}^{-1} E_{12})^4 = 1, \\
&O_1 E_{ij} O_1 E_{ij}^{-1} = 1 \ \text{if } i, j \neq 1, \\
&(O_1 E_{1j})^2 = 1 \ \text{if } j \neq 1, \\
&(O_1 E_{i1})^2 = 1 \ \text{if } i \neq 1, \\
&O_1^2 = 1.
\end{aligned}$$

Proof. (a) is clear.
(b) We have

$$1 \to \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(2, \mathbb{Z}) \xrightarrow{\det} \{-1, 1\} \to 1,$$

where $\{-1, 1\} \cong \langle g \mid g^2 = 1 \rangle$. We now apply Proposition 1.3. Since $\det(O_1) = -1$, we conclude that $\mathrm{GL}(2, \mathbb{Z})$ is generated by E_{12}, E_{21} and O_1. According to Proposition 1.3 the defining relations corresponding to these generators are:

1. The relations of $\mathrm{SL}(2,\mathbb{Z})$ (see Proposition 1.2).

2. $O_1^2 = 1$.

3. The conjugates of the generators E_{12} and E_{21} by O_1 and O_1^{-1} expressed in terms of E_{12} and E_{21}. Since $O_1 = O_1^{-1}$ it suffices to consider $O_1E_{12}O_1$ and $O_1E_{21}O_1$ and express both matrices in terms of E_{ij}:

$$\begin{aligned} O_1E_{12}O_1 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = E_{12}^{-1}, \end{aligned}$$

$$\begin{aligned} O_1E_{21}O_1 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = E_{21}^{-1}. \end{aligned}$$

(c) We have

$$1 \longrightarrow \mathrm{SL}(n,\mathbb{Z}) \longrightarrow \mathrm{GL}(n,\mathbb{Z}) \stackrel{\det}{\longrightarrow} \{-1,1\} \longrightarrow 1,$$

where $\{-1,1\} \cong \langle g \mid g^2 = 1\rangle$. We apply Proposition 1.3 again. By the same argument as above $\mathrm{GL}(n,\mathbb{Z})$ is generated by O_1 and E_{ij} $(i \neq j)$. The defining relations corresponding to these generators are:

1. The relations of $\mathrm{SL}(n,\mathbb{Z})$ (see Proposition 1.2).

2. $O_1^2 = 1$.

3. The conjugates of the generators E_{ij} $(i \neq j)$ by O_1 and O_1^{-1} expressed in terms of E_{ij}. As before, it suffices to consider $O_1E_{ij}O_1$. We show that $O_1E_{ij}O_1 = E_{ij}$ if $i,j \neq 1$:

   For this let us write E_{ij} and O_1 in the following form

$$E_{ij} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right), \quad O_1 = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I_{n-1} \end{array}\right),$$

where A is a $(n-1)\times(n-1)$-matrix and I_{n-1} the $(n-1)\times(n-1)$-identity matrix. Thus, we have

$$\begin{aligned}
O_1 E_{ij} O_1 &= \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I_{n-1} \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right)\left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I_{n-1} \end{array}\right) \\
&= \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & A \end{array}\right)\left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I_{n-1} \end{array}\right) \\
&= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right) = E_{ij}.
\end{aligned}$$

Finally, a short calculation (similar to the one in part (b)) yields

$$\begin{aligned}
O_1 E_{1j} O_1 &= E_{1j}^{-1} \text{ if } j \neq 1 \text{ and} \\
O_1 E_{i1} O_1 &= E_{i1}^{-1} \text{ if } i \neq 1.
\end{aligned}$$

□

In Chapter 4 we will need a special presentation of $\mathrm{GL}(n,\mathbb{Z})$. We obtain this presentation from Proposition 1.4 by applying Tietze transformations. For this let $O_i := \mathrm{diag}(1,\dots,1,-1,1,\dots,1)$ be the diagonal matrix with a -1 at the (i,i)-th position (see Notation). We add these elements to the set of generators from Proposition 1.4 and get the following presentation.

Proposition 1.5 (Second Presentation of $\mathrm{GL}(n,\mathbb{Z})$)**.**

(a) $\mathrm{GL}(1,\mathbb{Z}) = \langle O_1 \mid O_1^2 = 1\rangle = C_2$.

(b) $\mathrm{GL}(2,\mathbb{Z})$ *has a finite presentation with four generators* E_{12}, E_{21}, O_1 *and* O_2 *subject to the following relations*

$$\begin{aligned}
&1.)\ E_{12}E_{21}^{-1}E_{12}E_{21}E_{12}^{-1}E_{21} = 1, \\
&2.)\ (E_{12}E_{21}^{-1}E_{12})^4 = 1, \\
&3.)\ (O_1E_{12})^2 = 1, \\
&4.)\ (O_1E_{21})^2 = 1, \\
&5.)\ O_1^2 = 1, \\
&6.)\ E_{12}^{-1}E_{21}^{2}O_1E_{12}E_{21}^{-2}O_2^{-1} = 1.
\end{aligned}$$

(c) *For $n \geq 3$ the group $\mathrm{GL}(n, \mathbb{Z})$ has a finite presentation with $n(n-1)+n$ generators E_{ij} and O_i subject to the following relations*

1.) $[E_{ij}, E_{kl}] = 1$ *if* $j \neq k$, $i \neq l$,
2.) $[E_{ij}, E_{jk}]E_{ik}^{-1} = 1$ *if* i, j, k *are pairwise distinct,*
3.) $(E_{12}E_{21}^{-1}E_{12})^4 = 1$,
4.) $(O_1E_{1j})^2 = 1$ *if* $j \neq 1$,
5.) $(O_1E_{i1})^2 = 1$ *if* $i \neq 1$,
6.) $O_1E_{ij}O_1E_{ij}^{-1} = 1$ *if* $i, j \neq 1$,
7.) $O_1^2 = 1$,
8.) $E_{1j}^{-1}E_{j1}^2O_1E_{1j}E_{j1}^{-2}O_j^{-1} = 1$ *if* $j \neq 1$.

Proof. Part (a) is clear.
For (b) it suffices to express the new generator O_2 in terms of the other generators. Thus, we just have to show that $O_2 = E_{12}^{-1}E_{21}^2O_1E_{12}E_{21}^{-2}$:

$$\begin{aligned}
&\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
= &\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
= &\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

For (c) we have to show that $E_{1j}^{-1}E_{j1}^2O_1E_{1j}E_{j1}^{-2}O_j^{-1} = 1$ if $j \neq 1$. The proof is analogous to the case $n = 2$. □

1.3 Some facts about finitely presented groups

Let G and H be two finitely presented groups and let $\varphi : G \twoheadrightarrow H$ be a surjective homomorphism. We are now interested in the kernel of φ. If H is a finite group, then the index of $\ker(\varphi)$ in G is finite. By the following proposition, we conclude that in this case $\ker(\varphi)$ is also finitely presented.

Proposition 1.6. *Let G be a finitely presented group and K a subgroup of finite index in G. Then K is also finitely presented.*

Proof. See [16] Chapter 2.3. $\square$

In the case that the index of a subgroup K in a group G is infinite, it is not easy to decide whether or not K is finitely presented. Actually, it is already difficult to decide whether K is finitely generated. However, in the rest of this chapter we supply some results, which we are going to use later to prove that the Torelli groups are finitely generated.

Let $G = \langle g_1, \dots, g_n \rangle$ be a finitely generated group, H a finitely presented group and $\varphi : G \twoheadrightarrow H$ a surjective homomorphism of G onto H. Define $h_i := \varphi(g_i)$ for $1 \leq i \leq n$. The set $\{h_i \mid 1 \leq i \leq n\}$ is then a set of generators of H. Since H is finitely presented, it follows by Proposition 1.1 that there exist defining relations $R_1, \dots, R_m$ such that

$$H = \langle h_1, \dots, h_n \mid R_1(h_1, \dots, h_n), \dots, R_m(h_1, \dots, h_n) \rangle.$$

Notice, that the following diagram commutes

$$\begin{tikzcd} & G \arrow[dd, two heads, "\varphi"] \\ F_n \arrow[ur, two heads, "g"] \arrow[dr, two heads, "h"'] & \\ & H \end{tikzcd}$$

where $F_n = \langle x_1, \dots, x_n \rangle$ is the free group on n generators and $g : F_n \to G$ and $h : F_n \to H$ are the homomorphisms which send x_i to g_i and x_i to h_i, respectively.

Proposition 1.7. *Let G, H and $\varphi : G \twoheadrightarrow H$ be as above. Then we have*

$$\ker(\varphi) = \mathrm{NC}\left(R_1(g_1, \dots, g_n), \dots, R_m(g_1, \dots, g_n)\right),$$

where $\mathrm{NC}(R_1(g_1, \dots, g_n), \dots, R_m(g_1, \dots, g_n))$ *denotes the normal closure of* $R_1(g_1, \dots, g_n), \dots, R_m(g_1, \dots, g_n)$ *in G.*

Proof. Since

$$\varphi(R_j(g_1, \ldots, g_n)) = R_j(\varphi(g_1), \ldots, \varphi(g_n)) = R_j(h_1, \ldots, h_n)$$

for $1 \leq j \leq m$, it is clear, that $\mathrm{NC}\left(R_1(g_1, \ldots, g_n), \ldots, R_m(g_1, \ldots, g_n)\right) \subseteq \ker(\varphi)$. We now show the inverse inclusion

$$\ker(\varphi) \subseteq \mathrm{NC}\left(R_1(g_1, \ldots, g_n), \ldots, R_m(g_1, \ldots, g_n)\right).$$

Let $g_0 \in \ker(\varphi)$. Remember that the following diagram commutes.

$$\begin{tikzcd} & G \arrow[dd, two heads, "\varphi"] \\ F_n \arrow[ur, two heads, "g"] \arrow[dr, "h"'] & \\ & H \end{tikzcd}$$

Since $g : F_n \twoheadrightarrow G$ is onto, we find a word $w(x_1, \ldots, x_n) \in F_n$ such that $g(w(x_1, \ldots, x_n)) = g_0$. It follows that

$$1 = \varphi(g_0) = \varphi(g(w(x_1, \ldots, x_n))) = h(w(x_1, \ldots, x_n))$$

and we conclude that $w(x_1, \ldots, x_n) \in \ker(h)$. Hence

$$w(x_1, \ldots, x_n) \in \mathrm{NC}(R_1(x_1, \ldots, x_n), \ldots, R_m(x_1, \ldots, x_n)).$$

This means that

$$g_0 = g(w(x_1, \ldots, x_n)) \in \mathrm{NC}(R_1(g_1, \ldots, g_n), \ldots, R_m(g_1, \ldots, g_n)).$$

□

Corollary 1.8. *Let G be finitely generated abelian group. Further let H and $\pi : G \twoheadrightarrow H$ be as above. Then we have*

$$\ker(\varphi) = \langle R_1(g_1, \ldots, g_n), \ldots, R_m(g_1, \ldots, g_n)\rangle.$$

In particular $\ker(\varphi)$ is finitely generated as a group.

Proof. This is clear, since in an abelian group the normal closure of a subgroup is just the same subgroup. □

Proposition 1.9. *Let G, H and $\varphi : G \twoheadrightarrow H$ be as above and let $K \leq G$ be a subgroup of G with the following properties:*

1.) *K is a normal subgroup of G,*

2.) *$R_j(g_1, \ldots, g_n) \in K$ for $1 \leq j \leq m$ and*

3.) *$K \subseteq \ker(\varphi)$.*

Then we have $K = \ker(\varphi)$.

Proof. By 3.) it is sufficient to show that $\ker(\varphi) \subseteq K$:

$$\ker(\varphi) \overset{\text{Prop. 1.7}}{=} \mathrm{NC}\left(R_1(g_1, \ldots, g_n), \ldots, R_m(g_1, \ldots, g_n)\right) \overset{1.)\,+\,2.)}{\subseteq} K.$$

□

The most difficult part when applying Proposition 1.9 might be to verify, whether or not K is a normal subgroup of G. But if K is finitely generated as a group, this can be done with the help of the following lemma.

Lemma 1.10. *Let $G = \langle g_1, \ldots, g_s \rangle$ be a finitely generated group and let $K = \langle u_1, \ldots, u_t \rangle$ be a finitely generated subgroup of G. Then K is a normal subgroup of G if and only if*

$$g_i u_j g_i^{-1} \in K \quad \text{and} \quad g_i^{-1} u_j g_i \in K$$

for $1 \leq i \leq s$ and $1 \leq j \leq t$.

Proof. If K is a normal subgroup of G then we obviously have $g_i u_j g_i^{-1} \in K$ and $g_i^{-1} u_j g_i \in K$. Hence we have to show that if $g_i u_j g_i^{-1} \in K$ and $g_i^{-1} u_j g_i \in K$ for $1 \leq i \leq s$ and $1 \leq j \leq t$, then K is a normal subgroup of G.

Since $g_i u_j g_i^{-1} \in K$ and $g_i^{-1} u_j g_i \in K$, their inverses are also in K

$$g_i u_j^{-1} g_i^{-1} \in K, \quad g_i^{-1} u_j^{-1} g_i \in K. \tag{1.3}$$

Let $g = g_{i_1}^{e_{i_1}} \cdot \ldots \cdot g_{i_n}^{e_{i_n}}$ be an element of G and $u = u_{j_1}^{d_{j_1}} \cdot \ldots \cdot u_{j_m}^{d_{j_m}}$ be an element of K with $e_{i_j}, d_{j_k} \in \{-1, 1\}$. We have to show, that $gug^{-1} \in K$. Since

$$gug^{-1} = gu_{j_1}^{d_{j_1}} g^{-1} \cdot \ldots \cdot gu_{j_m}^{d_{j_m}} g^{-1}$$

it suffices to show that $gu_j^\delta g^{-1} \in K$ for each $j \in \{1,\ldots,t\}$ and $\delta \in \{-1,1\}$. We do this by induction on the length n of g:

In case $n = 1$ the assertion follows directly from the assumptions and (1.3). Now suppose that the assertion holds for n and let $g = g_{i_1}^{e_{i_1}} \cdot \ldots \cdot g_{i_n}^{e_{i_n}} \cdot g_{i_{n+1}}^{e_{i_{n+1}}}$ be of length $n+1$. Define $h := g_{i_2}^{e_{i_2}} \cdot \ldots \cdot g_{i_{n+1}}^{e_{i_{n+1}}}$. Because h is of length n we can apply the induction hypothesis and get $hu_j^\delta h^{-1} \in K$, say $hu_j^\delta h^{-1} = u_{k_1}^{d_{k_1}} \cdot \ldots \cdot u_{k_m}^{d_{k_m}}$ with $d_{k_l} \in \{-1,1\}$. Then we have

$$\begin{aligned}
gu_j^\delta g^{-1} &= g_{i_1}^{e_{i_1}} \cdot hu_j^\delta h^{-1} \cdot g_{i_1}^{-e_{i_1}} \\
&= g_{i_1}^{e_{i_1}} \cdot \left(u_{k_1}^{d_{k_1}} \cdot \ldots \cdot u_{k_m}^{d_{k_m}}\right) \cdot g_{i_1}^{-e_{i_1}} \\
&= g_{i_1}^{e_{i_1}} \cdot u_{k_1}^{d_{k_1}} \cdot g_{i_1}^{-e_{i_1}} \cdot \ldots \cdot g_{i_1}^{e_{i_1}} u_{k_m}^{d_{k_m}} \cdot g_{i_1}^{-e_{i_1}} \\
&= \left(g_{i_1}^{e_{i_1}} \cdot u_{k_1}^{d_{k_1}} \cdot g_{i_1}^{-e_{i_1}}\right) \cdot \ldots \cdot \left(g_{i_1}^{e_{i_1}} u_{k_m}^{d_{k_m}} \cdot g_{i_1}^{-e_{i_1}}\right).
\end{aligned}$$

This is in K by assumption and (1.3). $\square$

Chapter 2

Commutator Calculus

In this chapter we first recall some elementary definitions and facts about commutator calculus. After introducing commutator subgroups, we define the lower central series

$$G = \gamma_0(G) \geq \gamma_1(G) \geq \gamma_2(G) \geq \dots$$

of a group G. After this, we study the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ for $i \geq 0$. For example, we give a set of generators for these quotients if G is a finitely generated group. At the end of the chapter, we show that if

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} H \longrightarrow 1$$

is an exact sequence, then $\gamma_i(K)/\gamma_{i+1}(K)$ carries the structure of an H-module. We will apply this to the generalized Torelli groups K_n in Chapter 6.

We start by recalling the following definition.

Definition 2.1. Let G be a group and let $g_1, g_2, \dots \in G$. The commutator $[g_1, g_2]$ is defined by

$$[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}.$$

More generally we inductively define

$$[g_1, \dots, g_n] := [g_1, [g_2, \dots, g_n]]$$

for $n \geq 3$. Commutators of the form $[g_1, \dots, g_n]$ are called *simple commutators.* $\diamond$

Lemma 2.2. *Let G be a group and $a, b, c \in G$. Then the following commutator identities hold:*

a) $[a,b]^{-1} = [b,a]$,

b) $[ab,c] = [a,[b,c]] \cdot [b,c] \cdot [a,c] = [a,b,c] \cdot [b,c] \cdot [a,c]$,

c) $[a,bc] = [a,b] \cdot [b,[a,c]] \cdot [a,c] = [a,b] \cdot [b,a,c] \cdot [a,c]$.

Proof. Part a) is obvious. The proof of b) and c) is given by the following short calculation:

$$\begin{aligned}
[ab,c] &= abcb^{-1}a^{-1}c^{-1} \\
&= abcb^{-1}c^{-1}a^{-1}cbc^{-1}b^{-1}bcb^{-1}c^{-1}aca^{-1}c^{-1} \\
&= [a,[b,c]] \cdot [b,c] \cdot [a,c],
\end{aligned}$$

$$\begin{aligned}
[a,bc] &= abca^{-1}c^{-1}b^{-1} \\
&= aba^{-1}b^{-1}baca^{-1}c^{-1}b^{-1}cac^{-1}a^{-1}aca^{-1}c^{-1} \\
&= [a,b] \cdot [b,[a,c]] \cdot [a,c].
\end{aligned}$$

□

Definition 2.3. Let G be a group and let $G_1, G_2, \ldots$ be non-empty subsets of G. Define

$$[G_1, G_2] := \langle [g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2 \rangle$$

to be the *commutator subgroup of* G_1 *and* G_2. More generally, we inductively define

$$[G_1, \ldots, G_n] := [G_1, [G_2, \ldots, G_n]]$$

for $n \geq 3$. ◇

Remark 2.4. By Lemma 2.2 part a), we have

$$[G_1, G_2] = [G_2, G_1].$$

Definition 2.5. Let G be a group. We define

$$\gamma_0(G) := G \quad \text{and} \quad \gamma_k(G) := [G, \gamma_{k-1}(G)] \ \text{ for } \ k \geq 1.$$

The resulting series

$$G = \gamma_0(G) \geq \gamma_1(G) \geq \gamma_2(G) \geq \ldots$$

is called the *lower central series* of G. ◇

By definition, the subgroups $\gamma_k(G)$ are normal in G. Actually they are fully invariant subgroups of G, i.e. they are closed under every endomorphism of G. Notice that $\gamma_k(G)/\gamma_{k+1}(G)$ lies in the center of $G/\gamma_{k+1}(G)$. Hence $\gamma_k(G)/\gamma_{k+1}(G)$ is an abelian group.

The following proposition is taken from [8] Chapter 10:

Proposition 2.6. *Let G be a finitely generated group with generators $x_1, \ldots, x_r$, then $\gamma_k(G)/\gamma_{k+1}(G)$ is generated by the simple commutators $[y_1, y_2, \ldots, y_{k+1}] \mod \gamma_{k+1}(G)$, where $y_i \in \{x_1, \ldots, x_r\}$.*

Lemma 2.7. *Let G be a group, $[g_1, \ldots, g_n] \in \gamma_{n-1}(G)$ and $m \in \mathbb{N}$. Then*

$$[g_1, [g_2, \ldots, g_n]^m] \equiv [g_1, [g_2, \ldots, g_n]]^m \mod \gamma_n(G).$$

Proof. Induction on m:
The case $m = 1$ is obvious. Suppose that the assertion holds for m. We have

$$\begin{aligned}
& [g_1, [g_2, \ldots, g_n]^{(m+1)}] = [g_1, [g_2, \ldots, g_n] \cdot [g_2, \ldots, g_n]^m] \\
\overset{\text{L. 2.2}}{=} \; & [g_1, [g_2, \ldots, g_n]] \cdot [[g_2, \ldots, g_n], [g_1, [g_2, \ldots, g_n]^m]] \cdot [g_1, [g_2, \ldots, g_n]^m] \\
\equiv \; & [g_1, [g_2, \ldots, g_n]] \cdot [g_1, [g_2, \ldots, g_n]^m] \\
\overset{\text{by Ind.}}{\equiv} \; & [g_1, [g_2, \ldots, g_n]]^{(m+1)} \mod \gamma_n(G).
\end{aligned}$$

□

Lemma 2.8. *Let G be a group and $[g_1, [g_2, \ldots, g_n]] \in \gamma_{n-1}(G)$. If*

$$[g_2, \ldots, g_n]^m \equiv 0 \mod \gamma_{n-1}(G)$$

for some $m \in \mathbb{N}$, then

$$[g_1, [g_2, \ldots, g_n]]^m \equiv 0 \mod \gamma_n(G).$$

Proof. By assumption, we have $[g_2, \ldots, g_n]^m \in \gamma_{n-1}(G)$. It follows that

$$[g_1, [g_2, \ldots, g_n]]^m \overset{\text{Lem. 2.7}}{\equiv} [g_1, \underbrace{[g_2, \ldots, g_n]^m}_{\in \gamma_{n-1}(G)}] \equiv 0 \mod \gamma_n(G).$$

□

Lemma 2.9. *Let G be a group and $g_1, g_2 \in G$. Then we have*

$$[g_1^{-1}, g_2] \equiv [g_2, g_1] \mod \gamma_2(G) \quad \text{and} \quad [g_1, g_2^{-1}] \equiv [g_2, g_1] \mod \gamma_2(G).$$

Proof. Consider following equation

$$\begin{aligned} 1 &= [g_1 g_1^{-1}, g_2] \overset{\text{L. 2.2}}{=} [g_1, [g_1^{-1}, g_2]] \cdot [g_1^{-1}, g_2] \cdot [g_1, g_2] \\ &\equiv [g_1^{-1}, g_2] \cdot [g_1, g_2] \mod \gamma_2(G). \end{aligned}$$

It follows that

$$[g_1^{-1}, g_2] \equiv [g_1, g_2]^{-1} \overset{\text{L. 2.2}}{=} [g_2, g_1] \mod \gamma_2(G).$$

The proof of the second congruence is analogue. □

The next proposition and the corollaries will be very useful in Chapter 6, where we study quotients of the lower central series of the generalized Torelli group K_n.

Proposition 2.10. *Let G be a group. Then there are surjective homomorphisms*

$$\begin{aligned} \varepsilon_i : G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\gamma_i(G)/\gamma_{i+1}(G)) &\twoheadrightarrow \gamma_{i+1}(G)/\gamma_{i+2}(G) \\ (gG') \otimes_{\mathbb{Z}} (a\gamma_{i+1}(G)) &\mapsto [g, a]\gamma_{i+2}(G). \end{aligned}$$

Proof. For a proof see [15] Chapter 5. □

Corollary 2.11. *Let G be a group. If there is some $m \in \mathbb{N}$ such that $\gamma_m(G)/\gamma_{m+1}(G) = 0$, then $\gamma_k(G)/\gamma_{k+1}(G) = 0$ for all $k \geq m$.*

Proof. We show that $\gamma_{m+1}(G)/\gamma_{m+2}(G) = 0$. The corollary follows then by induction. Consider

$$\varepsilon_m : G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\gamma_m(G)/\gamma_{m+1}(G)) \twoheadrightarrow \gamma_{m+1}(G)/\gamma_{m+2}(G).$$

Since $\gamma_m(G)/\gamma_{m+1}(G) = 0$, we obtain

$$\varepsilon_m : G^{\mathrm{ab}} \otimes_{\mathbb{Z}} 0 \twoheadrightarrow \gamma_{m+1}(G)/\gamma_{m+2}(G).$$

Hence $\gamma_{m+1}(G)/\gamma_{m+2}(G) = 0$. □

Corollary 2.12. *Let G be a group with the property that G^{ab} is finitely generated. If there is some $m \in \mathbb{N}$ such that $\gamma_m(G)/\gamma_{m+1}(G)$ is finite, then $\gamma_k(G)/\gamma_{k+1}(G)$ is finite for all $k \geq m$.*

Proof. It suffices to show that $\gamma_{m+1}(G)/\gamma_{m+2}(G)$ is finite. For this consider

$$\varepsilon_m : G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\gamma_m(G)/\gamma_{m+1}(G)) \twoheadrightarrow \gamma_{m+1}(G)/\gamma_{m+2}(G).$$

By assumption, the group G^{ab} is a finitely generated abelian group, i.e. $G^{\mathrm{ab}} \cong \mathbb{Z}^r \oplus \mathrm{Tors}(G^{\mathrm{ab}})$. Since $F := \gamma_m(G)/\gamma_{m+1}(G)$ is finite, we get

$$\begin{aligned} & G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\gamma_m(G)/\gamma_{m+1}(G)) \cong (\mathbb{Z}^r \oplus \mathrm{Tors}(G^{\mathrm{ab}})) \otimes_{\mathbb{Z}} F \\ \cong \quad & (\mathbb{Z}^r \otimes_{\mathbb{Z}} F) \oplus \left(\mathrm{Tors}(G^{\mathrm{ab}}) \otimes F\right) \cong F^r \oplus \left(\mathrm{Tors}(G^{\mathrm{ab}}) \otimes F\right), \end{aligned}$$

which shows that $G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\gamma_m(G)/\gamma_{m+1}(G))$ is also finite. Since ε_m is onto this implies that $\gamma_{m+1}(G)/\gamma_{m+2}(G)$ is finite. □

Corollary 2.13. *Let G be a group with the property that G^{ab} is finitely generated with n generators. If there is some $m \in \mathbb{N}$ such that*

$$\gamma_m(G)/\gamma_{m+1}(G) \cong (\mathbb{Z}/2\mathbb{Z})^t,$$

then $\gamma_{m+i}(G)/\gamma_{m+i+1}(G) \cong (\mathbb{Z}/2\mathbb{Z})^{b_i}$ for all $i \geq 1$ and

$$0 \leq b_i \leq t \cdot n^k.$$

Proof. By Proposition 2.10, there is a surjective homomorphism

$$\varepsilon_i : \; G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\gamma_i(G)/\gamma_{i+1}(G)) \twoheadrightarrow \gamma_{i+1}(G)/\gamma_{i+2}(G).$$

By assumption we have

$$G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\gamma_i(G)/\gamma_{i+1}(G)) \cong G^{\mathrm{ab}} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})^t \cong (G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z})^t,$$

which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{b'_1}$ with

$$0 \leq b'_1 \leq n \cdot t.$$

Since the kernel of ε_i is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^{b'_1}$, we obtain $\ker(\varepsilon_i) \cong (\mathbb{Z}/2\mathbb{Z})^{\widetilde{b_1}}$ with $0 \leq \widetilde{b_1} \leq b'_1$. Hence

$$\gamma_{i+1}(G)/\gamma_{i+2}(G) \cong (\mathbb{Z}/2\mathbb{Z})^{b'_1}/(\mathbb{Z}/2\mathbb{Z})^{\widetilde{b_1}} \cong (\mathbb{Z}/2\mathbb{Z})^{b_1}$$

with

$$0 \leq b_1 \leq n \cdot t.$$

The lemma follows then by induction. □

Definition 2.14. Let G be a group.

a) G is called *nilpotent*, if $\gamma_{i_0}(G) = 1$ for some i_0.

b) G is called *residually nilpotent*, if

$$\bigcap_{i=0}^{\infty} \gamma_i(G) = 1.$$

◇

Remark 2.15. It follows that the residually nilpotent groups are exactly those groups G which satisfy the following condition:
For each $g \in G$, $g \neq 1$, there exists a normal subgroup N_g of G such that $g \notin N_g$ and G/N_g is nilpotent.

Proposition 2.16. *Let G be a group with the following two properties:*

1. *$\gamma_i(G)/\gamma_{i+1}(G)$ is torsion-free for all $i \geq 0$,*

2. *G is residually nilpotent.*

Then it follows that G is also torsion-free.

Proof. Assume that G is not torsion-free, i.e. there exists an element $g \in G \setminus \{1\}$ and $n \in \mathbb{N}$, $n \geq 2$, such that $g^n = 1$. We show by induction on i that $g \in \gamma_i(G)$ for all $i \geq 0$.
The case $i = 0$ is clear, since $G = \gamma_0(G)$. Now suppose that $g \in \gamma_i(G)$. We have to show that $g \in \gamma_{i+1}(G)$. By assumption, we have

$$g^n \equiv 1 \mod \gamma_{i+1}(G).$$

But $\gamma_i(G)/\gamma_{i+1}(G)$ is torsion-free. Hence

$$g \equiv 1 \mod \gamma_{i+1}(G),$$

i.e. $g \in \gamma_{i+1}(G)$.

Thus, we showed that $g \in \gamma_i(G)$ for all $i \geq 0$. So $g \in \bigcap_{i=0}^{\infty} \gamma_i(G) = \{1\}$, contradiction. □

For the rest of this chapter let G and H be groups and

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} H \longrightarrow 1$$

an exact sequence. We want to define a group action of H on the abelian group $\gamma_i(K)/\gamma_{i+1}(K)$. To ease the notation, set $\gamma_i := \gamma_i(K)$. Let $h \in H$ and $g \in G$ with $\varphi(g) = h$. Then we define $* : H \times \gamma_i/\gamma_{i+1} \to \gamma_i/\gamma_{i+1}$ by

$$h * (k \cdot \gamma_{i+1}) := gkg^{-1} \cdot \gamma_{i+1}.$$

Proposition 2.17. *The map $* : H \times \gamma_i/\gamma_{i+1} \to \gamma_i/\gamma_{i+1}$ defined above gives us a well defined action of H on γ_i/γ_{i+1}, i.e. the abelian group γ_i/γ_{i+1} becomes an H-module.*

Proof. For the complete proof, let $k \in \gamma_i$, $h \in H$ and $g \in G$ with $\varphi(g) = h$. We divide the proof into two steps. In the first step we show that the map $* : H \times \gamma_i/\gamma_{i+1} \to \gamma_i/\gamma_{i+1}$ is well defined. In the second step we prove that the map $* : H \times \gamma_i/\gamma_{i+1} \to \gamma_i/\gamma_{i+1}$ gives us an action of H on γ_i/γ_{i+1}.

1.) The map $* : H \times \gamma_i/\gamma_{i+1} \to \gamma_i/\gamma_{i+1}$ is well defined:

a) $xkx^{-1} \in \gamma_i$ for all $x \in G$:
Let $i_x : G \to G$ be the inner automorphism, which sends g to xgx^{-1} for all $g \in G$. Since K is a normal subgroup, the restriction $i_x|_K : K \to G$ is an endomorphism of K. In fact $i_x|_K \in \mathrm{Aut}(K)$. The inverse is given by $i_{x^{-1}}|_K$. It follows that

$$i_x|_K(\gamma_i) \leq \gamma_i$$

for all $i \in \mathbb{N}$, because γ_i is a fully invariant subgroup of K. Hence we proved that $xkx^{-1} \in \gamma_i$.

b) $h * (k_1 \cdot \gamma_{i+1}) = h * (k_2 \cdot \gamma_{i+1})$ for $k_1 \cdot \gamma_{i+1} = k_2 \cdot \gamma_{i+1}$:
Let $k_1, k_2 \in \gamma_i$ with $k_1 \cdot \gamma_{i+1} = k_2 \cdot \gamma_{i+1}$, i.e. $k_2^{-1}k_1 \in \gamma_{i+1}$. By a) we have $gk_2^{-1}k_1g^{-1} \in \gamma_{i+1}$. Hence

$$\begin{aligned} h * (k_2 \cdot \gamma_{i+1}) &= gk_2g^{-1} \cdot \gamma_{i+1} = gk_2g^{-1}gk_2^{-1}k_1g \cdot \gamma_{i+1} \\ &= gk_1g^{-1} \cdot \gamma_{i+1} = h * (k_1 \cdot \gamma_{i+1}). \end{aligned}$$

c) $g_1kg_1^{-1} \cdot \gamma_{i+1} = g_2kg_2^{-1} \cdot \gamma_{i+1}$ for $g_1, g_2 \in G$ with $\varphi(g_1) = \varphi(g_2)$:
Let $g_1, g_2 \in G$ with $\varphi(g_1) = \varphi(g_2)$. It follows that $\varphi(g_2^{-1}g_1) = 1$, i.e. $g_2^{-1}g_1 \in \ker(\varphi) = K$. The element $[g_1^{-1}g_2, k^{-1}]$ lies in γ_{i+1}. By a) we conclude that

$$g_2k^{-1}g_2^{-1}g_1kg_1^{-1} = g_1[g_1^{-1}g_2, k^{-1}]g_1^{-1} \in \gamma_{i+1}.$$

Thus

$$\begin{aligned} g_2kg_2^{-1} \cdot \gamma_{i+1} &= g_2kg_2^{-1}g_2k^{-1}g_2^{-1}g_1kg_1^{-1} \cdot \gamma_{i+1} \\ &= g_1kg_1^{-1} \cdot \gamma_{i+1}. \end{aligned}$$

2.) The map $* : H \times \gamma_i/\gamma_{i+1} \to \gamma_i/\gamma_{i+1}$ gives us an action of H on γ_i/γ_{i+1}:

a) $1_H * (k \cdot \gamma_{i+1}) = k \cdot \gamma_{i+1}$:
Since $\varphi(1_G) = 1_H$, we have

$$1_H * (k \cdot \gamma_{i+1}) = 1_G \cdot k \cdot 1_G^{-1} \cdot \gamma_{i+1} = k \cdot \gamma_{i+1}.$$

b) $(h_1h_2) * (k \cdot \gamma_{i+1}) = (h_1 * (h_2 * (k \cdot \gamma_{i+1}))$ for $h_1, h_2 \in H$:
Let $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. It follows that $\varphi(g_1g_2) = h_1h_2$. Hence

$$\begin{aligned} (h_1h_2) * (k \cdot \gamma_{i+1}) &= g_1g_2kg_2^{-1}g_1^{-1} \cdot \gamma_{i+1} \\ &= h_1 * (g_2kg_2^{-1} \cdot \gamma_{i+1}) \\ &= (h_1 * (h_2 * (k \cdot \gamma_{i+1})). \end{aligned}$$

c) $h * (k_1 \cdot \gamma_{i+1} + k_2 \cdot \gamma_{i+1}) = h * (k_1 \cdot \gamma_{i+1}) + h * (k_2 \cdot \gamma_{i+1})$ for $k_1, k_2 \in \gamma_i$:
Let $k_1, k_2 \in \gamma_i$. Then we have

$$\begin{aligned} h * (k_1 \cdot \gamma_{i+1} + k_2 \cdot \gamma_{i+1}) &= h * (k_1k_2 \cdot \gamma_{i+1}) \\ &= gk_1k_2g^{-1} \cdot \gamma_{i+1} \\ &= gk_1g^{-1}gk_2g^{-1} \cdot \gamma_{i+1} \\ &= gk_1g^{-1} \cdot \gamma_{i+1} + gk_2g^{-1} \cdot \gamma_{i+1} \\ &= h * (k_1 \cdot \gamma_{i+1}) + h * (k_2 \cdot \gamma_{i+1}). \end{aligned}$$

$\square$

Chapter 3

The classical Torelli Groups

Let F_n be the free group on n generators. A classical representation of $\mathrm{Aut}(F_n)$ is given by

$$\rho_1 : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/F_n') \cong \mathrm{GL}(n, \mathbb{Z}),$$

where F_n' is the commutator subgroup of F_n and $\rho_1(\varphi)$ is the automorphism of the abelian group F_n/F_n' induced by $\varphi \in \mathrm{Aut}(F_n)$. In Section 3.1 we summarize well-known facts about the kernel of ρ_1, which is called the *classical Torelli group* $\mathrm{IA}(F_n)$. In Section 3.2 we introduce more representations of $\mathrm{Aut}(F_n)$, which lead to *generalized Torelli groups* $\mathrm{IA}_i(F_n)$. Furthermore, we compare the series given by the $\mathrm{IA}_i(F_n)$ with the lower central series $\gamma_i(\mathrm{IA}(F_n))$.

3.1 Fundamentals

Let F_n be the free group generated by $x_1, \ldots, x_n$ $(n \geq 2)$ and $\mathrm{Aut}(F_n)$ its group of automorphisms. Let us introduce the following elements of $\mathrm{Aut}(F_n)$. Our convention here is that values not given are identical to the argument.

- $\lambda_{ij} : \{x_i \mapsto x_j x_i\}$ for $1 \leq i, j \leq n$ with $i \neq j$,
- $\nu_{ij} : \{x_i \mapsto x_i x_j\}$ for $1 \leq i, j \leq n$ with $i \neq j$,
- $\sigma_i : \{x_i \mapsto x_i^{-1}\}$ for $1 \leq i \leq n$,
- $\pi_{ij} : \{x_i \mapsto x_j,\ x_j \mapsto x_i\}$ for $1 \leq i, j \leq n$ with $i \neq j$,
- $\kappa_{ijk} : \{x_i \mapsto x_i x_j x_k x_j^{-1} x_k^{-1}\}$ for $1 \leq i, j, k \leq n$ with i, j, k pairwise distinct,

- $\text{kon}_{ij} : \{x_i \mapsto x_j x_i x_j^{-1}\}$ for $1 \leq i, j \leq n$ with $i \neq j$.

A theorem of Nielsen (see [14]) says that $\text{Aut}(F_n)$ is generated by the automorphisms λ_{ij}, ν_{ij}, π_{ij} for $1 \leq i, j \leq n$ with $i \neq j$ and σ_i for $1 \leq i \leq n$. In [14] Nielsen did not only show that $\text{Aut}(F_n)$ is finitely generated, he actually proved that $\text{Aut}(F_n)$ is finitely presented by constructing a finite set of defining relations. The finite presentation of $\text{Aut}(F_2)$ is given in the next proposition.

Proposition 3.1. *The automorphism group* $\text{Aut}(F_2)$ *has the following finite presentation*

$$\begin{aligned}\text{Aut}(F_2) = \langle \pi_{12}, \sigma_1, \nu_{12} \quad | \quad & \pi_{12}^2 = 1,\ \sigma_1^2 = 1,\ (\sigma_1 \circ \pi_{12})^4 = 1, \\ & \sigma_1^{-1} \circ \nu_{12}^{-1} \circ \sigma_1^{-1} \circ \nu_{12}^{-1} \circ \sigma_1 \circ \nu_{12} \circ \sigma_1 \circ \nu_{12} = 1, \\ & (\nu_{12} \circ \pi_{12} \circ \sigma_1 \circ \pi_{12})^2 = 1, \\ & (\sigma_1 \circ \pi_{12} \circ \nu_{12})^3 = 1\rangle.\end{aligned}$$

Proof. See [16] Chapter 3.5. □

A classical representation of $\text{Aut}(F_n)$ is given by

$$\rho_1 : \text{Aut}(F_n) \to \text{Aut}(F_n/F_n') \cong \text{GL}(n, \mathbb{Z}),$$

where F_n' is the commutator subgroup of F_n and $\rho_1(\varphi)$ is the automorphism of the abelian group F_n/F_n' induced by $\varphi \in \text{Aut}(F_n)$. Let us consider the images of the generators of $\text{Aut}(F_n)$ under ρ_1:

$$\begin{aligned} \rho_1(\lambda_{ij}) &= E_{ji}, & \rho_1(\nu_{ij}) &= E_{ji}, \\ \rho_1(\sigma_i) &= O_i, & \rho_1(\pi_{ij}) &= P_{ij}. \end{aligned}$$

For the definition of E_{ij}, O_i and P_{ij} see the Notation at the beginning. Because E_{ij} and O_i generate $\text{GL}(n, \mathbb{Z})$ by Proposition 1.5, we see that ρ_1 is onto. The kernel of ρ_1 is called the *classical Torelli group* and is denoted by $\text{IA}(F_n)$. Thus we have an exact sequence

$$1 \to \text{IA}(F_n) \to \text{Aut}(F_n) \to \text{GL}(n, \mathbb{Z}) \to 1. \tag{3.1}$$

Since $\text{Aut}(F_n)$ is finitely generated and $\text{GL}(n, \mathbb{Z})$ is finitely presented, it is clear by Proposition 1.7 that $\text{IA}(F_n)$ is finitely generated as a normal subgroup of $\mathrm{Aut}(F_n)$. By (3.1) the group $\mathrm{IA}(F_n)$ has infinite index in $\mathrm{Aut}(F_n)$. Hence there is no reason why $\mathrm{IA}(F_n)$ should be finitely generated as a group. Indeed our intuition tells us that it should be infinitly generated. But in contrast a theorem of Nielsen and Magnus ([13],[11]) asserts the opposite, namely that the classical Torelli group $\mathrm{IA}(F_n)$ is a finitely generated group. Actually explicit generators are given:

Theorem 3.2 (Nielsen/Magnus).

(a) *The group* $\mathrm{IA}(F_2)$ *is generated by the automorphisms*

$$\mathrm{kon}_{12} : \{x_1 \mapsto x_2 x_1 x_2^{-1}\} \quad \text{and} \quad \mathrm{kon}_{21} : \{x_2 \mapsto x_1 x_2 x_1^{-1}\}.$$

In particular $\mathrm{IA}(F_2) = \mathrm{Inn}(F_2)$.

(b) *For* $n \geq 3$ *the group* $\mathrm{IA}(F_n)$ *is generated by the automorphisms*

$$\mathrm{kon}_{ij} : \{x_i \mapsto x_j x_i x_j^{-1}\} \quad (i \neq j),$$

$$\kappa_{ijk} : \{x_i \mapsto x_i x_j x_k x_j^{-1} x_k^{-1}\} \quad (i, j, k \text{ pairwise distinct}).$$

The idea of the proof is to apply Proposition 1.9 and Lemma 1.10 (see [13] and [11]).

Define now $\mathrm{Aut}^+(F_n)$ to be the subgroup of $\mathrm{Aut}(F_n)$ whose image under ρ_1 lies in $\mathrm{SL}(n, \mathbb{Z})$, i.e.

$$\mathrm{Aut}^+(F_n) := \{\varphi \in \mathrm{Aut}(F_n) \mid \det(\rho_1(\varphi)) = 1\} = \ker(\det \circ \rho_1).$$

The subgroup $\mathrm{Aut}^+(F_n)$ is a normal subgroup of index two in $\mathrm{Aut}(F_n)$. By Nielsen (see [16]) $\mathrm{Aut}^+(F_n)$ is generated by the automorphisms λ_{ij} and ν_{ij} for $i = 1, \dots n$ with $i \neq j$. Hence $\rho_1 : \mathrm{Aut}^+(F_n) \to \mathrm{SL}(n, \mathbb{Z})$ is onto and we obtain the exact sequence

$$1 \to \mathrm{IA}(F_n) \to \mathrm{Aut}^+(F_n) \to \mathrm{SL}(n, \mathbb{Z}) \to 1.$$

3.2 Series of $\mathrm{IA}(F_n)$

Let us consider the lower central series of the group $\mathrm{IA}(F_n)$

$$\mathrm{IA}(F_n) = \gamma_0(\mathrm{IA}(F_n)) \geq \gamma_1(\mathrm{IA}(F_n)) \geq \gamma_2(\mathrm{IA}(F_n)) \geq \dots \ . \tag{3.2}$$

There is another central series of $\mathrm{IA}(F_n)$, which is more studied than the lower central series (3.2). Let us now construct this series.
For this let

$$F_n = \gamma_0(F_n) \geq \gamma_1(F_n) \geq \gamma_2(F_n) \geq \dots$$

be the lower central series of the free group F_n. Since the group $\gamma_i(F_n)$ is a characteristic subgroup of F_n we obtain homomorphisms

$$\rho_i : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/\gamma_i(F_n)),$$

where $\rho_i(\varphi)$ is the automorphism of the abelian group $F_n/\gamma_i(F_n)$ induced by $\varphi \in \mathrm{Aut}(F_n)$. Notice that for $i = 1$ this coincides with the classical representation ρ_1. Define

$$\mathrm{IA}_i(F_n) := \ker(\rho_i).$$

The automorphisms in $\mathrm{IA}_i(F_n)$ are those, which induce the identity on $F_n/\gamma_i(F_n)$. The groups $\mathrm{IA}_i(F_n)$ are called *generalized Torelli groups.* In fact, we have $\mathrm{IA}_1(F_n) = \mathrm{IA}(F_n)$.

By this construction, we obtain the following central series (see [1])

$$\mathrm{IA}(F_n) = \mathrm{IA}_1(F_n) \geq \mathrm{IA}_2(F_n) \geq \mathrm{IA}_3(F_n) \geq \dots \ . \tag{3.3}$$

The next proposition collects some results about this much investigated series.

Proposition 3.3.

a) $\mathrm{IA}_i(F_n)/\mathrm{IA}_{i+1}(F_n)$ *is torsion-free abelian* (*see* [1]).

b) $\mathrm{IA}_2(F_n) = \gamma_1(\mathrm{IA}_1(F_n))$ (*see* [3]).

c) $\bigcap_{i=0}^{\infty} \mathrm{IA}_i(F_n) = 1$ (*see* [1]).

d) $\mathrm{IA}_i(F_n) \cap \mathrm{Inn}(F_n) = \gamma_{i-1}(\mathrm{Inn}(F_n)) \cong \gamma_{i-1}(F_n)$ (*see* [6]).

e) $\gamma_i(\mathrm{IA}(F_n)) \leq \mathrm{IA}_{i+1}(F_n)$ (*see* [1]).

By this proposition, we see that the two central series (3.2) and (3.3) fit together in the following way

$$\begin{array}{ccccccccc}
\mathrm{IA}_1(F_n) & \geq & \mathrm{IA}_2(F_n) & \geq & \mathrm{IA}_3(F_n) & \geq & \mathrm{IA}_4(F_n) & \geq & \dots \\
\| & & \| & & \vee\!| & & \vee\!| & & \\
\gamma_0(\mathrm{IA}(F_n)) & \geq & \gamma_1(\mathrm{IA}(F_n)) & \geq & \gamma_2(\mathrm{IA}(F_n)) & \geq & \gamma_3(\mathrm{IA}(F_n)) & \geq & \dots
\end{array}$$

In the case $n = 2$ Andreadakis showed in [1] that these two series coincide, i.e. $\mathrm{IA}_i(F_2) = \gamma_{i-1}(\mathrm{IA}(F_2))$ for all $i \in \mathbb{N}$. This leads to the following conjecture.

Conjecture 3.4. *Let $n \geq 3$. Then we have*

$$\mathrm{IA}_i(F_n) = \gamma_{i-1}(\mathrm{IA}(F_n))$$

for all $i \in \mathbb{N}$.

From Proposition 3.3 we obtain the following corollary.

Corollary 3.5. *Let $n \geq 2$. The classical Torelli group $\mathrm{IA}(F_n)$ is torsion-free and residually nilpotent.*

Proof. By Proposition 3.3 part c) and e) we have $\bigcap_{i=0}^{\infty} \mathrm{IA}_i(F_n) = 1$ and $\gamma_i(\mathrm{IA}(F_n)) \leq \mathrm{IA}_{i+1}(F_n)$. But then it follows that

$$\bigcap_{i=0}^{\infty} \gamma_i(\mathrm{IA}(F_n)) = 1,$$

i.e. $\mathrm{IA}(F_n)$ is residually nilpotent. Now we apply Proposition 2.16 to see that $\mathrm{IA}(F_n)$ is torsion-free. □

Formanek constructs in [6] the following $\mathrm{Aut}(F_n)$-equivarient homomorphisms

$$\begin{array}{rrcl}
\varepsilon_i : & \mathrm{IA}_i(F_n)/\,\mathrm{IA}_{i+1}(F_n) & \to & \mathrm{Hom}_{\mathbb{Z}}(F_n/[F_n,F_n], \gamma_i(F_n)/\gamma_{i+1}(F_n)) \\
& \Phi \cdot \mathrm{IA}_{i+1}(F_n) & \mapsto & (x \cdot [F_n,F_n] \mapsto \Phi(x)x^{-1} \cdot \gamma_{i+1}(F_n)),
\end{array}$$

where the action of $\mathrm{Aut}(F_n)$ on the groups $\mathrm{IA}_i(F_n)/\,\mathrm{IA}_{i+1}(F_n)$ and $\mathrm{Hom}_{\mathbb{Z}}(F_n/[F_n,F_n], \gamma_i(F_n)/\gamma_{i+1}(F_n))$ is given by conjugation. The action is trivial when restricted to $\mathrm{IA}(F_n)$. Thus the ε_i are $\mathrm{Aut}(F_n)/\,\mathrm{IA}(F_n)$-homomorphisms, i.e. $\mathrm{GL}(n,\mathbb{Z})$-homomorphisms. The construction of these $\mathrm{GL}(n,\mathbb{Z})$-homomorphisms is similar to the construction of the so-called Johnson homomorphisms in the theory of mapping class groups (see [9]).

In the case $i = 1$ the homomorphism ε_1 is an ismorphism (see [3],[6]):

$$\begin{aligned} \mathrm{IA}_1(F_n)/\,\mathrm{IA}_2(F_n) &\cong \mathrm{Hom}_{\mathbb{Z}}(F_n/[F_n,F_n],\gamma_1(F_n)/\gamma_2(F_n)) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n,\Lambda_2(\mathbb{Z}^n)), \end{aligned}$$

where $\Lambda_2(\mathbb{Z}^n)$ is the second exterior power of $\mathbb{Z}^n$. Since we have $\mathrm{IA}_2(F_n) = \gamma_1(\mathrm{IA}(F_n))$ by Proposition 3.3, we obtain

$$\mathrm{IA}(F_n)^{\mathrm{ab}} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n,\Lambda_2(\mathbb{Z}^n)).$$

After tensoring with $\mathbb{C}$, we obtain

$$\mathrm{IA}(F_n)^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n \oplus V_n$$

as a $\mathrm{GL}(n,\mathbb{C})$-module, where V_n is a certain $\mathrm{GL}(n,\mathbb{C})$-module of dimension $\dim_{\mathbb{C}}(V_n) = n(n+1)(n-2)/2$ which is irreducible even as $\mathrm{SL}(n,\mathbb{C})$-module (see [6]).

Chapter 4

Generalized Torelli Groups

This chapter is concerned with the main subject of the thesis. In Section 4.1 we describe the construction of the representations

$$\rho_{G,\pi} : \Gamma(G,\pi) \to \mathcal{G}_{G,\pi}(\mathbb{Z}),$$

where $\Gamma(G,\pi)$ is a subgroup of $\mathrm{Aut}(F_n)$ with finite index. This construction is introduced by F. Grunewald und A. Lubotzky in [5]. Section 4.2 deals with the special case $G = C_2$. We introduce the representation

$$\sigma_{-1} : \Gamma^+(C_2,\pi) \to \mathrm{GL}(n-1,\mathbb{Z}).$$

We show that the map σ_{-1} is surjective by analysing the images of the generators of $\Gamma^+(C_2,\pi)$. Hence the kernel K_n of σ_{-1} fits into the following exact sequence

$$1 \to K_n \to \Gamma^+(C_2,\pi) \to \mathrm{GL}(n-1,\mathbb{Z}) \to 1.$$

Our main theorem states that K_n is finitly generated as a group. Generators of K_n are given explicitly. This is done in Section 4.3. Note that our main theorem corresponds to the theorem of Nielsen and Magnus (see Theorem 3.2).

4.1 Construction of the representation $\rho_{G,\pi}$

The representations, which we describe here, are introduced by F. Grunewald und A. Lubotzky in [5].

Let G be a finite group and $\pi : F_n \twoheadrightarrow G$ a surjective homomorphism of the free group F_n onto G, i.e. π is a presentation of G. Moreover, let $R := \ker(\pi)$ be the kernel of π. Then R is a finitely generated free group.

By the formula of Reidemeister and Schreier (see [16] Chapter 2.4) we obtain that R is free on

$$|G| \cdot (n-1) + 1$$

generators. Now we define

$$\Gamma(R) := \{\varphi \in \mathrm{Aut}(F_n) \mid \varphi(R) = R\} \leq \mathrm{Aut}(F_n)$$

and

$$\Gamma(G, \pi) := \{\varphi \in \Gamma(R) \mid \varphi \text{ induces the identity on } F_n/R\} \leq \mathrm{Aut}(F_n).$$

Both subgroups $\Gamma(R)$ and $\Gamma(G, \pi)$ have finite index in $\mathrm{Aut}(F_n)$ (see [2]). Define further

$$\bar{R} := R/[R, R] = R^{\mathrm{ab}}.$$

The action of F_n on R by conjugation leads to an action of the group G on $\bar{R}$. The group $\bar{R}$ is called the *relation module*. Every automorphism $\varphi \in \Gamma(R)$ induces a linear automorphism $\bar{\varphi} \in \mathrm{Aut}(\bar{R})$. By a result of Gaschütz (see [7]), we have

$$\Gamma(G, \pi) = \{\varphi \in \Gamma(R) \mid \bar{\varphi} : \bar{R} \to \bar{R} \text{ is } G\text{-equivariant}\}.$$

The relation module $\bar{R}$ is a finitely generated free abelian group. Let $t := |G| \cdot (n-1) + 1$ denote the $\mathbb{Z}$-rank of $\bar{R}$. We define

$$\mathcal{G}_{G,\pi} := \mathrm{Aut}_G(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}) \leq \mathrm{GL}(t, \mathbb{C}).$$

The group $\mathcal{G}_{G,\pi}$ is the centraliser of the group G acting on $\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}$ through matrices with rational entries. We set

$$\mathcal{G}_{G,\pi}(\mathbb{Z}) := \{\Phi \in \mathcal{G}_{G,\pi} \mid \Phi(\bar{R}) = \bar{R}\}.$$

Choosing a $\mathbb{Z}$-basis of $\bar{R}$ we obtain an integral linear representation

$$\begin{array}{rcl} \rho_{G,\pi} : \Gamma(G, \pi) & \to & \mathcal{G}_{G,\pi}(\mathbb{Z}) \\ \varphi & \mapsto & \bar{\varphi}. \end{array}$$

4.2 THE REPRESENTATION σ_{-1}

The next constructions are taken from [5] Section 6. Let F_n $(n \geq 2)$ be the free group generated by $x, y_1, \ldots, y_{n-1}$. Later we will see that the generator x will play a special role. That is the reason why we denote the generators this way. Let us introduce the following elements of $\mathrm{Aut}(F_n)$. Our convention is that values not given are identical to the argument.

- $\delta_i : \{x \mapsto y_i x\}$ and $\varepsilon_i : \{x \mapsto x y_i\}$ for $i = 1, \ldots n-1$,
- $\varphi_i : \{y_i \mapsto x y_i\}$ and $\psi_i : \{y_i \mapsto y_i x\}$ for $i = 1, \ldots n-1$,
- $\lambda_{ij} : \{y_i \mapsto y_j y_i\}$ and $\nu_{ij} : \{y_i \mapsto y_i y_j\}$ for $i = 1, \ldots n-1$ with $i \neq j$.

A theorem of Nielsen (see Chapter 3) asserts that these elements generate $\mathrm{Aut}^+(F_n)$. Let us introduce further

- $\kappa_{jk} : \{x \mapsto x[y_j, y_k]\}$, $\kappa_{ijk} : \{y_i \mapsto y_i[y_j, y_k]\}$ for $1 \leq i, j, k \leq n-1$ with i, j, k pairwise distinct,
- $\tau_{ij} : \{y_i \mapsto y_i[x, y_j]\}$ for $1 \leq i, j \leq n-1$ with $i \neq j$,
- $\mathrm{kon}_{ix} : \{y_i \mapsto x y_i x^{-1}\}$, $\mathrm{kon}_{xi} : \{x \mapsto y_i x y_i^{-1}\}$ and $\mathrm{kon}_{ij} : \{y_i \mapsto y_j y_i y_j^{-1}\}$ for $1 \leq i, j \leq n-1$ with $i \neq j$.

The set consisting of the κ_{jk}, κ_{ijk}, τ_{ij}, kon_{ix}, kon_{xi} and kon_{ij} generates the group $\mathrm{IA}(F_n)$ by Theorem 3.2.

Now let $G := C_2$ the cyclic group of order two generated by g and $\pi : F_n \to C_2$ be the following presentation

$$\pi(x) := g, \pi(y_1) = 1, \ldots, \pi(y_{n-1}) = 1.$$

The kernel R of this presentation consists exactly of those elements of F_n with an even number of x's. By the formula of Reidemeister and Schreier this is a free group of rank $2n-1$. Free generators are given by

$$x^2, y_1, \ldots, y_{n-1}, x y_1 x^{-1}, \ldots, x y_{n-1} x^{-1}.$$

The corresponding relation module $\bar{R}$ has then the following $\mathbb{Z}$-basis

$$\overline{x^2}, \overline{y_1}, \ldots, \overline{y_{n-1}}, \overline{x y_1 x^{-1}}, \ldots, \overline{x y_{n-1} x^{-1}}.$$

By the construction described in Section 4.1 we obtain the integral linear representation

$$\rho_{C_2,\pi} : \Gamma(C_2, \pi) \to \mathrm{GL}(\bar{R}) \cong \mathrm{GL}(2n-1, \mathbb{Z}).$$

Set now

$$\Gamma^+(C_2, \pi) := \{\varphi \in \Gamma(C_2, \pi) \mid \det(\rho_1(\varphi)) = 1\}.$$

Lemma 4.1. *The index of* $\Gamma^+(C_2, \pi)$ *in* $\mathrm{Aut}^+(F_n)$ *is* $2^n - 1$.

Proof. See [5], [2]. □

The restriction of $\rho_{C_2,\pi}$ on $\Gamma^+(C_2, \pi)$ leads to the integral linear representation

$$\rho_{C_2,\pi} : \Gamma^+(C_2, \pi) \to \mathrm{GL}(\bar{R}) \cong \mathrm{GL}(2n-1, \mathbb{Z}).$$

The $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$ decomposes as $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} = V_1 \oplus V_{-1}$, where V_1, V_{-1} are the ± 1 eigenspaces of g respectively. Set $\bar{R}_1 := \bar{R} \cap V_1$ and $\bar{R}_{-1} := \bar{R} \cap V_{-1}$. Introduce

$$v_i := \overline{y_i} + \overline{x y_i x^{-1}}, \; w_i := \overline{y_i} - \overline{x y_i x^{-1}} \; (i = 1, \dots, n-1).$$

Then $\overline{x^2}, v_1, \dots, v_{n-1}$ is a $\mathbb{Z}$-basis of $\bar{R}_1$ and $w_1, \dots, w_{n-1}$ is a $\mathbb{Z}$-basis of $\bar{R}_{-1}$. Since $\Gamma(C_2, \pi)$ leaves $\bar{R}_1$ and $\bar{R}_{-1}$ invariant, we obtain, with the above $\mathbb{Z}$-basis chosen, representations

$$\sigma_1 : \Gamma^+(C_2, \pi) \to \mathrm{GL}(n, \mathbb{Z}), \quad \sigma_{-1} : \Gamma^+(C_2, \pi) \to \mathrm{GL}(n-1, \mathbb{Z}).$$

The representation σ_1 is equivalent to ρ_1 restricted to $\Gamma^+(C_2, \pi)$. A proof for the case $n = 2$ is in [2], but the general case is analogous.

In contrast the map σ_{-1} somewhat less expected. The goal of this chapter is to understand the map σ_{-1}. We especially want to study the kernel of σ_{-1}, which we call a *generalized Torelli group*.

We adopt from [5] the following proposition, which presents us a set of generators for $\Gamma^+(C_2, \pi)$. To give the generators of $\Gamma^+(C_2, \pi)$ is an important feature, which is not possible for an arbitrary group G.

Proposition 4.2. *Let $n \geq 2$ be a natural number. The group $\Gamma^+(C_2, \pi)$ is generated by the automorphisms*

- δ_i, ε_i, φ_i^2, ψ_i^2, kon_{ix}, kon_{xi} $(1 \leq i \leq n-1)$,
- λ_{ij}, ν_{ij}, κ_{ij}, τ_{ij}, kon_{ij} $(1 \leq i, j \leq n-1, i \neq j)$,
- κ_{ijk} $(1 \leq i, j, k \leq n-1,\ i, j, k$ pairwise distinct$)$.

Lemma 4.3. *Let $n \geq 2$. The group $\mathrm{IA}(F_n)$ is contained in $\Gamma^+(C_2, \pi)$.*

Proof. This is clear by Theorem 3.2 and the generators of $\Gamma^+(C_2, \pi)$ given in Proposition 4.2. □

If we take a close look on the set of generators for $\Gamma^+(C_2, \pi)$ given in Proposition 4.2, we see that there are some redundant generators, i.e. we can express some generators in terms of others, which yields the following set of generators.

Corollary 4.4. *Generators for $\Gamma^+(C_2, \pi)$ are*

- *for $n = 2$ the automorphisms ε_1, ψ_1^2 and kon_{1x},*
- *for $n \geq 3$ the automorphisms ε_i, ψ_i^2, kon_{ix} $(i = 1, \ldots, n-1)$ and ν_{ij} $(i, j = 1, \ldots, n-1,\ i \neq j)$.*

Proof. We are going to prove the following relations:
For $n \geq 2$ and $1 \leq i \leq n-1$:

$$\begin{aligned}
\delta_i &= \mathrm{kon}_{ix}^{-1} \circ \varepsilon_i \circ \mathrm{kon}_{ix} \\
\mathrm{kon}_{xi} &= \delta_i \circ \varepsilon_i^{-1}, \\
\varphi_i^2 &= \mathrm{kon}_{ix}^2 \circ \psi_i^2.
\end{aligned}$$

For $n \geq 3$ and $1 \leq i, j \leq 2$, $i \neq j$:

$$\begin{aligned}
\mathrm{kon}_{ij} &= [\mathrm{kon}_{ix}^{-1}, \delta_j], \\
\lambda_{ij} &= \mathrm{kon}_{ij} \circ \nu_{ij}, \\
\tau_{ij} &= \mathrm{kon}_{ix}^{-1} \circ \nu_{ij} \circ \mathrm{kon}_{ix} \circ \nu_{ij}^{-1}, \\
\kappa_{ij} &= \varepsilon_i \circ \varepsilon_j \circ \varepsilon_i^{-1} \circ \varepsilon_j^{-1}.
\end{aligned}$$

For $n \geq 4$ and $1 \leq i, j, k \leq n-1$, i, j, k pairwise distinct:

$$\kappa_{ijk} \;\;=\;\; \nu_{ij} \circ \nu_{ik} \circ \nu_{ij}^{-1} \circ \nu_{ik}^{-1}.$$

Let us now prove these formulas in detail:

- $\delta_i = \mathrm{kon}_{ix}^{-1} \circ \varepsilon_i \circ \mathrm{kon}_{ix}$:

$$\left\{ \begin{array}{lllllll} x & \overset{\mathrm{kon}_{ix}}{\mapsto} & x & \overset{\varepsilon_i}{\mapsto} & xy_i & \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} & y_i x \\ y_i & \overset{\mathrm{kon}_{ix}}{\mapsto} & xy_i x^{-1} & \overset{\varepsilon_i}{\mapsto} & xy_i x^{-1} & \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} & y_i \end{array} \right\},$$

- $\mathrm{kon}_{xi} = \delta_i \circ \varepsilon_i^{-1}$:

$$\left\{ \; x \;\; \overset{\varepsilon_i^{-1}}{\mapsto} \;\; xy_i^{-1} \;\; \overset{\delta_i}{\mapsto} \;\; y_i x y_i^{-1} \; \right\},$$

- $\varphi_i^2 = \mathrm{kon}_{ix}^2 \circ \psi_i^2$:

$$\left\{ \; y_i \;\; \overset{\psi_i^2}{\mapsto} \;\; y_i x^2 \;\; \overset{\mathrm{kon}_{ix}^2}{\mapsto} \;\; x^2 y_i \; \right\},$$

- $\mathrm{kon}_{ij} = \mathrm{kon}_{ix}^{-1} \circ \delta_j \circ \mathrm{kon}_{ix} \circ \delta_j^{-1}$:

$$\left\{ \begin{array}{lllllll} x & \overset{\delta_j^{-1}}{\mapsto} & y_j^{-1} x & \overset{\mathrm{kon}_{ix}}{\mapsto} & y_j^{-1} x & \overset{\delta_j}{\mapsto} & x \\ y_i & \overset{\delta_j^{-1}}{\mapsto} & y_i & \overset{\mathrm{kon}_{ix}}{\mapsto} & xy_i x^{-1} & \overset{\delta_j}{\mapsto} & y_j x y_i x^{-1} y_j^{-1} \end{array} \right.$$
$$\left. \begin{array}{ll} \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} & x \\ \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} & y_j y_i y_j^{-1} \end{array} \right\},$$

- $\lambda_{ij} = \mathrm{kon}_{ij} \circ \nu_{ij}$:

$$\left\{ \; y_i \;\; \overset{\nu_{ij}}{\mapsto} \;\; y_i y_j \;\; \overset{\mathrm{kon}_{ij}}{\mapsto} \;\; y_j y_i \; \right\},$$

- $\tau_{ij} = \mathrm{kon}_{ix}^{-1} \circ \nu_{ij} \circ \mathrm{kon}_{ix} \circ \nu_{ij}^{-1}$:

$$\left\{ \begin{array}{l} y_i \overset{\nu_{ij}^{-1}}{\mapsto} y_i y_j^{-1} \overset{\mathrm{kon}_{ix}}{\mapsto} x y_i x^{-1} y_j^{-1} \overset{\nu_{ij}}{\mapsto} x y_i y_j x^{-1} y_j^{-1} \\ \quad \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} y_i x y_j x^{-1} y_j^{-1} \end{array} \right\},$$

- $\kappa_{ij} = \varepsilon_i \circ \varepsilon_j \circ \varepsilon_i^{-1} \circ \varepsilon_j^{-1}$:

$$\left\{ \begin{array}{l} x \overset{\varepsilon_j^{-1}}{\mapsto} x y_j^{-1} \overset{\varepsilon_i^{-1}}{\mapsto} x y_i^{-1} y_j^{-1} \overset{\varepsilon_j}{\mapsto} x y_j y_i^{-1} y_j^{-1} \\ \quad \overset{\varepsilon_i}{\mapsto} x y_i y_j y_i^{-1} y_j^{-1} \end{array} \right\},$$

- $\kappa_{ijk} = \nu_{ij} \circ \nu_{ik} \circ \nu_{ij}^{-1} \circ \nu_{ik}^{-1}$:

$$\left\{ \begin{array}{l} y_i \overset{\nu_{ik}^{-1}}{\mapsto} y_i y_k^{-1} \overset{\nu_{ij}^{-1}}{\mapsto} y_i y_j^{-1} y_k^{-1} \overset{\nu_{ik}}{\mapsto} y_i y_k y_j^{-1} y_k^{-1} \\ \quad \overset{\nu_{ij}}{\mapsto} y_i y_j y_k y_j^{-1} y_k^{-1} \end{array} \right\}.$$

□

Now that we have found this improved generator set of $\Gamma^+(C_2, \pi)$, let us consider the images of these generators under σ_{-1}.

Proposition 4.5. *We have*

$$\begin{array}{ll} \sigma_{-1}(\varepsilon_i) = I_{n-1}, & \sigma_{-1}(\nu_{ij}) = E_{ji} \quad (i \neq j), \\ \sigma_{-1}(\psi_i^2) = I_{n-1}, & \sigma_{-1}(\mathrm{kon}_{ix}) = O_i \end{array}$$

for $1 \leq i, j \leq n-1$. *In particular* σ_{-1} *is surjective onto* $\mathrm{GL}(n-1, \mathbb{Z})$.

Proof. We consider the images of σ_{-1} with respect to the $\mathbb{Z}$-basis $w_1, \ldots, w_{n-1}$ of R_{-1}. We have

- $\sigma_{-1}(\varepsilon_i) = I_{n-1}$:

$$\begin{aligned} \sigma_{-1}(\varepsilon_i)(w_k) &= \overline{\varepsilon_i(y_k)} - \overline{\varepsilon_i(x y_k x^{-1})} = \overline{y_k} - \overline{x y_i y_k y_i^{-1} x^{-1}} \\ &= \overline{y_k} - \left(\overline{x y_i x^{-1}} + \overline{x y_k x^{-1}} - \overline{x y_i x^{-1}} \right) \\ &= \overline{y_k} - \overline{x y_k x^{-1}} = w_k \quad \text{for all } k. \end{aligned}$$

- $\sigma_{-1}(\nu_{ij}) = E_{ji}$:

$$\begin{aligned}\sigma_{-1}(\nu_{ij})(w_k) &= \overline{\nu_{ij}(y_k)} - \overline{\nu_{ij}(xy_kx^{-1})} = \overline{y_k} - \overline{xy_kx^{-1}} \\ &= w_k \quad \text{for all } k \neq i,\end{aligned}$$

$$\begin{aligned}\sigma_{-1}(\nu_{ij})(w_i) &= \overline{\nu_{ij}(y_i)} - \overline{\nu_{ij}(xy_ix^{-1})} = \overline{y_iy_j} - \overline{xy_iy_jx^{-1}} \\ &= \overline{y_i} + \overline{y_j} - \left(\overline{xy_ix^{-1}} + \overline{xy_jx^{-1}}\right) = w_i + w_j.\end{aligned}$$

- $\sigma_{-1}(\psi_i^2) = I_{n-1}$:

$$\begin{aligned}\sigma_{-1}(\psi_i^2)(w_k) &= \overline{\psi_i^2(y_k)} - \overline{\psi_i^2(xy_kx^{-1})} = \overline{y_k} - \overline{xy_kx^{-1}} \\ &= w_k \quad \text{for all } k \neq i,\end{aligned}$$

$$\begin{aligned}\sigma_{-1}(\psi_i^2)(w_i) &= \overline{\psi_i^2(y_i)} - \overline{\psi_i^2(xy_ix^{-1})} = \overline{y_ix^2} - \overline{xy_ix} \\ &= \overline{y_i} + \overline{x^2} - \left(\overline{xy_ix^{-1}} + \overline{x^2}\right) = w_i.\end{aligned}$$

- $\sigma_{-1}(\mathrm{kon}_{ix}) = O_i$:

$$\begin{aligned}\sigma_{-1}(\mathrm{kon}_{ix})(w_j) &= \overline{\mathrm{kon}_{ix}(y_k)} - \overline{\mathrm{kon}_{ix}(xy_kx^{-1})} = \overline{y_k} - \overline{xy_kx^{-1}} \\ &= w_k \quad \text{for all } k \neq i,\end{aligned}$$

$$\begin{aligned}\sigma_{-1}(\mathrm{kon}_{ix})(w_i) &= \overline{\mathrm{kon}_{ix}(y_i)} - \overline{\mathrm{kon}_{ix}(xy_ix^{-1})} = \overline{xy_ix^{-1}} - \overline{x^2y_ix^{-2}} \\ &= \overline{xy_ix^{-1}} - \left(\overline{x^2} + \overline{y_i} - \overline{x^2}\right) = -w_i.\end{aligned}$$

The surjectivity of σ_{-1} follows directly from Proposition 1.5. □

4.3 The kernel of σ_{-1}

The kernel of σ_{-1} can be considered as a generalization of $\mathrm{IA}(F_n)$. Hence we call $\ker(\sigma_{-1})$ a *generalized Torelli group*. By Proposition 4.5 we obtain the following exact sequence

$$1 \to \ker(\sigma_{-1}) \to \Gamma^+(C_2, \pi) \xrightarrow{\sigma_{-1}} \mathrm{GL}(n-1, \mathbb{Z}) \to 1.$$

By this sequence we see that the index of $\ker(\sigma_{-1})$ in $\Gamma^+(C_2,\pi)$ is infinite for $n \geq 3$. For $n = 2$, the index of $\ker(\sigma_{-1})$ in $\Gamma^+(C_2,\pi)$ is two and it follows by Proposition 1.6 that, in this case, $\ker(\sigma_{-1})$ is a finitely generated group. But there is no obvious reason why $\ker(\sigma_{-1})$ should be a finitely generated group for $n \geq 3$. However the result of Nielsen and Magnus, which says that $\mathrm{IA}(F_n)$ is finitely generated as a group (Theorem 3.2), makes the finite generation of $\ker(\sigma_{-1})$ more likely. Indeed we are going to prove that $\ker(\sigma_{-1})$ is finitely generated as a group for all $n \geq 2$ (see Theorem 4.14).

Definition 4.6. Let $n \geq 2$. Define $K_n \leq \Gamma^+(C_2, \pi)$ to be the subgroup generated by the following automorphisms

$$\begin{array}{ll} \varepsilon_i : \{x \mapsto xy_i\}, & \delta_i : \{x \mapsto y_i x\}, \\ \psi_i^2 : \{y_i \mapsto y_i x^2\}. & \end{array}$$

for $i = 1, \ldots, n-1$. ◇

Notice that K_n is a finitely generated group by definition. The next goal will be to prove that the group K_n is the kernel of σ_{-1} for all $n \geq 2$. To prove this we will apply Proposition 1.9. We know by Corollary 4.4 the generators of $\Gamma^+(C_2, \pi)$ and by Proposition 4.5 the images of these generators under σ_{-1}, namely

$$\begin{array}{ll} \sigma_{-1}(\varepsilon_i) = I_{n-1}, & \sigma_{-1}(\nu_{ij}) = E_{ji}, \\ \sigma_{-1}(\psi_i^2) = I_{n-1}, & \sigma_{-1}(\mathrm{kon}_{ix}) = O_i. \end{array}$$

A presentation of $\mathrm{GL}(n-1, \mathbb{Z})$ in terms of these generators is given in Proposition 1.5. So our strategy is the following:

- Show that the group K_n is a normal subgroup in $\Gamma^+(C_2, \pi)$ (see Lemma 4.12).
- Let $R(E_{ij}, O_k)$ be the set relations of $\mathrm{GL}(n-1, \mathbb{Z})$ as given in Proposition 1.5. Then show that $R(\nu_{ji}, \mathrm{kon}_{kx}) \in K_n$ (see Lemma 4.13).
- Show that $K_n \leq \ker(\sigma_{-1})$ (see Lemma 4.7).

Then we can apply Proposition 1.9 and conclude that $K_n = \ker(\sigma_{-1})$.

Lemma 4.7. *We have*

$$K_n \leq \ker(\sigma_{-1}).$$

Proof. We have by Proposition 4.5

$$\sigma_{-1}(\varepsilon_i) = I_{n-1} \quad \text{and} \quad \sigma_{-1}(\psi_i^2) = I_{n-1}.$$

Hence it suffices to show that $\sigma_{-1}(\delta_i) = I_{n-1}$:

$$\begin{array}{rcl} \sigma_{-1}(\delta_i) & \overset{\text{Cor. 4.4}}{=} & \sigma_{-1}(\mathrm{kon}_{ix}^{-1} \circ \varepsilon_i \circ \mathrm{kon}_{ix}) \\ & = & \sigma_{-1}(\mathrm{kon}_{ix})^{-1} \cdot \sigma_{-1}(\varepsilon_i) \cdot \sigma_{-1}(\mathrm{kon}_{ix}) = I_{n-1}. \end{array}$$

□

The proofs of the Lemmas 4.12 and 4.13 become easier, if we introduce some more elements in K_n. This will be done in Definition 4.8 and Definition 4.10.

Definition 4.8. Let $n \geq 2$. Define α_i and β_i for $1 \leq i \leq n-1$ to be the following elements of $\mathrm{Aut}(F_n)$

$$\alpha_i : \left\{ \begin{array}{ccc} x & \mapsto & x^{-1} \\ y_i & \mapsto & x y_i^{-1} x^{-1} \end{array} \right\}, \qquad \beta_i : \left\{ \begin{array}{ccc} x & \mapsto & x^{-1} \\ y_i & \mapsto & x^{-1} y_i^{-1} x \end{array} \right\}.$$

◇

Proposition 4.9. *Let $n \geq 2$. Then the automorphisms α_i and β_i ($i = 1, \ldots, n-1$) are in K_n and satisfy $\alpha_i^2 = \mathrm{id}$ and $\beta_i^2 = \mathrm{id}$. In particular, K_n is not torsion-free.*

Proof. We have

$$\begin{array}{rcl} \alpha_i & = & \psi_i^{-2} \circ \varepsilon_i \circ \psi_i^{-2} \circ \delta_i, \\ \beta_i & = & \psi_i^{2} \circ \delta_i^{-1} \circ \psi_i^{2} \circ \varepsilon_i^{-1}. \end{array}$$

We prove these formulas now in detail:

- $\alpha_i = \psi_i^{-2} \circ \varepsilon_i \circ \psi_i^{-2} \circ \delta_i$:

$$\left\{ \begin{array}{ccccccccc} x & \overset{\delta_i}{\mapsto} & y_i x & \overset{\psi_i^{-2}}{\mapsto} & y_i x^{-1} & \overset{\varepsilon_i}{\mapsto} & x^{-1} & \overset{\psi_i^{-2}}{\mapsto} & x^{-1} \\ y_i & \overset{\delta_i}{\mapsto} & y_i & \overset{\psi_i^{-2}}{\mapsto} & y_i x^{-2} & \overset{\varepsilon_i}{\mapsto} & x^{-1} y_i^{-1} x^{-1} & \overset{\psi_i^{-2}}{\mapsto} & x y_i^{-1} x^{-1} \end{array} \right\},$$

- $\beta_i = \psi_i^2 \circ \delta_i^{-1} \circ \psi_i^2 \circ \varepsilon_i^{-1}$:

$$\left\{ \begin{array}{lllllllll} x & \overset{\varepsilon_i^{-1}}{\mapsto} & xy_i^{-1} & \overset{\psi_i^2}{\mapsto} & x^{-1}y_i^{-1} & \overset{\delta_i^{-1}}{\mapsto} & x^{-1} & \overset{\psi_i^2}{\mapsto} & x^{-1} \\ y_i & \overset{\varepsilon_i^{-1}}{\mapsto} & y_i & \overset{\psi_i^2}{\mapsto} & y_i x^2 & \overset{\delta_i^{-1}}{\mapsto} & xy_i^{-1}x & \overset{\psi_i^2}{\mapsto} & x^{-1}y_i^{-1}x \end{array} \right\}.$$

Finally we show that $\alpha_i^2 = \mathrm{id}$ and $\beta_i^2 = \mathrm{id}$:

$$\left\{ \begin{array}{lllll} x & \overset{\alpha_i}{\mapsto} & x^{-1} & \overset{\alpha_i}{\mapsto} & x \\ y_i & \overset{\alpha_i}{\mapsto} & xy_i^{-1}x^{-1} & \overset{\alpha_i}{\mapsto} & x^{-1}(xy_i^{-1}x^{-1})^{-1}x = y_i \end{array} \right\},$$

$$\left\{ \begin{array}{lllll} x & \overset{\beta_i}{\mapsto} & x^{-1} & \overset{\beta_i}{\mapsto} & x \\ y_i & \overset{\beta_i}{\mapsto} & x^{-1}y_i^{-1}x & \overset{\beta_i}{\mapsto} & x(x^{-1}y_i^{-1}x)^{-1}x^{-1} = y_i \end{array} \right\}.$$

□

Definition 4.10. Let us define some more elements in $\mathrm{Aut}(F_n)$ for $n \geq 3$.

$$\zeta_{ij}^a : \{y_i \mapsto y_i y_j x y_j x^{-1}\}, \qquad \zeta_{ij}^b : \{y_i \mapsto x^{-1} y_j x y_j y_i\},$$
$$\zeta_{ij}^c : \{y_i \mapsto y_i y_j x^{-1} y_j x\}, \qquad \zeta_{ij}^d : \{y_i \mapsto x y_j x^{-1} y_j y_i\},$$

$$\mathrm{kon}_{ijx}^- : \left\{ \begin{array}{lll} y_i & \mapsto & xy_i^{-1}x^{-1} \\ y_j & \mapsto & xy_j^{-1}x^{-1} \end{array} \right\}.$$

◇

Lemma 4.11. *The following automorphisms are in K_n*

- *for $n \geq 2$ and $1 \leq i \leq n-1$:*

$$\mathrm{kon}_{xi} : \{x \mapsto y_i x y_i^{-1}\}, \qquad \varphi_i^2 : \{y_i \mapsto x^2 y_i\},$$
$$\mathrm{kon}_{ix}^2 : \{y_i \mapsto x^2 y_i x^{-2}\},$$

- *for $n \geq 3$ and $1 \leq i, j \leq n-1$, $i \neq j$:*

$$\kappa_{jk} : \{x \mapsto x y_j y_k y_j^{-1} y_k^{-1}\}, \qquad \mathrm{kon}_{ij} : \{y_i \mapsto y_j y_i y_j^{-1}\},$$
$$\zeta_{ij}^a : \{y_i \mapsto y_i y_j x y_j x^{-1}\}, \qquad \zeta_{ij}^b : \{y_i \mapsto x^{-1} y_j x y_j y_i\},$$
$$\zeta_{ij}^c : \{y_i \mapsto y_i y_j x^{-1} y_j x\}, \qquad \zeta_{ij}^d : \{y_i \mapsto x y_j x^{-1} y_j y_i\},$$

$$\mathrm{kon}_{ijx}^- : \left\{ \begin{array}{lll} y_i & \mapsto & xy_i^{-1}x^{-1} \\ y_j & \mapsto & xy_j^{-1}x^{-1} \end{array} \right\},$$

- *for $n \geq 4$ and $1 \leq i,j,k \leq n-1$, i,j,k pairwise distinct:*

$$\kappa_{ijk} : \{y_i \mapsto y_i y_j y_k y_j^{-1} y_k^{-1}\}.$$

Proof. The following relations, which we are going to prove below, hold in K_n.
For $n \geq 2$ and $1 \leq i \leq n-1$:

$$\begin{aligned}
\mathrm{kon}_{xi} &= \delta_i \circ \varepsilon_i^{-1}, \\
\mathrm{kon}_{ix}^2 &= \alpha_i \circ \beta_i, \\
\varphi_i^2 &= \mathrm{kon}_{ix}^2 \circ \psi_i^2.
\end{aligned}$$

For $n \geq 3$ and $1 \leq i,j \leq n-1$, $i \neq j$:

$$\begin{aligned}
\mathrm{kon}_{ijx}^- &= \alpha_i \circ \beta_j, \\
\kappa_{ij} &= \varepsilon_i \circ \varepsilon_j \circ \varepsilon_i^{-1} \circ \varepsilon_j^{-1}, \\
\mathrm{kon}_{ij} &= \beta_i \circ \varepsilon_j^{-1} \circ \beta_i \circ \delta_j^{-1}, \\
\zeta_{ij}^c &= \varepsilon_j^{-1} \circ \psi_i^{-2} \circ \varepsilon_j \circ \psi_i^2, \\
\zeta_{ij}^a &= \mathrm{kon}_{ij}^{-1} \circ \psi_i^2 \circ \mathrm{kon}_{ij} \circ \zeta_{ij}^c \circ \psi_i^{-2}, \\
\zeta_{ij}^b &= \alpha_i \circ (\zeta_{ij}^c)^{-1} \circ \alpha_i, \\
\zeta_{ij}^d &= \beta_i \circ (\zeta_{ij}^a)^{-1} \circ \beta_i.
\end{aligned}$$

For $n \geq 4$ and $1 \leq i,j,k \leq n-1$ and i,j,k pairwise distinct:

$$\begin{aligned}
\kappa_{ijk} = {} & \mathrm{kon}_{xj} \circ \varepsilon_j \circ (\mathrm{kon}_{ijx}^-)^{-1} \circ \varepsilon_j \circ \mathrm{kon}_{ijx}^- \circ \mathrm{kon}_{ik}^{-1} \circ \zeta_{ik}^d \circ (\mathrm{kon}_{ijx}^-)^{-1} \circ \\
& \varepsilon_j^{-1} \circ \mathrm{kon}_{ijx}^- \circ (\zeta_{ik}^d)^{-1} \circ \mathrm{kon}_{ik} \circ \varepsilon_j^{-1} \circ \mathrm{kon}_{xj}^{-1}.
\end{aligned}$$

We prove now all these formulas in detail:

- $\mathrm{kon}_{xi} = \delta_i \circ \varepsilon_i^{-1}$:
  See proof of Corollary 4.4,

- $\mathrm{kon}_{ix}^2 = \alpha_i \circ \beta_i$:

$$\left\{ \begin{array}{lclcl} x & \overset{\beta_i}{\mapsto} & x^{-1} & \overset{\alpha_i}{\mapsto} & x \\ y_i & \overset{\beta_i}{\mapsto} & x^{-1} y_i^{-1} x & \overset{\alpha_i}{\mapsto} & x(x y_i^{-1} x^{-1})^{-1} x^{-1} = x^2 y_i x^{-2} \end{array} \right\},$$

- $\varphi_i^2 = \mathrm{kon}_{ix}^2 \circ \psi_i^2$:
  See proof of Corollary 4.4,

- $\mathrm{kon}_{ijx}^{-} = \alpha_i \circ \beta_j$:
  $$\left\{\begin{array}{lllll} x & \stackrel{\beta_j}{\mapsto} & x^{-1} & \stackrel{\alpha_i}{\mapsto} & x \\ y_i & \stackrel{\beta_j}{\mapsto} & y_i & \stackrel{\alpha_i}{\mapsto} & xy_i^{-1}x^{-1} \\ y_j & \stackrel{\beta_j}{\mapsto} & x^{-1}y_j^{-1}x & \stackrel{\alpha_i}{\mapsto} & xy_j^{-1}x^{-1} \end{array}\right\},$$

- $\kappa_{ij} = \varepsilon_i \circ \varepsilon_j \circ \varepsilon_i^{-1} \circ \varepsilon_j^{-1}$:
  See proof of Corollary 4.4,

- $\mathrm{kon}_{ij} = \beta_i \circ \varepsilon_j^{-1} \circ \beta_i \circ \delta_j^{-1}$:
  $$\left\{\begin{array}{lllllllll} x & \stackrel{\delta_j^{-1}}{\mapsto} & y_j^{-1}x & \stackrel{\beta_i}{\mapsto} & y_j^{-1}x^{-1} & \stackrel{\varepsilon_j^{-1}}{\mapsto} & x^{-1} & \stackrel{\beta_i}{\mapsto} & x \\ y_i & \stackrel{\delta_j^{-1}}{\mapsto} & y_i & \stackrel{\beta_i}{\mapsto} & x^{-1}y_i^{-1}x & \stackrel{\varepsilon_j^{-1}}{\mapsto} & y_jx^{-1}y_i^{-1}xy_j^{-1} & \stackrel{\beta_i}{\mapsto} & y_jy_iy_j^{-1} \end{array}\right\},$$

- $\zeta_{ij}^c = \varepsilon_j^{-1} \circ \psi_i^{-2} \circ \varepsilon_j \circ \psi_i^2$:
  $$\left\{\begin{array}{lllllllll} x & \stackrel{\psi_i^2}{\mapsto} & x & \stackrel{\varepsilon_j}{\mapsto} & xy_j & \stackrel{\psi_i^{-2}}{\mapsto} & xy_j & \stackrel{\varepsilon_j^{-1}}{\mapsto} & x \\ y_i & \stackrel{\psi_i^2}{\mapsto} & y_ix^2 & \stackrel{\varepsilon_j}{\mapsto} & y_ixy_jxy_j & \stackrel{\psi_i^{-2}}{\mapsto} & y_ix^{-1}y_jxy_j & \stackrel{\varepsilon_j^{-1}}{\mapsto} & y_iy_jx^{-1}y_jx \end{array}\right\},$$

- $\zeta_{ij}^a = \mathrm{kon}_{ij}^{-1} \circ \psi_i^2 \circ \mathrm{kon}_{ij} \circ \zeta_{ij}^c \circ \psi_i^{-2}$:
  $$\Big\{\; y_i \stackrel{\psi_i^{-2}}{\mapsto} y_ix^{-2} \stackrel{\zeta_{ij}^c}{\mapsto} y_iy_jx^{-1}y_jx^{-1} \stackrel{\mathrm{kon}_{ij}}{\mapsto} y_jy_ix^{-1}y_jx^{-1}$$
  $$\stackrel{\psi_i^2}{\mapsto} y_jy_ixy_jx^{-1} \stackrel{\mathrm{kon}_{ij}^{-1}}{\mapsto} y_iy_jxy_jx^{-1} \;\Big\},$$

- $\zeta_{ij}^b = \alpha_i \circ (\zeta_{ij}^c)^{-1} \circ \alpha_i$:
  $$\left\{\begin{array}{lllllll} x & \stackrel{\alpha_i}{\mapsto} & x^{-1} & \stackrel{(\zeta_{ij}^c)^{-1}}{\mapsto} & x^{-1} & \stackrel{\alpha_i}{\mapsto} & x \\ y_i & \stackrel{\alpha_i}{\mapsto} & xy_i^{-1}x^{-1} & \stackrel{(\zeta_{ij}^c)^{-1}}{\mapsto} & xy_jx^{-1}y_jxy_i^{-1}x^{-1} & \stackrel{\alpha_i}{\mapsto} & x^{-1}y_jxy_jy_i \end{array}\right\},$$

- $\zeta^d_{ij} = \beta_i \circ (\zeta^a_{ij})^{-1} \circ \beta_i$:

$$\left\{\begin{array}{lllllll} x & \overset{\beta_i}{\longmapsto} & x^{-1} & \overset{(\zeta^a_{ij})^{-1}}{\longmapsto} & x^{-1} & \overset{\beta_i}{\longmapsto} & x \\ y_i & \overset{\beta_i}{\longmapsto} & x^{-1}y_i^{-1}x & \overset{(\zeta^a_{ij})^{-1}}{\longmapsto} & x^{-1}y_jxy_jx^{-1}y_i^{-1}x & \overset{\beta_i}{\longmapsto} & xy_jx^{-1}y_jy_i \end{array}\right\},$$

- $\kappa_{ijk} = \mathrm{kon}_{ij}^{-1} \circ \mathrm{kon}_{ik}^{-1} \circ \zeta^d_{ik} \circ (\mathrm{kon}^-_{ijx})^{-1} \circ \varepsilon_j^{-1} \circ \mathrm{kon}^-_{ijx} \circ (\zeta^d_{ik})^{-1} \circ \mathrm{kon}_{ik} \circ \delta_j^{-1}$:

$$\left\{\begin{array}{lllllll} x & \overset{\delta_j^{-1}}{\longmapsto} & y_j^{-1}x & \overset{\mathrm{kon}_{ik}}{\longmapsto} & y_j^{-1}x & \overset{(\zeta^d_{ik})^{-1}}{\longmapsto} & y_j^{-1}x \\ y_i & \overset{\delta_j^{-1}}{\longmapsto} & y_i & \overset{\mathrm{kon}_{ik}}{\longmapsto} & y_ky_iy_k^{-1} & \overset{(\zeta^d_{ik})^{-1}}{\longmapsto} & xy_k^{-1}x^{-1}y_iy_k^{-1} \\ y_j & \overset{\delta_j^{-1}}{\longmapsto} & y_j & \overset{\mathrm{kon}_{ik}}{\longmapsto} & y_j & \overset{(\zeta^d_{ik})^{-1}}{\longmapsto} & y_j \end{array}\right.$$

$$\begin{array}{llll} \overset{\mathrm{kon}^-_{ijx}}{\longmapsto} & xy_j & \overset{\varepsilon_j^{-1}}{\longmapsto} & x \\ \overset{\mathrm{kon}^-_{ijx}}{\longmapsto} & xy_k^{-1}y_i^{-1}x^{-1}y_k^{-1} & \overset{\varepsilon_j^{-1}}{\longmapsto} & xy_j^{-1}y_k^{-1}y_i^{-1}y_jx^{-1}y_k^{-1} \\ \overset{\mathrm{kon}^-_{ijx}}{\longmapsto} & xy_j^{-1}x^{-1} & \overset{\varepsilon_j^{-1}}{\longmapsto} & xy_j^{-1}x^{-1} \end{array}$$

$$\begin{array}{llll} \overset{(\mathrm{kon}^-_{ijx})^{-1}}{\longmapsto} & x & \overset{\zeta^d_{ik}}{\longmapsto} & x \\ \overset{(\mathrm{kon}^-_{ijx})^{-1}}{\longmapsto} & y_jxy_k^{-1}x^{-1}y_iy_j^{-1}y_k^{-1} & \overset{\zeta^d_{ik}}{\longmapsto} & y_jy_ky_iy_j^{-1}y_k^{-1} \\ \overset{(\mathrm{kon}^-_{ijx})^{-1}}{\longmapsto} & y_j & \overset{\zeta^d_{ik}}{\longmapsto} & y_j \end{array}$$

$$\left.\begin{array}{llll} \overset{\mathrm{kon}_{ik}^{-1}}{\longmapsto} & x & \overset{\mathrm{kon}_{ij}^{-1}}{\longmapsto} & x \\ \overset{\mathrm{kon}_{ik}^{-1}}{\longmapsto} & y_jy_iy_ky_j^{-1}y_k^{-1} & \overset{\mathrm{kon}_{ij}^{-1}}{\longmapsto} & y_iy_jy_ky_j^{-1}y_k^{-1} \\ \overset{\mathrm{kon}_{ik}^{-1}}{\longmapsto} & y_j & \overset{\mathrm{kon}_{ij}^{-1}}{\longmapsto} & y_j \end{array}\right\}.$$

□

Lemma 4.12. *The group K_n is a normal subgroup in $\Gamma^+(C_2, \pi)$.*

Proof. By Lemma 1.10 it suffices to conjugate the generators of K_n with the generators of $\Gamma^+(C_2, \pi)$ and their inverses. The generators of K_n are by definition ε_i, δ_i and ψ_i^2 $(1 \leq i \leq n-1)$ and the generators of $\Gamma^+(C_2, \pi)$ are, by Corollary 4.4, the automorphisms ε_i, ψ_i^2, kon_{ix} $(1 \leq i \leq n-1)$ and ν_{ij} $(1 \leq i, j \leq n-1,\ i \neq j)$. Since ε_i and ψ_i^2 are automorphisms in

K_n as well as in $\Gamma^+(C_2,\pi)$, it suffices to conjugate with ν_{ij}, ν_{ij}^{-1}, kon_{ix} and kon_{ix}^{-1}.

In order to reduce the formulas in the proof, let us note here:

- We do not need to conjugate with kon_{ix}^{-1}. To see this, suppose that $\mathrm{kon}_{ix} \circ \alpha \circ \mathrm{kon}_{ix}^{-1} \in K_n$ for some $\alpha \in K_n$. Since $\mathrm{kon}_{ix}^2 \in K_n$ for $n \geq 2$, by Lemma 4.11, it follows that

$$\mathrm{kon}_{ix}^{-1} \circ \alpha \circ \mathrm{kon}_{ix} = \underbrace{\mathrm{kon}_{ix}^{-2}}_{\in K_n} \circ \big(\underbrace{\mathrm{kon}_{ix} \circ \alpha \circ \mathrm{kon}_{ix}^{-1}}_{\in K_n} \big) \circ \underbrace{\mathrm{kon}_{ix}^{2}}_{\in K_n}$$

  is also in K_n.

- Notice that the automorphisms ε_i, δ_i and ψ_i^2 commute with ν_{jk} if $j \neq i$ and $k \neq i$. Hence we do not need to conjugate ε_i, δ_i and ψ_i^2 with ν_{jk} if $j \neq i$ and $k \neq i$.

First we list the results and prove the formulas below.

1.)
$$\begin{aligned}
&\nu_{ij} \circ \varepsilon_i \circ \nu_{ij}^{-1} = \varepsilon_i \circ \varepsilon_j \in K_n,\\
&\nu_{ji} \circ \varepsilon_i \circ \nu_{ji}^{-1} = \varepsilon_i \in K_n,\\
&\nu_{ij}^{-1} \circ \varepsilon_i \circ \nu_{ij} = \varepsilon_i \circ \varepsilon_j^{-1} \in K_n,\\
&\nu_{ji}^{-1} \circ \varepsilon_i \circ \nu_{ji} = \varepsilon_i \in K_n,\\
&\mathrm{kon}_{ix} \circ \varepsilon_i \circ \mathrm{kon}_{ix}^{-1} = \mathrm{kon}_{ix}^2 \circ \delta_i \circ \mathrm{kon}_{ix}^{-2} \in K_n,\\
&\mathrm{kon}_{jx} \circ \varepsilon_i \circ \mathrm{kon}_{jx}^{-1} = \varepsilon_i \circ \mathrm{kon}_{ji}^{-1} \in K_n,
\end{aligned}$$

2.)
$$\begin{aligned}
&\nu_{ij} \circ \delta_i \circ \nu_{ij}^{-1} = \delta_j \circ \delta_i \in K_n,\\
&\nu_{ji} \circ \delta_i \circ \nu_{ji}^{-1} = \delta_i \in K_n,\\
&\nu_{ij}^{-1} \circ \delta_i \circ \nu_{ij} = \delta_j^{-1} \circ \delta_i \in K_n,\\
&\nu_{ji}^{-1} \circ \delta_i \circ \nu_{ji} = \delta_i \in K_n,\\
&\mathrm{kon}_{ix} \circ \delta_i \circ \mathrm{kon}_{ix}^{-1} = \varepsilon_i \in K_n,\\
&\mathrm{kon}_{jx} \circ \delta_i \circ \mathrm{kon}_{jx}^{-1} = \delta_i \circ \alpha_j \circ \mathrm{kon}_{ji}^{-1} \circ \alpha_j \in K_n,
\end{aligned}$$

3.)
$$\begin{aligned}
&\nu_{ij} \circ \psi_i^2 \circ \nu_{ij}^{-1} = \mathrm{kon}_{ij}^{-1} \circ \psi_i^2 \circ \mathrm{kon}_{ij} \in K_n,\\
&\nu_{ji} \circ \psi_i^2 \circ \nu_{ji}^{-1} = \psi_i^2 \circ \mathrm{kon}_{ji}^{-1} \circ \psi_j^{-2} \circ \mathrm{kon}_{ji} \in K_n,\\
&\nu_{ij}^{-1} \circ \psi_i^2 \circ \nu_{ij} = \mathrm{kon}_{ij} \circ \psi_i^2 \circ \mathrm{kon}_{ij}^{-1} \in K_n,\\
&\nu_{ji}^{-1} \circ \psi_i^2 \circ \nu_{ji} = \psi_j^2 \circ \psi_i^2 \in K_n,\\
&\mathrm{kon}_{ix} \circ \psi_i^2 \circ \mathrm{kon}_{ix}^{-1} = \psi_i^2 \in K_n,\\
&\mathrm{kon}_{jx} \circ \psi_i^2 \circ \mathrm{kon}_{jx}^{-1} = \psi_i^2 \in K_n.
\end{aligned}$$

We check these formulas now in detail:

- $\nu_{ij} \circ \varepsilon_i \circ \nu_{ij}^{-1} = \varepsilon_i \circ \varepsilon_j$:
$$\left\{\begin{array}{lllllll} x & \overset{\nu_{ij}^{-1}}{\mapsto} & x & \overset{\varepsilon_i}{\mapsto} & xy_i & \overset{\nu_{ij}}{\mapsto} & xy_iy_j \\ y_i & \overset{\nu_{ij}^{-1}}{\mapsto} & y_iy_j^{-1} & \overset{\varepsilon_i}{\mapsto} & y_iy_j^{-1} & \overset{\nu_{ij}}{\mapsto} & y_i \end{array}\right\},$$

- $\nu_{ji} \circ \varepsilon_i \circ \nu_{ji}^{-1} = \varepsilon_i$:
$$\left\{\begin{array}{lllllll} x & \overset{\nu_{ji}^{-1}}{\mapsto} & x & \overset{\varepsilon_i}{\mapsto} & xy_i & \overset{\nu_{ji}}{\mapsto} & xy_i \\ y_j & \overset{\nu_{ji}^{-1}}{\mapsto} & y_jy_i^{-1} & \overset{\varepsilon_i}{\mapsto} & y_jy_i^{-1} & \overset{\nu_{ji}}{\mapsto} & y_j \end{array}\right\},$$

- $\nu_{ij}^{-1} \circ \varepsilon_i \circ \nu_{ij} = \varepsilon_i \circ \varepsilon_j^{-1}$:
$$\left\{\begin{array}{lllllll} x & \overset{\nu_{ij}}{\mapsto} & x & \overset{\varepsilon_i}{\mapsto} & xy_i & \overset{\nu_{ij}^{-1}}{\mapsto} & xy_iy_j^{-1} \\ y_i & \overset{\nu_{ij}}{\mapsto} & y_iy_j & \overset{\varepsilon_i}{\mapsto} & y_iy_j & \overset{\nu_{ij}^{-1}}{\mapsto} & y_i \end{array}\right\},$$

- $\nu_{ji}^{-1} \circ \varepsilon_i \circ \nu_{ji} = \varepsilon_i$:
$$\left\{\begin{array}{lllllll} x & \overset{\nu_{ji}}{\mapsto} & x & \overset{\varepsilon_i}{\mapsto} & xy_i & \overset{\nu_{ji}^{-1}}{\mapsto} & xy_i \\ y_j & \overset{\nu_{ji}}{\mapsto} & y_jy_i & \overset{\varepsilon_i}{\mapsto} & y_jy_i & \overset{\nu_{ji}^{-1}}{\mapsto} & y_j \end{array}\right\},$$

- $\mathrm{kon}_{ix} \circ \varepsilon_i \circ \mathrm{kon}_{ix}^{-1} = \mathrm{kon}_{ix}^2 \circ \delta_i \circ \mathrm{kon}_{ix}^{-2}$:
$$\left\{\begin{array}{lllllll} x & \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} & x & \overset{\varepsilon_i}{\mapsto} & xy_i & \overset{\mathrm{kon}_{ix}}{\mapsto} & x^2y_ix^{-1} \\ y_i & \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} & x^{-1}y_ix & \overset{\varepsilon_i}{\mapsto} & y_i^{-1}x^{-1}y_ixy_i & \overset{\mathrm{kon}_{ix}}{\mapsto} & xy_i^{-1}x^{-1}y_ixy_ix^{-1} \end{array}\right\}$$

$$= \left\{\begin{array}{lllll} x & \overset{\mathrm{kon}_{ix}^{-2}}{\mapsto} & x & \overset{\delta_i}{\mapsto} & y_ix \\ y_i & \overset{\mathrm{kon}_{ix}^{-2}}{\mapsto} & x^{-2}y_ix^2 & \overset{\delta_i}{\mapsto} & x^{-1}y_i^{-1}x^{-1}y_ixy_ix \end{array}\right.$$

$$\left.\begin{array}{ll} \overset{\mathrm{kon}_{ix}^{2}}{\mapsto} & x^2y_ix^{-1} \\ \overset{\mathrm{kon}_{ix}^{2}}{\mapsto} & xy_i^{-1}x^{-1}y_ixy_ix^{-1} \end{array}\right\},$$

- $\mathrm{kon}_{jx} \circ \varepsilon_i \circ \mathrm{kon}_{jx}^{-1} = \varepsilon_i \circ \mathrm{kon}_{ji}^{-1}$:
$$\left\{ \begin{array}{lllllll} x & \overset{\mathrm{kon}_{jx}^{-1}}{\longmapsto} & x & \overset{\varepsilon_i}{\longmapsto} & xy_i & \overset{\mathrm{kon}_{jx}}{\longmapsto} & xy_i \\ y_j & \overset{\mathrm{kon}_{jx}^{-1}}{\longmapsto} & x^{-1}y_jx & \overset{\varepsilon_i}{\longmapsto} & y_i^{-1}x^{-1}y_jxy_i & \overset{\mathrm{kon}_{jx}}{\longmapsto} & y_i^{-1}y_jy_i \end{array} \right\}$$
$$= \left\{ \begin{array}{lllll} x & \overset{\mathrm{kon}_{ji}^{-1}}{\longmapsto} & x & \overset{\varepsilon_i}{\longmapsto} & xy_i \\ y_j & \overset{\mathrm{kon}_{ji}^{-1}}{\longmapsto} & y_i^{-1}y_jy_i & \overset{\varepsilon_i}{\longmapsto} & y_i^{-1}y_jy_i \end{array} \right\},$$

- $\nu_{ij} \circ \delta_i \circ \nu_{ij}^{-1} = \delta_j \circ \delta_i$:
$$\left\{ \begin{array}{lllllll} x & \overset{\nu_{ij}^{-1}}{\longmapsto} & x & \overset{\delta_i}{\longmapsto} & y_ix & \overset{\nu_{ij}}{\longmapsto} & y_iy_jx \\ y_i & \overset{\nu_{ij}^{-1}}{\longmapsto} & y_iy_j^{-1} & \overset{\delta_i}{\longmapsto} & y_iy_j^{-1} & \overset{\nu_{ij}}{\longmapsto} & y_i \end{array} \right\},$$

- $\nu_{ji} \circ \delta_i \circ \nu_{ji}^{-1} = \delta_i$:
$$\left\{ \begin{array}{lllllll} x & \overset{\nu_{ji}^{-1}}{\longmapsto} & x & \overset{\delta_i}{\longmapsto} & y_ix & \overset{\nu_{ji}}{\longmapsto} & y_ix \\ y_j & \overset{\nu_{ji}^{-1}}{\longmapsto} & y_jy_i^{-1} & \overset{\delta_i}{\longmapsto} & y_jy_i^{-1} & \overset{\nu_{ji}}{\longmapsto} & y_j \end{array} \right\},$$

- $\nu_{ij}^{-1} \circ \delta_i \circ \nu_{ij} = \delta_j^{-1} \circ \delta_i$:
$$\left\{ \begin{array}{lllllll} x & \overset{\nu_{ij}}{\longmapsto} & x & \overset{\delta_i}{\longmapsto} & y_ix & \overset{\nu_{ij}^{-1}}{\longmapsto} & y_iy_j^{-1}x \\ y_i & \overset{\nu_{ij}}{\longmapsto} & y_iy_j & \overset{\delta_i}{\longmapsto} & y_iy_j & \overset{\nu_{ij}^{-1}}{\longmapsto} & y_i \end{array} \right\},$$

- $\nu_{ji}^{-1} \circ \delta_i \circ \nu_{ji} = \delta_i$:
$$\left\{ \begin{array}{lllllll} x & \overset{\nu_{ji}}{\longmapsto} & x & \overset{\delta_i}{\longmapsto} & y_ix & \overset{\nu_{ji}^{-1}}{\longmapsto} & y_ix \\ y_j & \overset{\nu_{ji}}{\longmapsto} & y_jy_i & \overset{\delta_i}{\longmapsto} & y_jy_i & \overset{\nu_{ji}^{-1}}{\longmapsto} & y_j \end{array} \right\},$$

- $\mathrm{kon}_{ix} \circ \delta_i \circ \mathrm{kon}_{ix}^{-1} = \varepsilon_i$:
$$\left\{ \begin{array}{lllllll} x & \overset{\mathrm{kon}_{ix}^{-1}}{\longmapsto} & x & \overset{\delta_i}{\longmapsto} & y_ix & \overset{\mathrm{kon}_{ix}}{\longmapsto} & xy_i \\ y_i & \overset{\mathrm{kon}_{ix}^{-1}}{\longmapsto} & x^{-1}y_ix & \overset{\delta_i}{\longmapsto} & x^{-1}y_ix & \overset{\mathrm{kon}_{ix}}{\longmapsto} & y_i \end{array} \right\},$$

- $\mathrm{kon}_{jx} \circ \delta_i \circ \mathrm{kon}_{jx}^{-1} = \delta_i \circ \alpha_j \circ \mathrm{kon}_{ji}^{-1} \circ \alpha_j$:

$$\left\{\begin{array}{lllllll} x & \overset{\mathrm{kon}_{jx}^{-1}}{\mapsto} & x & \overset{\delta_i}{\mapsto} & y_i x & \overset{\mathrm{kon}_{jx}}{\mapsto} & y_i x \\ y_j & \overset{\mathrm{kon}_{jx}^{-1}}{\mapsto} & x^{-1}y_jx & \overset{\delta_i}{\mapsto} & x^{-1}y_i^{-1}y_jy_ix & \overset{\mathrm{kon}_{jx}}{\mapsto} & x^{-1}y_i^{-1}xy_jx^{-1}y_ix \end{array}\right\}$$

$$= \left\{\begin{array}{lllll} x & \overset{\alpha_j}{\mapsto} & x^{-1} & \overset{\mathrm{kon}_{ji}^{-1}}{\mapsto} & x^{-1} \\ y_j & \overset{\alpha_j}{\mapsto} & xy_j^{-1}x^{-1} & \overset{\mathrm{kon}_{ji}^{-1}}{\mapsto} & xy_i^{-1}y_j^{-1}y_ix^{-1} \end{array}\right.$$

$$\left.\begin{array}{llll} \overset{\alpha_j}{\mapsto} & x & \overset{\delta_i}{\mapsto} & y_ix \\ \overset{\alpha_j}{\mapsto} & x^{-1}y_i^{-1}xy_jx^{-1}y_ix & \overset{\delta_i}{\mapsto} & x^{-1}y_i^{-1}xy_jx^{-1}y_ix \end{array}\right\},$$

- $\nu_{ij} \circ \psi_i^2 \circ \nu_{ij}^{-1} = \mathrm{kon}_{ij}^{-1} \circ \psi_i^2 \circ \mathrm{kon}_{ij}$:

$$\left\{\begin{array}{lllllll} y_i & \overset{\nu_{ij}^{-1}}{\mapsto} & y_iy_j^{-1} & \overset{\psi_i^2}{\mapsto} & y_ix^2y_j^{-1} & \overset{\nu_{ij}}{\mapsto} & y_iy_jx^2y_j^{-1} \end{array}\right\} =$$

$$\left\{\begin{array}{lllllll} y_i & \overset{\mathrm{kon}_{ij}}{\mapsto} & y_jy_iy_j^{-1} & \overset{\psi_i^2}{\mapsto} & y_jy_ix^2y_j^{-1} & \overset{\mathrm{kon}_{ij}^{-1}}{\mapsto} & y_iy_jx^2y_j^{-1} \end{array}\right\},$$

- $\nu_{ji} \circ \psi_i^2 \circ \nu_{ji}^{-1} = \psi_i^2 \circ \mathrm{kon}_{ji}^{-1} \circ \psi_j^{-2} \circ \mathrm{kon}_{ji}$:

$$\left\{\begin{array}{lllllll} y_i & \overset{\nu_{ji}^{-1}}{\mapsto} & y_i & \overset{\psi_i^2}{\mapsto} & y_ix^2 & \overset{\nu_{ji}}{\mapsto} & y_ix^2 \\ y_j & \overset{\nu_{ji}^{-1}}{\mapsto} & y_jy_i^{-1} & \overset{\psi_i^2}{\mapsto} & y_jx^{-2}y_i^{-1} & \overset{\nu_{ji}}{\mapsto} & y_jy_ix^{-2}y_i^{-1} \end{array}\right\} =$$

$$\left\{\begin{array}{lllllll} y_i & \overset{\mathrm{kon}_{ji}}{\mapsto} & y_i & \overset{\psi_j^{-2}}{\mapsto} & y_i & \overset{\mathrm{kon}_{ji}^{-1}}{\mapsto} & y_i \\ y_j & \overset{\mathrm{kon}_{ji}}{\mapsto} & y_iy_jy_i^{-1} & \overset{\psi_j^{-2}}{\mapsto} & y_iy_jx^{-2}y_i^{-1} & \overset{\mathrm{kon}_{ji}^{-1}}{\mapsto} & y_jy_ix^{-2}y_i^{-1} \end{array}\right.$$

$$\left.\begin{array}{ll} \overset{\psi_i^2}{\mapsto} & y_ix^2 \\ \overset{\psi_i^2}{\mapsto} & y_jy_ix^{-2}y_i^{-1} \end{array}\right\},$$

- $\nu_{ij}^{-1} \circ \psi_i^2 \circ \nu_{ij} = \mathrm{kon}_{ij} \circ \psi_i^2 \circ \mathrm{kon}_{ij}^{-1}$:

$$\left\{\begin{array}{lllllll} y_i & \overset{\nu_{ij}}{\mapsto} & y_iy_j & \overset{\psi_i^2}{\mapsto} & y_ix^2y_j & \overset{\nu_{ij}^{-1}}{\mapsto} & y_iy_j^{-1}x^2y_j \end{array}\right\} =$$

$$\left\{\begin{array}{lllllll} y_i & \overset{\mathrm{kon}_{ij}^{-1}}{\mapsto} & y_j^{-1}y_iy_j & \overset{\psi_i^2}{\mapsto} & y_j^{-1}y_ix^2y_j & \overset{\mathrm{kon}_{ij}}{\mapsto} & y_iy_j^{-1}x^2y_j \end{array}\right\},$$

- $\nu_{ji}^{-1} \circ \psi_i^2 \circ \nu_{ji} = \psi_j^2 \circ \psi_i^2$:

$$\left\{ \begin{array}{ccccccc} y_i & \overset{\nu_{ji}}{\mapsto} & y_i & \overset{\psi_i^2}{\mapsto} & y_i x^2 & \overset{\nu_{ji}^{-1}}{\mapsto} & y_i x^2 \\ y_j & \overset{\nu_{ji}}{\mapsto} & y_j y_i & \overset{\psi_i^2}{\mapsto} & y_j y_i x^2 & \overset{\nu_{ji}^{-1}}{\mapsto} & y_j x^2 \end{array} \right\},$$

- $\mathrm{kon}_{ix} \circ \psi_i^2 \circ \mathrm{kon}_{ix}^{-1} = \psi_i^2$:

$$\left\{ \begin{array}{ccccccc} y_i & \overset{\mathrm{kon}_{ix}^{-1}}{\mapsto} & x^{-1} y_i x & \overset{\psi_i^2}{\mapsto} & x^{-1} y_i x^3 & \overset{\mathrm{kon}_{ix}}{\mapsto} & y_i x^2 \end{array} \right\},$$

- $\mathrm{kon}_{jx} \circ \psi_i^2 \circ \mathrm{kon}_{jx}^{-1} = \psi_i^2$:

$$\left\{ \begin{array}{ccccccc} y_i & \overset{\mathrm{kon}_{jx}^{-1}}{\mapsto} & y_i & \overset{\psi_i^2}{\mapsto} & y_i x^2 & \overset{\mathrm{kon}_{jx}}{\mapsto} & y_i x^2 \\ y_j & \overset{\mathrm{kon}_{jx}^{-1}}{\mapsto} & x^{-1} y_j x & \overset{\psi_i^2}{\mapsto} & x^{-1} y_j x & \overset{\mathrm{kon}_{jx}}{\mapsto} & y_j \end{array} \right\}.$$

□

Lemma 4.13. *Let* $n \geq 2$ *and let* $\mathrm{GL}(n-1, \mathbb{Z})$ *be presented as in Proposition* 1.5 *and let* $R(E_{ij}, O_k)$ *be the corresponding set of relations. Then*

$$R(\nu_{ji}, \mathrm{kon}_{kx}) \in K_n.$$

Proof. Let us begin with the case $n = 2$. But there is only one relation in the group $\mathrm{GL}(1, \mathbb{Z})$, namely $O_1^2 = 1$. Thus we just have to show that $\mathrm{kon}_{1x}^2 \in K_2$. But this is clear by Lemma 4.11.

Let us now consider the case $n = 3$. According to Proposition 1.5 the group $\mathrm{GL}(2, \mathbb{Z})$ is generated by E_{12}, E_{21}, O_1 and O_2 subject to the following relations

1.) $E_{12} E_{21}^{-1} E_{12} E_{21} E_{12}^{-1} E_{21} = 1$,
2.) $(E_{12} E_{21}^{-1} E_{12})^4 = 1$,
3.) $(O_1 E_{12})^2 = 1$,
4.) $(O_1 E_{21})^2 = 1$,
5.) $O_1^2 = 1$,
6.) $E_{12}^{-1} E_{21}^2 O_1 E_{12} E_{21}^{-2} O_2^{-1} = 1$.

Hence we have to show

1.) $\nu_{21} \circ \nu_{12}^{-1} \circ \nu_{21} \circ \nu_{12} \circ \nu_{21}^{-1} \circ \nu_{12} \in K_3$,
2.) $(\nu_{21} \circ \nu_{12}^{-1} \circ \nu_{21})^4 \in K_3$,

3.) $(\mathrm{kon}_{1x} \circ \nu_{21})^2 \in K_3$,
4.) $(\mathrm{kon}_{1x} \circ \nu_{12})^2 \in K_3$,
5.) $\mathrm{kon}_{1x}^2 \in K_3$,
6.) $\nu_{21}^{-1} \circ \nu_{12}^2 \circ \mathrm{kon}_{1x} \circ \nu_{21} \circ \nu_{12}^{-2} \circ \mathrm{kon}_{2x}^{-1} \in K_3$.

1.) $\nu_{21} \circ \nu_{12}^{-1} \circ \nu_{21} \circ \nu_{12} \circ \nu_{21}^{-1} \circ \nu_{12} = \mathrm{id} \in K_3$:

$$\left\{ \begin{array}{lllllllll} y_1 & \overset{\nu_{12}}{\mapsto} & y_1 y_2 & \overset{\nu_{21}^{-1}}{\mapsto} & y_1 y_2 y_1^{-1} & \overset{\nu_{12}}{\mapsto} & y_1 y_2 y_1^{-1} & \overset{\nu_{21}}{\mapsto} & y_1 y_2 \\ y_2 & \overset{\nu_{12}}{\mapsto} & y_2 & \overset{\nu_{21}^{-1}}{\mapsto} & y_2 y_1^{-1} & \overset{\nu_{12}}{\mapsto} & y_1^{-1} & \overset{\nu_{21}}{\mapsto} & y_1^{-1} \end{array} \right.$$

$$\left. \begin{array}{llll} \overset{\nu_{12}^{-1}}{\mapsto} & y_1 & \overset{\nu_{21}}{\mapsto} & y_1 \\ \overset{\nu_{12}^{-1}}{\mapsto} & y_2 y_1^{-1} & \overset{\nu_{21}}{\mapsto} & y_2 \end{array} \right\}.$$

2.) $(\nu_{21} \circ \nu_{12}^{-1} \circ \nu_{21})^4 = \mathrm{kon}_{12} \circ \mathrm{kon}_{21} \circ \mathrm{kon}_{12}^{-1} \circ \mathrm{kon}_{21}^{-1} \in K_3$:

In order to compute $(\nu_{21} \circ \nu_{12}^{-1} \circ \nu_{21})^4$ first we define $\chi := \nu_{21} \circ \nu_{12}^{-1} \circ \nu_{21}$. Then we have:

$$\chi = \left\{ \begin{array}{lllllll} y_1 & \overset{\nu_{21}}{\mapsto} & y_1 & \overset{\nu_{12}^{-1}}{\mapsto} & y_1 y_2^{-1} & \overset{\nu_{21}}{\mapsto} & y_2^{-1} \\ y_2 & \overset{\nu_{21}}{\mapsto} & y_2 y_1 & \overset{\nu_{12}^{-1}}{\mapsto} & y_2 y_1 y_2^{-1} & \overset{\nu_{21}}{\mapsto} & y_2 y_1 y_2^{-1} \end{array} \right\},$$

$$\chi^2 = \left\{ \begin{array}{lllll} y_1 & \overset{\chi}{\mapsto} & y_2^{-1} & \overset{\chi}{\mapsto} & y_2 y_1^{-1} y_2^{-1} \\ y_2 & \overset{\chi}{\mapsto} & y_2 y_1 y_2^{-1} & \overset{\chi}{\mapsto} & y_2 y_1 y_2^{-1} y_1^{-1} y_2^{-1} \end{array} \right\},$$

$$\chi^4 = \left\{ \begin{array}{lllll} y_1 & \overset{\chi^2}{\mapsto} & y_2 y_1^{-1} y_2^{-1} & \overset{\chi^2}{\mapsto} & y_2 y_1 y_2^{-1} y_1 y_2 y_1^{-1} y_2^{-1} \\ y_2 & \overset{\chi^2}{\mapsto} & y_2 y_1 y_2^{-1} y_1^{-1} y_2^{-1} & \overset{\chi^2}{\mapsto} & y_2 y_1 y_2^{-1} y_1^{-1} y_2 y_1 y_2 y_1^{-1} y_2^{-1} \end{array} \right\}.$$

In the last step we show that $\chi^4 = \mathrm{kon}_{12} \circ \mathrm{kon}_{21} \circ \mathrm{kon}_{12}^{-1} \circ \mathrm{kon}_{21}^{-1}$:

$$\left\{ \begin{array}{lllllll} y_1 & \overset{\mathrm{kon}_{21}^{-1}}{\mapsto} & y_1 & \overset{\mathrm{kon}_{12}^{-1}}{\mapsto} & y_2^{-1} y_1 y_2 & \overset{\mathrm{kon}_{21}}{\mapsto} & \\ y_2 & \overset{\mathrm{kon}_{21}^{-1}}{\mapsto} & y_1^{-1} y_2 y_1 & \overset{\mathrm{kon}_{12}^{-1}}{\mapsto} & y_2^{-1} y_1^{-1} y_2 y_1 y_2 & \overset{\mathrm{kon}_{21}}{\mapsto} & \end{array} \right.$$

$$\left.\begin{array}{lcl} y_1y_2^{-1}y_1y_2y_1^{-1} & \overset{\mathrm{kon}_{12}}{\mapsto} & y_2y_1y_2^{-1}y_1y_2y_1^{-1}y_2^{-1} \\ y_1y_2^{-1}y_1^{-1}y_2y_1y_2y_1^{-1} & \overset{\mathrm{kon}_{12}}{\mapsto} & y_2y_1y_2^{-1}y_1^{-1}y_2y_1y_2y_1^{-1}y_2^{-1} \end{array}\right\}.$$

3.) $\mathrm{kon}_{1x} \circ \nu_{21} \circ \mathrm{kon}_{1x} \circ \nu_{21} \in K_3$:
In order to show this let us be a little bit more general. For this let $n \geq 3$. We are going to show now

$$\mathrm{kon}_{1x} \circ \nu_{j1} \circ \mathrm{kon}_{1x} \circ \nu_{j1} = \mathrm{kon}_{1x}^2 \circ \mathrm{kon}_{j1} \circ \zeta_{j1}^c \circ \mathrm{kon}_{j1}^{-1} \in K_n$$

for $2 \leq j \leq n-1$:

$$\left\{\begin{array}{lclclcl} y_1 & \overset{\nu_{j1}}{\mapsto} & y_1 & \overset{\mathrm{kon}_{1x}}{\mapsto} & xy_1x^{-1} & \overset{\nu_{j1}}{\mapsto} & xy_1x^{-1} \\ y_j & \overset{\nu_{j1}}{\mapsto} & y_jy_1 & \overset{\mathrm{kon}_{1x}}{\mapsto} & y_jxy_1x^{-1} & \overset{\nu_{j1}}{\mapsto} & y_jy_1xy_1x^{-1} \end{array}\right.$$

$$\left.\begin{array}{cl} \overset{\mathrm{kon}_{1x}}{\mapsto} & x^2y_1x^{-2} \\ \overset{\mathrm{kon}_{1x}}{\mapsto} & y_jxy_1xy_1x^{-2} \end{array}\right\}$$

$$= \left\{\begin{array}{lclclcl} y_1 & \overset{\mathrm{kon}_{j1}^{-1}}{\mapsto} & y_1 & \overset{\zeta_{j1}^c}{\mapsto} & y_1 & \overset{\mathrm{kon}_{j1}}{\mapsto} & y_1 \\ y_j & \overset{\mathrm{kon}_{j1}^{-1}}{\mapsto} & y_1^{-1}y_jy_1 & \overset{\zeta_{j1}^c}{\mapsto} & y_1^{-1}y_jy_1x^{-1}y_1xy_1 & \overset{\mathrm{kon}_{j1}}{\mapsto} & y_jx^{-1}y_1xy_1 \end{array}\right.$$

$$\left.\begin{array}{cl} \overset{\mathrm{kon}_{1x}^2}{\mapsto} & x^2y_1x^{-2} \\ \overset{\mathrm{kon}_{1x}^2}{\mapsto} & y_jxy_1xy_1x^{-2} \end{array}\right\}.$$

4.) $\mathrm{kon}_{1x} \circ \nu_{12} \circ \mathrm{kon}_{1x} \circ \nu_{12} \in K_3$:
Let us be more general again. We show for $n \geq 3$ that

$$\mathrm{kon}_{1x} \circ \nu_{1i} \circ \mathrm{kon}_{1x} \circ \nu_{1i} = \mathrm{kon}_{1x}^2 \circ \mathrm{kon}_{1i} \circ \zeta_{1i}^a \circ \mathrm{kon}_{1i}^{-1} \in K_n$$

for $2 \leq i \leq n-1$:

$$\left\{\begin{array}{lclclcl} y_1 & \overset{\nu_{1i}}{\mapsto} & y_1y_i & \overset{\mathrm{kon}_{1x}}{\mapsto} & xy_1x^{-1}y_i & \overset{\nu_{1i}}{\mapsto} & xy_1y_ix^{-1}y_i \end{array}\right.$$

$$\left.\begin{array}{cl} \overset{\mathrm{kon}_{1x}}{\mapsto} & x^2y_1x^{-1}y_ix^{-1}y_i \end{array}\right\}$$

$$= \left\{ \begin{array}{l} y_1 \overset{\mathrm{kon}_{1i}^{-1}}{\longmapsto} y_i^{-1}y_1y_i \overset{\zeta_{1i}^a}{\longmapsto} y_i^{-1}y_1y_ixy_ix^{-1}y_i \overset{\mathrm{kon}_{1i}}{\longmapsto} y_1xy_ix^{-1}y_i \\ \qquad \overset{\mathrm{kon}_{1x}^2}{\longmapsto} x^2y_1x^{-1}y_ix^{-1}y_i \end{array} \right\}.$$

5.) $\mathrm{kon}_{1x}^2 \in K_3$: This is clear by the case $n = 2$.

6.) $\nu_{21}^{-1} \circ \nu_{12}^2 \circ \mathrm{kon}_{1x} \circ \nu_{21} \circ \nu_{12}^{-2} \circ \mathrm{kon}_{2x}^{-1} \in K_3$:
Let us here be more general again. We want to show that

$$\nu_{j1}^{-1} \circ \nu_{1j}^2 \circ \mathrm{kon}_{1x} \circ \nu_{j1} \circ \nu_{1j}^{-2} \circ \mathrm{kon}_{jx}^{-1} \in K_n$$

for $n \geq 3$ and $2 \leq j \leq n-1$. Checking this formula is the most complicated part of the complete proof. For this we mention first that the following equation holds

$$\begin{aligned} & \mathrm{kon}_{1x}^{-1} \circ \nu_{1j}^{-2} \circ \nu_{j1} \circ \mathrm{kon}_{jx} \circ \nu_{1j}^2 \circ \nu_{j1}^{-1} \\ = \; & \beta_1 \circ \mathrm{kon}_{1j}^2 \circ \beta_1 \circ \mathrm{kon}_{1x}^{-2} \circ \varepsilon_1 \circ \psi_1^{-2} \circ \delta_j^{-1} \circ \varphi_1^2 \circ \mathrm{kon}_{1j}^{-1} \circ \delta_1 \circ \mathrm{kon}_{j1} \circ \\ & \varphi_j^{-2} \circ \mathrm{kon}_{j1}^{-1} \circ \zeta_{j1}^b \circ \psi_j^2 \circ \mathrm{kon}_{1jx}^{-} \circ (\zeta_{1j}^a)^{-1} \circ \delta_j \in K_n \end{aligned}$$

for $2 \leq j \leq n-1$. For the proof see the next two pages:

$$\mathrm{kon}_{1x}^{-1} \circ \nu_{1j}^{-2} \circ \nu_{j1} \circ \mathrm{kon}_{jx} \circ \nu_{1j}^{2} \circ \nu_{j1}^{-1} =$$

$$\left\{\begin{array}{lllllllllll} y_1 & \overset{\nu_{j1}^{-1}}{\longmapsto} & y_1 & \overset{\nu_{1j}^{2}}{\longmapsto} & y_1 y_j^2 & \overset{\mathrm{kon}_{jx}}{\longmapsto} & y_1 x y_j^2 x^{-1} & \overset{\nu_{j1}}{\longmapsto} & y_1 x y_j y_1 y_j y_1 x^{-1} & \overset{\nu_{1j}^{-2}}{\longmapsto} & y_1 y_j^{-2} x y_j y_1 y_j^{-1} y_1 y_j^{-2} x^{-1} \\ y_j & \overset{\nu_{j1}^{-1}}{\longmapsto} & y_j y_1^{-1} & \overset{\nu_{1j}^{2}}{\longmapsto} & y_j^{-1} y_1^{-1} & \overset{\mathrm{kon}_{jx}}{\longmapsto} & x y_j^{-1} x^{-1} y_1^{-1} & \overset{\nu_{j1}}{\longmapsto} & x y_1^{-1} y_j^{-1} x^{-1} y_1^{-1} & \overset{\nu_{1j}^{-2}}{\longmapsto} & x y_j^2 y_1^{-1} y_j^{-1} x^{-1} y_j^2 y_1^{-1} \end{array}\right.$$

$$\left.\begin{array}{ll} \overset{\mathrm{kon}_{1x}^{-1}}{\longmapsto} & x^{-1} y_1 x y_j^{-2} x y_j x^{-1} y_1 x y_j^{-1} x^{-1} y_1 x y_j^{-2} x^{-1} \\ \overset{\mathrm{kon}_{1x}^{-1}}{\longmapsto} & x y_j^2 x^{-1} y_1^{-1} x y_j^{-1} x^{-1} y_j^2 x^{-1} y_1^{-1} x \end{array}\right\},$$

$$\beta_1 \circ \mathrm{kon}_{1j}^{2} \circ \beta_1 \circ \mathrm{kon}_{1x}^{-2} \circ \varepsilon_1 \circ \psi_1^{-2} \circ \delta_j^{-1} \circ \varphi_1^{2} \circ \mathrm{kon}_{1j}^{-1} \circ \delta_1 \circ \mathrm{kon}_{j1} \circ \varphi_j^{-2} \circ \mathrm{kon}_{j1}^{-1} \circ \zeta_{j1}^{b} \circ \psi_j^{2} \circ \mathrm{kon}_{1jx}^{-} \circ (\zeta_{1j}^{a})^{-1} \circ \delta_j =$$

$$\left\{\begin{array}{lllllllllll} x & \overset{\delta_j}{\longmapsto} & y_j x & \overset{(\zeta_{1j}^{a})^{-1}}{\longmapsto} & y_j x & \overset{\mathrm{kon}_{1jx}^{-}}{\longmapsto} & x y_j^{-1} & \overset{\psi_j^{2}}{\longmapsto} & x^{-1} y_j^{-1} & \overset{\zeta_{j1}^{b}}{\longmapsto} & x^{-1} y_j^{-1} y_1^{-1} x^{-1} y_1^{-1} x \\ y_1 & \overset{\delta_j}{\longmapsto} & y_1 & \overset{(\zeta_{1j}^{a})^{-1}}{\longmapsto} & y_1 x y_j^{-1} x^{-1} y_j^{-1} & \overset{\mathrm{kon}_{1jx}^{-}}{\longmapsto} & x y_1^{-1} x y_j x^{-1} y_j x^{-1} & \overset{\psi_j^{2}}{\longmapsto} & x y_1^{-1} x y_j x y_j x & \overset{\zeta_{j1}^{b}}{\longmapsto} & x^2 y_1 y_j y_1 x y_1 y_j x \\ y_j & \overset{\delta_j}{\longmapsto} & y_j & \overset{(\zeta_{1j}^{a})^{-1}}{\longmapsto} & y_j & \overset{\mathrm{kon}_{1jx}^{-}}{\longmapsto} & x y_j^{-1} x^{-1} & \overset{\psi_j^{2}}{\longmapsto} & x^{-1} y_j^{-1} x^{-1} & \overset{\zeta_{j1}^{b}}{\longmapsto} & x^{-1} y_j^{-1} y_1^{-1} x^{-1} y_1^{-1} \end{array}\right.$$

$$\begin{array}{llllllll} \overset{\mathrm{kon}_{j1}^{-1}}{\longmapsto} & x^{-1} y_1^{-1} y_j^{-1} x^{-1} y_1^{-1} x & \overset{\varphi_j^{-2}}{\longmapsto} & x^{-1} y_1^{-1} y_j^{-1} x y_1^{-1} x & \overset{\mathrm{kon}_{j1}}{\longmapsto} & x^{-1} y_j^{-1} y_1^{-1} x y_1^{-1} x & \overset{\delta_1}{\longmapsto} & x^{-1} y_1^{-1} y_j^{-1} x^2 \\ \overset{\mathrm{kon}_{j1}^{-1}}{\longmapsto} & x^2 y_j y_1^2 x y_j y_1 x & \overset{\varphi_j^{-2}}{\longmapsto} & y_j y_1^2 x^{-1} y_j y_1 x & \overset{\mathrm{kon}_{j1}}{\longmapsto} & y_1 y_j y_1 x^{-1} y_1 y_j x & \overset{\delta_1}{\longmapsto} & y_1 y_j y_1 x^{-1} y_j y_1 x \\ \overset{\mathrm{kon}_{j1}^{-1}}{\longmapsto} & x^{-1} y_1^{-1} y_j^{-1} x^{-1} y_1^{-1} & \overset{\varphi_j^{-2}}{\longmapsto} & x^{-1} y_1^{-1} y_j^{-1} x y_1^{-1} & \overset{\mathrm{kon}_{j1}}{\longmapsto} & x^{-1} y_j^{-1} y_1^{-1} x y_1^{-1} & \overset{\delta_1}{\longmapsto} & x^{-1} y_1^{-1} y_j^{-1} x y_1^{-1} \end{array}$$

$$
\left.
\begin{array}{llllll}
\overset{\mathrm{kon}_{1j}^{-1}}{\longmapsto} & x^{-1}y_j^{-1}y_1^{-1}x^2 & \overset{\varphi_1^2}{\longmapsto} & x^{-1}y_j^{-1}y_1^{-1} & \overset{\delta_j^{-1}}{\longmapsto} & x^{-1}y_1^{-1} \\
\overset{\mathrm{kon}_{1j}^{-1}}{\longmapsto} & y_j^{-1}y_1y_jy_1y_jx^{-1}y_1y_jx & \overset{\varphi_1^2}{\longmapsto} & y_j^{-1}x^2y_1y_jx^2y_1y_jxy_1y_jx & \overset{\delta_j^{-1}}{\longmapsto} & y_j^{-2}xy_j^{-1}xy_1xy_j^{-1}xy_1xy_1x \\
\overset{\mathrm{kon}_{1j}^{-1}}{\longmapsto} & x^{-1}y_j^{-1}y_1^{-1}xy_j^{-1}y_1^{-1}y_j & \overset{\varphi_1^2}{\longmapsto} & x^{-1}y_j^{-1}y_1^{-1}x^{-1}y_j^{-1}y_1^{-1}x^{-2}y_j & \overset{\delta_j^{-1}}{\longmapsto} & x^{-1}y_1^{-1}x^{-1}y_1^{-1}x^{-1}y_jx^{-1}y_j^2 \\
\overset{\psi_1^{-2}}{\longmapsto} & xy_1^{-1} & \overset{\varepsilon_1}{\longmapsto} & x & \overset{\mathrm{kon}_{1x}^{-2}}{\longmapsto} & x \\
\overset{\psi_1^{-2}}{\longmapsto} & y_j^{-2}xy_j^{-1}xy_1x^{-1}y_j^{-1}xy_1x^{-1}y_1x^{-1} & \overset{\varepsilon_1}{\longmapsto} & y_j^{-2}xy_1y_j^{-1}xy_1x^{-1}y_j^{-1}xy_1x^{-2} & \overset{\mathrm{kon}_{1x}^{-2}}{\longmapsto} & y_j^{-2}x^{-1}y_1x^2y_j^{-1}x^{-1}y_1xy_j^{-1}x^{-1}y_1 \\
\overset{\psi_1^{-2}}{\longmapsto} & xy_1^{-1}xy_1^{-1}x^{-1}y_jx^{-1}y_j^2 & \overset{\varepsilon_1}{\longmapsto} & x^2y_1^{-1}x^{-1}y_jy_1^{-1}x^{-1}y_j^2 & \overset{\mathrm{kon}_{1x}^{-2}}{\longmapsto} & y_1^{-1}xy_jx^{-2}y_1^{-1}xy_j^2 \\
\overset{\beta_1}{\longmapsto} & x^{-1} & \overset{\mathrm{kon}_{1j}^{2}}{\longmapsto} & x^{-1} & \overset{\beta_1}{\longmapsto} & x \\
\overset{\beta_1}{\longmapsto} & y_j^{-2}y_1^{-1}x^{-1}y_j^{-1}y_1^{-1}y_j^{-1}y_1^{-1}x & \overset{\mathrm{kon}_{1j}^{2}}{\longmapsto} & y_1^{-1}y_j^{-2}x^{-1}y_jy_1^{-1}y_j^{-1}y_1^{-1}y_j^{-2}x & \overset{\beta_1}{\longmapsto} & x^{-1}y_1xy_j^{-2}xy_jx^{-1}y_1xy_j^{-1}x^{-1}y_1xy_j^{-2}x^{-1} \\
\overset{\beta_1}{\longmapsto} & x^{-1}y_1y_jxy_1y_j^2 & \overset{\mathrm{kon}_{1j}^{2}}{\longmapsto} & x^{-1}y_j^2y_1y_j^{-1}xy_j^2y_1 & \overset{\beta_1}{\longmapsto} & xy_j^2x^{-1}y_1^{-1}xy_j^{-1}x^{-1}y_j^2x^{-1}y_1^{-1}x
\end{array}
\right\} .
$$

So we proved that

$$\mathrm{kon}_{1x}^{-1} \circ \nu_{1j}^{-2} \circ \nu_{j1} \circ \mathrm{kon}_{jx} \circ \nu_{1j}^{2} \circ \nu_{j1}^{-1} \in K_n. \tag{4.1}$$

But we have to show that

$$\nu_{j1}^{-1} \circ \nu_{1j}^{2} \circ \mathrm{kon}_{1x} \circ \nu_{j1} \circ \nu_{1j}^{-2} \circ \mathrm{kon}_{jx}^{-1} \in K_n.$$

The inverse of (4.1)

$$Y := \nu_{j1} \circ \nu_{1j}^{-2} \circ \mathrm{kon}_{jx}^{-1} \circ \nu_{j1}^{-1} \circ \nu_{1j}^{2} \circ \mathrm{kon}_{1x}$$

is also in K_n. Since K_n is a normal subgroup in $\Gamma^+(C_2, \pi)$ we have

$$X := \mathrm{kon}_{jx} \circ (\nu_{1j}^{2} \circ (\nu_{j1}^{-1} \circ Y \circ \nu_{j1}) \circ \nu_{1j}^{-2}) \circ \mathrm{kon}_{jx}^{-1} \in K_n.$$

But now we see that $X = \nu_{j1}^{-1} \circ \nu_{1j}^{2} \circ \mathrm{kon}_{1x} \circ \nu_{j1} \circ \nu_{1j}^{-2} \circ \mathrm{kon}_{jx}^{-1} \in K_n$. This is what we wanted to show.

Finally we consider the case $n \geq 4$. According to Proposition 1.5 the group $\mathrm{GL}(n-1, \mathbb{Z})$ is generated by E_{ij} and O_i $(1 \leq i, j \leq n-1,\ i \neq j)$ subject to the following relations

1.) $[E_{ij}, E_{kl}] = 1$ if $j \neq k$, $i \neq l$,
2.) $[E_{ij}, E_{jk}]E_{ik}^{-1} = 1$ if $i \neq j \neq k \neq i$,
3.) $(E_{12}E_{21}^{-1}E_{12})^4 = 1$,
4.) $(O_1E_{1j})^2 = 1$ if $j \neq 1$,
5.) $(O_1E_{i1})^2 = 1$ if $i \neq 1$,
6.) $O_1E_{ij}O_1E_{ij}^{-1} = 1$ if $i, j \neq 1$,
7.) $O_1^2 = 1$,
8.) $E_{1j}^{-1}E_{j1}^{2}O_1E_{1j}E_{j1}^{-2}O_j^{-1} = 1$ if $j \neq 1$.

This means we have to show

1.) $[\nu_{ji}, \nu_{lk}] \in K_n$ if $j \neq k$, $i \neq l$,
2.) $[\nu_{ji}, \nu_{kj}] \circ \nu_{ki}^{-1} \in K_n$ if $i \neq j \neq k \neq i$,
3.) $(\nu_{21} \circ \nu_{12}^{-1} \circ \nu_{21})^4 \in K_n$,
4.) $(\mathrm{kon}_{1x} \circ \nu_{j1})^2 \in K_n$ if $j \neq 1$,
5.) $(\mathrm{kon}_{1x} \circ \nu_{1i})^2 \in K_n$ if $i \neq 1$,
6.) $\mathrm{kon}_{1x} \circ \nu_{ji} \circ \mathrm{kon}_{1x} \circ \nu_{ji}^{-1} \in K_n$ if $i, j \neq 1$,
7.) $\mathrm{kon}_{1x}^2 \in K_n$,
8.) $\nu_{j1}^{-1} \circ \nu_{1j}^{2} \circ \mathrm{kon}_{1x} \circ \nu_{j1} \circ \nu_{1j}^{-2} \circ \mathrm{kon}_{jx}^{-1} \in K_n$ if $j \neq 1$.

In the case $n = 3$ we have proved already 3.), 4.), 5.), 7.) and 8.). So there are only 1.), 2.) and 6.) left to show:

1.) $\nu_{ji} \circ \nu_{lk} \circ \nu_{ji}^{-1} \circ \nu_{lk}^{-1}$ for $j \neq k, i \neq l$:

- $\nu_{ji} \circ \nu_{lk} \circ \nu_{ji}^{-1} \circ \nu_{lk}^{-1} = \mathrm{id} \in K_n$ for $j \neq l, i \neq k$:

$$\left\{ \begin{array}{lllllllll} y_l & \overset{\nu_{lk}^{-1}}{\mapsto} & y_l y_k^{-1} & \overset{\nu_{ji}^{-1}}{\mapsto} & y_l y_k^{-1} & \overset{\nu_{lk}}{\mapsto} & y_l & \overset{\nu_{ji}}{\mapsto} & y_l \\ y_j & \overset{\nu_{lk}^{-1}}{\mapsto} & y_j & \overset{\nu_{ji}^{-1}}{\mapsto} & y_j y_i^{-1} & \overset{\nu_{lk}}{\mapsto} & y_j y_i^{-1} & \overset{\nu_{ji}}{\mapsto} & y_j \end{array} \right\},$$

- $\nu_{ji} \circ \nu_{li} \circ \nu_{ji}^{-1} \circ \nu_{li}^{-1} = \mathrm{id} \in K_n$ for $j \neq l, i = k$:

$$\left\{ \begin{array}{lllllllll} y_l & \overset{\nu_{li}^{-1}}{\mapsto} & y_l y_i^{-1} & \overset{\nu_{ji}^{-1}}{\mapsto} & y_l y_i^{-1} & \overset{\nu_{li}}{\mapsto} & y_l & \overset{\nu_{ji}}{\mapsto} & y_l \\ y_j & \overset{\nu_{li}^{-1}}{\mapsto} & y_j & \overset{\nu_{ji}^{-1}}{\mapsto} & y_j y_i^{-1} & \overset{\nu_{li}}{\mapsto} & y_j y_i^{-1} & \overset{\nu_{ji}}{\mapsto} & y_j \end{array} \right\},$$

- $\nu_{ji} \circ \nu_{jk} \circ \nu_{ji}^{-1} \circ \nu_{jk}^{-1} = \kappa_{jik} \in K_n$ for $j = l, i \neq k$:

$$\Big\{ \; y_j \overset{\nu_{jk}^{-1}}{\mapsto} y_j y_k^{-1} \overset{\nu_{ji}^{-1}}{\mapsto} y_j y_i^{-1} y_k^{-1} \overset{\nu_{jk}}{\mapsto} y_j y_k y_i^{-1} y_k^{-1}$$

$$\overset{\nu_{ji}}{\mapsto} y_j y_i y_k y_i^{-1} y_k^{-1} \Big\},$$

- $\nu_{ji} \circ \nu_{ji} \circ \nu_{ji}^{-1} \circ \nu_{ji}^{-1} = \mathrm{id} \in K_n$ for $j = l, i = k$:
  This is clear!

2.) $\nu_{ji} \circ \nu_{kj} \circ \nu_{ji}^{-1} \circ \nu_{kj}^{-1} \circ \nu_{ki}^{-1} = \kappa_{kji} \in K_n$:

$$\left\{ \begin{array}{lllllll} y_k & \overset{\nu_{ki}^{-1}}{\mapsto} & y_k y_i^{-1} & \overset{\nu_{kj}^{-1}}{\mapsto} & y_k y_j^{-1} y_i^{-1} & \overset{\nu_{ji}^{-1}}{\mapsto} & y_k y_i y_j^{-1} y_i^{-1} \\ y_j & \overset{\nu_{ki}^{-1}}{\mapsto} & y_j & \overset{\nu_{kj}^{-1}}{\mapsto} & y_j & \overset{\nu_{ji}^{-1}}{\mapsto} & y_j y_i^{-1} \end{array} \right.$$

$$\left. \begin{array}{llll} \overset{\nu_{kj}}{\mapsto} & y_k y_j y_i y_j^{-1} y_i^{-1} & \overset{\nu_{ji}}{\mapsto} & y_k y_j y_i y_j^{-1} y_i^{-1} \\ \overset{\nu_{kj}}{\mapsto} & y_j y_i^{-1} & \overset{\nu_{ji}}{\mapsto} & y_j \end{array} \right\},$$

6.) $\mathrm{kon}_{1x} \circ \nu_{ji} \circ \mathrm{kon}_{1x} \circ \nu_{ji}^{-1} = \mathrm{kon}_{1x}^2 \in K_n$:

$$\left\{ \begin{array}{lllllllll} y_1 & \overset{\nu_{ji}^{-1}}{\mapsto} & y_1 & \overset{\mathrm{kon}_{1x}}{\mapsto} & x y_1 x^{-1} & \overset{\nu_{ji}}{\mapsto} & x y_1 x^{-1} & \overset{\mathrm{kon}_{1x}}{\mapsto} & x^2 y_1 x^{-2} \\ y_j & \overset{\nu_{ji}^{-1}}{\mapsto} & y_j y_i^{-1} & \overset{\mathrm{kon}_{1x}}{\mapsto} & y_j y_i^{-1} & \overset{\nu_{ji}}{\mapsto} & y_j & \overset{\mathrm{kon}_{1x}}{\mapsto} & y_j \end{array} \right\}.$$

$\square$

Theorem 4.14. *Let $n \geq 2$ and let $\sigma_{-1} : \Gamma^+(C_2, \pi) \to \mathrm{GL}(n-1, \mathbb{Z})$ be the map defined above and let $K_n \leq \Gamma^+(C_2, \pi)$ be the subgroup generated by the following automorphisms:*

$$\begin{array}{ll} \varepsilon_i : \{x \mapsto xy_i\}, & \delta_i : \{x \mapsto y_i x\}, \\ \psi_i^2 : \{y_i \mapsto y_i x^2\} & \end{array}$$

for $1 \leq i \leq n-1$. Then $\ker(\sigma_{-1}) = K_n$. In particular the generalized Torelli group $\ker(\sigma_{-1})$ is finitely generated as a group.

Proof. Apply Proposition 1.9 together with Lemma 4.12, Lemma 4.13 and Lemma 4.7. □

From now on we will write always K_n for the kernel of σ_{-1}. In the next corollary we give another set of generators for K_n, which is only a little bit different from that given in Theorem 4.14.

Corollary 4.15. *Let $n \geq 2$. The group K_n is generated by the following automorphisms:*

$$\varepsilon_i : \{x \mapsto xy_i\}, \qquad \psi_i^2 : \{y_i \mapsto y_i x^2\} \ ,$$

$$\alpha_i : \left\{ \begin{array}{ccc} x & \mapsto & x^{-1} \\ y_i & \mapsto & xy_i^{-1}x^{-1} \end{array} \right\}$$

for $1 \leq i \leq n-1$.

Proof. By Proposition 4.9 it is clear that $\alpha_i \in K_n$. Furthermore we have

$$\delta_i = \psi_i^2 \circ \varepsilon_i^{-1} \circ \psi_i^2 \circ \alpha_i,$$

which shows, together with Theorem 4.14, that K_n is generated by ε_i, ψ_i^2 and α_i for $1 \leq i \leq n-1$. □

Chapter 5

Some matrix groups

In this chapter we study some matrix groups, which will occur in Chapter 6 and Chapter 7. For this purpose, let $n \geq 2$ and define

$$\begin{aligned}
\Gamma_n(2) &:= \{M \in \mathrm{GL}(n,\mathbb{Z}) \mid M \equiv I_n \mod 2\} \\
\Gamma_n^1(2) &:= \left\{ M \in \mathrm{SL}(n,\mathbb{Z}) \mid M \equiv \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline * & & * & \end{array}\right) \mod 2 \right\}, \\
\widetilde{\Gamma_n^1}(2) &:= \left\{ M \in \mathrm{SL}(n,\mathbb{Z}) \mid M \equiv \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline * & & I_{n-1} & \end{array}\right) \mod 2 \right\}.
\end{aligned}$$

In this chapter we determine generators for these groups. The idea of the proof is to use the Euclidean algorithm in $\mathbb{Z}$.

5.1 A modified Euclidean algorithm

Recall the classical division algorithm, that is, for $a, b \in \mathbb{Z}$ there are $q, r \in \mathbb{Z}$ with

$$a = qb + r \quad \text{and} \quad |r| < |b|.$$

The next lemma will modify this algorithm a little bit. Actually it says that q can be chosen in $2\mathbb{Z}$.

Lemma 5.1 (Modified divsion algorithm)**.** *Let $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \notin b\mathbb{Z}$. Then we can find $q, r \in \mathbb{Z}$ with the following properties:*

1. *$a = qb + r$,*

2. *$q \in 2\mathbb{Z}$,*

3. $|r| < |b|$.

Proof. Let $a, b \in \mathbb{Z} \setminus \{0\}$. By the classical division algorithm there are $q', r' \in \mathbb{Z}$ with

$$a = q'b + r' \quad \text{and} \quad |r'| < |b|.$$

The case $r' = 0$ is impossible since $a \notin b\mathbb{Z}$. Hence $r' \neq 0$ and we have $0 < |r'| < |b|$. If q' is even we stop here and set $r := r'$ and $q := q'$.

If q' is odd we consider the four cases:
If r' and b are both positive, we have $0 < r' < b$. Subtraction of b yields $-b < r' - b < 0$. Set now $r := r' - b$ and $q = q' + 1$. Then we have $|r| < |b|$, q is even and

$$qb + r = (q' + 1)b + r' - b = q'b + r' = a.$$

The other three cases are similar. □

We can now imitate the Euclidean algorithm with our modified division algorithm. For this let $r_0, r_1 \in \mathbb{Z} \setminus \{0\}$. If $r_0 \notin r_1\mathbb{Z}$ there are by Lemma 5.1 $a_0 \in 2\mathbb{Z}$ and $r_2 \in \mathbb{Z}$ with

$$r_0 = a_0 r_1 + r_2 \quad \text{and} \quad |r_2| < |r_1|.$$

If $r_1 \in r_2\mathbb{Z}$ we stop the process. Otherwise there are $a_1 \in 2\mathbb{Z}$ and $r_3 \in \mathbb{Z}$ with

$$r_1 = a_1 r_2 + r_3 \quad \text{and} \quad |r_3| < |r_2|.$$

We iterate this process. The sequence $|r_1| > |r_2| > |r_3| > \ldots > 0$ must stop after a finite number of steps, say after j iterations:

$$\begin{aligned} r_0 &= a_0 r_1 + r_2 \\ r_1 &= a_1 r_2 + r_3 \\ &\ \vdots \\ r_{j-1} &= a_{j-1} r_j + r_{j+1} \end{aligned} \tag{5.1}$$

with $r_i \in \mathbb{Z}$, $a_i \in 2\mathbb{Z}$ and

$$|r_1| > |r_2| > |r_3| > \ldots > |r_{j+1}| > 0.$$

Since the algorithm stops after j steps, we have

$$r_j \in r_{j+1}\mathbb{Z}.$$

We call this algorithm the *modified Euclidean algorithm.*

Lemma 5.2. *Let $r_0, r_1 \in \mathbb{Z} \setminus \{0\}$. If we apply the modified Euclidean algorithm to r_0 and r_1 and the algorithm stops after j steps then*

$$r_{j+1} = \pm \gcd(r_0, r_1).$$

Proof. The proof is analogous to the proof of the same result with the normal Euclidean algorithm (see for example [4]). $\square$

Together with (5.1) we can calculate $\gcd(r_0, r_1)$ in the following way:

$$\begin{aligned} r_2 &= r_0 - a_0 r_1 \\ r_3 &= r_1 - a_1 r_2 \\ &\vdots \\ \pm \gcd(r_0, r_1) = r_{j+1} &= r_{j-1} - a_{j-1} r_j, \end{aligned} \tag{5.2}$$

with $a_i \in 2\mathbb{Z}$. We remark that if we iterate this and use the fact that $\gcd(n_1, \ldots, n_{k-1}, n_k) = \gcd(\gcd(n_1, \ldots, n_{k-1}), n_k)$, we can use the modified Euclidean algorithm to compute the greatest common divisor of more than two integers.

5.2 Generators for the matrix groups

We apply now the modified Euclidean algorithm to find generators for the matrix groups mentioned above.

We start with the group

$$\Gamma_n(2) := \{M \in \mathrm{GL}(n, \mathbb{Z}) \mid M \equiv I_n \mod 2\}.$$

This group is called the *principal congruence subgroup of level two.* Note that the following sequence is exact

$$1 \to \Gamma_n(2) \to \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}/2\mathbb{Z}) \to 1.$$

For any $a \in \mathbb{Z}$ let $E_{ij}(a)$ be the identity matrix with an additional entry a in the (i,j)-th position, $i \neq j$. For $E_{ij}(1)$ we just write E_{ij}. Moreover let $O_i := \mathrm{diag}(1, \ldots, 1, -1, 1, \ldots, 1)$ the matrix with a -1 at the (i,i)-th position (see Notation).

Proposition 5.3. *Let $n \geq 2$. Then the group $\Gamma_n(2)$ is generated by the matrices $E_{ij}(2)$ for $1 \leq i, j \leq n$ with $i \neq j$ and O_i for $1 \leq i \leq n$.*

Proof. Define

$$G := \langle E_{ij}(2)\ (i, j = 1, \ldots, n; i \neq j),\ O_i\ (i = 1, \ldots, n) \rangle$$

to be the group generated by the matrices $E_{ij}(2)$ and O_i. We have to show that G equals $\Gamma_n(2) = \{M \in \mathrm{GL}(n, \mathbb{Z}) \mid M \equiv 1 \mod 2\}$. It is clear that G is a subgroup of $\Gamma_n(2)$. So we only have to show that every element in $\Gamma_n(2)$ can be written as a product of matrices in G, i.e. $G = \Gamma_n(2)$. Note that

$$E_{ij}(2a) = (E_{ij}(2))^a$$

for all $a \in \mathbb{Z}$. This means that all elementary matrices with an even entry in the (i, j)-th position are in G.
Now let $M = (a_{ij}) \in \Gamma_n(2)$. Since $\det(M) = \pm 1$ we have

$$\gcd(a_{11}, \ldots, a_{n1}) = 1.$$

We want now to compute $\gcd(a_{11}, \ldots, a_{n1})$ via the modified euclidean algorithm (see (5.2) and the remark) in the first column of M. For this notice that if we multiply the matrices $E_{1i}(2a)$ (resp. $E_{i1}(2a)$) with $i = 2, \ldots, n$ from the left to M, we add the $2a$-fold of the i-th row to the first row of M (resp. add the $2a$-fold of the first row to the i-th row of M). In this way we can transfer the modified Euclidean algorithm to M to compute $\pm \gcd(a_{11}, \ldots, a_{n1})$ (a good reference is [2]). Since $\gcd(a_{11}, \ldots, a_{n1}) = \pm 1$, we finally find $g_1 \in G$ (the product of all $E_{1i}(2a)$ and $E_{i1}(2a)$ needed for the modified Euclidean algorithm) with

$$g_1 \cdot M = \begin{pmatrix} \pm 1 & a'_{12} & \ldots & a'_{1n} \\ a'_{21} & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ a'_{n1} & * & \ldots & * \end{pmatrix}$$

with suitable a'_{i1}, $a'_{1i} \in 2\mathbb{Z}$. The ± 1 must occur in the upper left corner, since otherwise the matrix $g_1 \cdot M$ would not be in $\Gamma_n(2)$. In fact, we can assume that there is a $+1$, since otherwise we can just multiply with O_1. We now use the $+1$ to eliminate the a'_{i1}'s and a'_{1i}'s. In order to do this set

$$f_1 := E_{21}\left(-a'_{21}\right) \cdot \ldots \cdot E_{n1}\left(-a'_{n1}\right) \in G,$$

$$h_1 := E_{12}\left(-a'_{12}\right) \cdot \ldots \cdot E_{1n}\left(-a'_{1n}\right) \in G.$$

With these matrices we have

$$f_1 \cdot (g_1 \cdot M) \cdot h_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \ldots & * \end{pmatrix}.$$

We repeat this argument with the second row and column using the matrices $E_{2i}(2a)$, $E_{i2}(2a)$ and O_2 and so on. Finally we have f_i, g_i and $h_i \in G$ $(i = 1, \ldots, n)$ with

$$\prod_{i=0}^{n-1}(f_{n-i} \cdot g_{n-i}) \cdot M \cdot \prod_{j=1}^{n} h_j = I_n,$$

which is equivalent to

$$M = \left(\prod_{i=0}^{n-1}(f_{n-i} \cdot g_{n-i})\right)^{-1} \cdot \left(\prod_{j=1}^{n} h_j\right)^{-1} \in G.$$

This shows that every element in $\Gamma_n(2)$ can be written as a product of matrices in G, which completes the proof. □

Define now

$$\Gamma_n^+(2) := \{M \in \mathrm{SL}(n, \mathbb{Z}) \mid M \equiv 1 \mod 2\}.$$

Further let $O_{1i} = \mathrm{diag}(-1, 1, \ldots, 1, -1, 1, \ldots, 1)$ be the diagonal matrix with entry -1 in the $(1,1)$- and (i,i)-place for $i = 2, \ldots, n$ (see Notation). With the same argument as above we get the following corollary.

Corollary 5.4. *The group $\Gamma_n^+(2)$ is generated by the matrices $E_{ij}(2)$ for $i, j = 1, \ldots, n$ with $i \neq j$ and O_{1i} for $i = 2, \ldots, n$.*

We are now going to give generators for the following two subgroups of $\Gamma_n^+(2)$:

$$\Gamma_n^1(2) := \left\{ A \in \mathrm{SL}(n, \mathbb{Z}) \mid A \equiv \left(\begin{array}{c|ccc} 1 & 0 & \ldots & 0 \\ \hline * & & * & \end{array}\right) \mod 2 \right\},$$

$$\widetilde{\Gamma_n^1}(2) := \left\{ A \in \mathrm{SL}(n, \mathbb{Z}) \mid A \equiv \left(\begin{array}{c|ccc} 1 & 0 & \ldots & 0 \\ \hline * & & I_{n-1} & \end{array}\right) \mod 2 \right\}.$$

Proposition 5.5.

a) *The group* $\Gamma_n^1(2)$ *is generated by the* $(n-1)^2$ *matrices*

$$E_{1j}(2)\ (2 \leq j \leq n) \quad \text{and} \quad E_{ij}\ (2 \leq i \leq n, 1 \leq j \leq n, i \neq j).$$

b) *The group* $\widetilde{\Gamma_n^1}(2)$ *is generated by the* $2(n-1)$ *matrices*

$$E_{1j}(2)\ (2 \leq j \leq n) \quad \text{and} \quad E_{i1}\ (2 \leq i \leq n).$$

Proof. For a proof of part a) with $n \geq 3$ see [2]. The case $n = 2$ is included in part b) since we have

$$\Gamma_2^1(2) = \widetilde{\Gamma_2^1}(2).$$

We are now going to prove part b). First note that the matrices

$$E_{1j}(2)\ (2 \leq j \leq n) \quad \text{and} \quad E_{i1}\ (2 \leq i \leq n) \tag{5.3}$$

are in $\widetilde{\Gamma_n^1}(2)$ by definition. For the moment, we define G to be the subgroup of $\widetilde{\Gamma_n^1}(2)$ which is generated by the matrices given in (5.3). Our aim is to show that $\widetilde{\Gamma_n^1}(2) = G$.

Before doing this we list some other matrices which are in G:

- $E_{ij}(2) \in G$ for all $1 \leq i, j, \leq n$ $(i \neq j)$:
  The matrices $E_{1j}(2)$ $(2 \leq j \leq n)$ are by definition in G and the matrices $E_{i1}(2) = E_{i1}^2$ are also in G. Hence we have to show that $E_{ij}(2) \in G$ for $2 \leq i, j, \leq n$ $(i \neq j)$ and $n \geq 3$.

$$\begin{aligned} E_{ij}(2) &= E_{ij}^2 \\ &\overset{\text{Prop. 1.2 (c)}}{=} E_{ij}E_{1j}E_{ij}E_{1j}^{-1} \\ &= E_{ij}E_{1j}E_{ij}E_{1j}E_{1j}^{-1}E_{1j}^{-1} \\ &\overset{\text{Prop. 1.2 (c)}}{=} E_{ij}E_{1j}E_{i1}E_{1j}E_{i1}^{-1}E_{1j}^{-2} \\ &\overset{\text{Prop. 1.2 (c)}}{=} E_{i1}E_{1j}^2E_{i1}^{-1}E_{1j}^{-2} \in G. \end{aligned}$$

- $O_{1i} \in G$ for all $2 \leq i \leq n$:
  We have

$$O_{1i} = \left(E_{i1}^{-1} \cdot E_{1i}(2)\right)^2 \in G.$$

We will prove this just for $n = 2$ and $i = 2$. The other cases are analogue.

$$\begin{aligned}\left(E_{i1}^{-1} \cdot E_{1i}(2)\right)^2 &= \left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right)^2 \\ &= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}$$

Let $M = (a_{ij}) \in \widetilde{\Gamma_n^1}(2)$. Since $\det(M) = 1$, we have

$$\gcd(a_{11}, \ldots, a_{1n}) = 1.$$

For each $a \in \mathbb{Z}$ the matrices $E_{1i}(2a) = (E_{1i}(2))^a$ and $E_{i1}(2a) = E_{i1}^{2a}$ $(2 \leq i \leq n)$ are in G. If we multiply the matrices $E_{1i}(2a)$ (resp. $E_{i1}(2a)$) from the right to M, we add the $2a$-fold of the first column to the i-th column of M (resp. add the $2a$-fold of the i-th column to the first column of M). In this way we can transfer the modified Euclidean algorithm to the first row of M in order to compute $\pm\gcd(a_{11}, \ldots, a_{1n})$. Since $\gcd(a_{11}, \ldots, a_{1n}) = 1$ we finally find $g \in G$ (the product of all $E_{1i}(2a)$ and $E_{i1}(2a)$ needed for the modified Euclidean algorithm) with

$$M \cdot g = \begin{pmatrix} \pm 1 & a'_{12} & \ldots & a'_{1n} \\ a'_{21} & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ a'_{n1} & * & \ldots & * \end{pmatrix}$$

with suitable $a'_{i1} \in \mathbb{Z}$ and $a'_{1i} \in 2\mathbb{Z}$. The ± 1 must occur in the upper left corner, since otherwise the matrix $g_1 \cdot M$ would not be in $\widetilde{\Gamma_n^1}(2)$. We can assume that there is a $+1$, since otherwise we can just multiply with $O_{1,2}$.

We use now the $+1$ to eliminate the a'_{i1}'s and a'_{1i}'s. In order to do this set

$$\begin{aligned} f &:= E_{21}\left(-a'_{21}\right) \cdot \ldots \cdot E_{n1}\left(-a'_{n1}\right) \in G, \\ h &:= E_{12}\left(-a'_{12}\right) \cdot \ldots \cdot E_{1n}\left(-a'_{1n}\right) \in G. \end{aligned}$$

With these matrices we have

$$f \cdot (M \cdot g) \cdot h = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \ldots & * \end{pmatrix} =: A.$$

By Corollary 5.4 and its proof, the matrix A is a product of the matrices $E_{ij}(2)$ with $2 \leq i, j, \leq n$ $(i \neq j)$ and O_{1i} with $2 \leq i \leq n$. Hence $A \in G$ by the above remark.

It follows that

$$M = f^{-1} \cdot A \cdot h^{-1} \cdot g^{-1} \in G.$$

This shows that every element M in $\widetilde{\Gamma_n^1}(2)$ can be written as a product of matrices in G, which completes the proof. $\square$

Chapter 6

Lower central series quotients of K_n

Let K_n be the kernel of σ_{-1} (see Chapter 4) and

$$K_n = \gamma_0(K_n) \geq \gamma_1(K_n) \geq \gamma_2(K_n) \geq \dots$$

be the corresponding lower central series. In this chapter we study the quotients $\gamma_i(K_n)/\gamma_{i+1}(K_n)$ for $i \geq 0$. By Chapter 2 these are modules over $\mathrm{GL}(n-1,\mathbb{Z})$. In Section 6.1 we supply some facts about modules over $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{GL}(n,\mathbb{Z})$. Section 6.2 is concerned with $K_n^{\mathrm{ab}} = \gamma_0(K_n)/\gamma_1(K_n)$. For $n \geq 3$ we construct an epimorphism

$$\Phi_n : V_{n-1} \oplus M_{n-1} \twoheadrightarrow K_n^{\mathrm{ab}},$$

where $V_{n-1} \oplus M_{n-1}$ is a certain $\mathrm{GL}(n-1,\mathbb{Z})$-module with underlying abelian group $(\mathbb{Z}^{n-1} \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}) \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}$. The precise structure of $V_{n-1} \oplus M_{n-1}$ is described in Chapter 6.1. The special case $n = 2$ is discussed in Section 6.3. In this case it is possible to give a finite presentation of K_2 and identify the isomorphism type of K_2^{ab}. The last Section 6.4 is about the quotients $\gamma_i(K_n)/\gamma_{i+1}(K_n)$ for $i \geq 1$. Our second main theorem states the surprising fact that these quotients are finite groups of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{n,i}}$ with

$$0 \leq b_{n,i} \leq (3n-3)^{i-1} \cdot \big(3n^2 - 7n + 4\big)$$

(see Theorem 6.25).

6.1 Modules over $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{GL}(n,\mathbb{Z})$

Let $M = \mathbb{Z}^n$ be the $\mathrm{SL}(n,\mathbb{Z})$-module with the action given by matrix multiplication. To be more precise let $e_1, \dots e_n$ denote the standard basis of M. Then the $\mathrm{SL}(n,\mathbb{Z})$-action is given by

$$\begin{aligned} E_{ji}\cdot e_i &= e_i + e_j, \\ E_{jk}\cdot e_i &= e_i \quad \text{for } k \neq i, \\ E_{ji}^{-1}\cdot e_i &= e_i - e_j, \\ E_{jk}^{-1}\cdot e_i &= e_i \quad \text{for } k \neq i. \end{aligned}$$

We call this action on M the *standard* $\mathrm{SL}(n,\mathbb{Z})$*-action*.

Proposition 6.1. *Let $M = \mathbb{Z}^n$ be the $\mathrm{SL}(n,\mathbb{Z})$-module with the standard action and let $S \leq M$ be a submodule of M with $S \neq 0$. Then the index of S in M is finite.*

Proof. Let $e_1, \ldots e_n$ be the standard basis of M and let $v \in S$ with $v \neq 0$. Then there are $a_1, \ldots, a_n \in \mathbb{Z}$ with

$$v = \sum_{i=1}^{n} a_i e_i \quad (a_j \neq 0 \ \text{ for some } j).$$

For $1 \leq k \leq n$, $k \neq j$, consider now

$$\begin{aligned} (E_{kj}\cdot v) - v &= \left(\sum_{i=1}^{n} a_i (E_{kj}\cdot e_i)\right) - v \\ &= \left(\sum_{i=1}^{n} a_i e_i\right) + a_j e_k - v = a_j e_k. \end{aligned}$$

Thus $a_j e_k \in S$ for $1 \leq k \leq n$, $k \neq j$. Since

$$\begin{aligned} (E_{j1}\cdot a_j e_1) - a_j e_1 &= a_j (E_{j1}\cdot e_1) - a_j e_1 \\ &= a_j(e_1 + e_j) - a_j e_1 = a_j e_j, \end{aligned}$$

we see that $a_j e_k \in S$ for all $1 \leq k \leq n$. Hence the index of S in M is at most $(a_j)^n < \infty$. □

Let now $M = (\mathbb{Z}/2\mathbb{Z})^n$ be the $\mathrm{GL}(n,\mathbb{Z})$-module with the action given by matrix multiplication. If $e_1, \ldots, e_n$ denote the standard generators of

M, this means

$$\begin{aligned}
E_{ji} \cdot e_i &= e_i + e_j, \\
E_{jk} \cdot e_i &= e_i \quad \text{for } k \neq i, \\
E_{ji}^{-1} \cdot e_i &= e_i - e_j = e_i + e_j, \\
E_{jk}^{-1} \cdot e_i &= e_i \quad \text{for } k \neq i, \\
O_i \cdot e_i &= -e_i = e_i, \\
O_j \cdot e_i &= e_j \quad \text{for } j \neq i.
\end{aligned}$$

We call this action on M the *standard* $\mathrm{GL}(n,\mathbb{Z})$*-action.*

Furthermore, we consider the $\mathrm{GL}(n,\mathbb{Z})$-action on M given by

$$A \cdot x := (A^{-1})^t x$$

for $x \in M$ and $A \in \mathrm{GL}(n,\mathbb{Z})$. We call this action on M *the dual standard* $\mathrm{GL}(n,\mathbb{Z})$*-action.* In terms of the generators $e_1, \dots e_n$ the dual standard $\mathrm{GL}(n,\mathbb{Z})$-action is given by

$$\begin{aligned}
E_{ij} \cdot e_i &= (E_{ij}^{-1})^t \cdot e_i = E_{ji}^{-1} = e_i - e_j = e_i + e_j, \\
E_{kj} \cdot e_i &= (E_{kj}^{-1})^t \cdot e_i = E_{jk}^{-1} = e_i \quad \text{for } k \neq i, \\
E_{ij}^{-1} \cdot e_i &= (E_{ij})^t \cdot e_i = E_{ji} = e_i + e_j, \\
E_{kj}^{-1} \cdot e_i &= (E_{kj})^t \cdot e_i = E_{jk} = e_i \quad \text{for } k \neq i, \\
O_i \cdot e_i &= (O_i^{-1})^t \cdot e_i = O_i \cdot e_i = e_i, \\
O_j \cdot e_i &= (O_j^{-1})^t \cdot e_i = O_j \cdot e_i = e_j \quad \text{for } j \neq i.
\end{aligned}$$

Proposition 6.2. *Let $M = (\mathbb{Z}/2\mathbb{Z})^n$ be the $\mathrm{GL}(n,\mathbb{Z})$-module with the standard action. Then M is also a $\mathrm{GL}(n,\mathbb{Z}/2\mathbb{Z})$-module with the action induced by standard $\mathrm{GL}(n,\mathbb{Z})$- action. The same holds for the dual standard action.*

Proof. Let $e_1, \dots, e_n$ denote the standard generators of M. By the exactness of the sequence

$$1 \to \Gamma_n(2) \to \mathrm{GL}(n,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{Z}/2\mathbb{Z}) \to 1$$

it suffices to show that $\Gamma_n(2)$ acts trivial on M. By Proposition 5.3

$$\Gamma_n(2) = \langle E_{ij}(2)(1 \leq i, j \leq n\ ,\ i \neq j), O_i(1 \leq i \leq n) \rangle.$$

Hence it suffices to show, that $E_{ij}(2)$ and O_i act trivial on the generators:

$$\begin{aligned}
E_{ji}(2)\cdot e_i &= E_{ji}^2\cdot e_i = E_{ji}\cdot(e_i+e_j) = e_i + 2e_j = e_i,\\
E_{jk}(2)\cdot e_i &= E_{jk}^2\cdot e_i = e_i \quad \text{for } k\neq i,\\
O_j\cdot e_i &= e_i \quad \text{for all } j.
\end{aligned}$$

The proof with the dual standard action is analogous. □

Proposition 6.3. *Let $M = (\mathbb{Z}/2\mathbb{Z})^n$ be the $\mathrm{GL}(n,\mathbb{Z})$-module with the standard action. Then M is irreducible as a $\mathrm{GL}(n,\mathbb{Z})$-module. Actually M is irreducible as a $\mathrm{SL}(n,\mathbb{Z})$-module. The same holds for M together with the dual standard action.*

Proof. Let $e_1,\ldots,e_n$ denote the standard generators of M. The action of the elementary matrices E_{ij} on $e_1,\ldots,e_n$ is then given by

$$\begin{aligned}
E_{ji}\cdot e_i &= e_i + e_j\\
E_{jk}\cdot e_i &= e_i \quad \text{for } k\neq i.
\end{aligned}$$

Assume now there is a submodule $S\leq M$ with $S\neq 0$. We have to show that $S = M$. Since $S\neq 0$, there exists an element $v\in S$, $v\neq 0$, say

$$v = \sum_{i=1}^n a_i e_i$$

with $a_i\in\{0,1\}$ and $a_j = 1$ for some j. We have

$$(E_{kj}\cdot v) - v = \left(\sum_{i=1}^n a_i(E_{kj}\cdot e_i)\right) - v = \left(\sum_{i=1}^n a_i e_i\right) + e_k - v = e_k \in S,$$

for all $k\neq j$. But then

$$(E_{j1}\cdot e_1) - e_1 = (e_1+e_j) - e_1 = e_j \in S.$$

Hence $S = M$.
The proof for M together with the dual standard action is analogous. □

Finally, we will define another $\mathrm{GL}(n,\mathbb{Z})$-module, which will arise in the next section. We are going to specify the action of $\mathrm{GL}(n,\mathbb{Z})$ on this new module in terms of generators and then extend it linearly. Before doing this let us take a look at the general situation:

Let G be finitely presented group and A be a finitely generated abelian group, say

$$\begin{aligned} G = \langle g_1, \ldots, g_n \quad &| \quad R_1, \ldots, R_m \rangle, \\ A = \langle e_1, \ldots, e_s, d_1, \ldots, d_t, \quad &| \quad [e_i, e_j] = 0, [e_i, d_j] = 0, [d_i, d_j] = 0, \\ & \quad\;\; a_1 \cdot d_1 = 0, \ldots, a_t \cdot d_t = 0 \rangle \end{aligned}$$

with $a_1, \ldots, a_t \in \mathbb{N} \setminus \{1\}$. The e_i are then the free generators of A and the d_i are the torsion generators. In order to define an action of G on A we proceed in the following way:
First we define the action in terms of the generators

$$g_i \cdot e_j := v_{ij} \in A, \qquad g_i^{-1} \cdot e_j := \widetilde{v_{ij}} \in A,$$

$$g_i \cdot d_j := w_{ij} \in A, \qquad g_i^{-1} \cdot d_j := \widetilde{w_{ij}} \in A.$$

Of course there are restrictions for this definitions (see below). Then we extend the action linearly, i.e. for $x = \sum_{k=1}^{s} b_k e_k + \sum_{j=1}^{t} c_j d_j$ $(b_k, c_j \in \mathbb{Z})$ we define

$$g_i^{\varepsilon} \cdot x := \sum_{k=1}^{s} b_k \, (g_i^{\varepsilon} \cdot e_k) + \sum_{j=1}^{t} c_j \, (g_i^{\varepsilon} \cdot d_j) \qquad (\varepsilon \in \{-1, 1\})$$

and for $g = g_{i_1}^{\varepsilon_1} \ldots g_{i_k}^{\varepsilon_k}$ $(\varepsilon_j \in \{-1, 1\})$

$$g \cdot x := g_{i_1}^{\varepsilon_1} \cdot \big(\cdots (g_{i_k}^{e_k} \cdot x) \ldots \big) .$$

Note that these definitions lead to a well defined action of G on A if and only if the following relations are satisfied

a) $g_i \cdot (g_i^{-1} \cdot e_j) = e_j, \quad g_i^{-1} \cdot (g_i \cdot e_j) = e_j$ and
$g_i \cdot (g_i^{-1} \cdot d_j) = d_j, \quad g_i^{-1} \cdot (g_i \cdot d_j) = d_j,$

b) $a_j \cdot (g_i^{\varepsilon} \cdot d_j) = 0$ for all $1 \leq j \leq t \quad (\varepsilon \in \{-1, 1\}),$

c) $R_i \cdot e_j = e_j$ and $R_i \cdot d_j = d_j$ for $1 \leq i \leq m$.

Remark 6.4. *Let $x = \sum_{k=1}^{s} b_k e_k + \sum_{j=1}^{t} c_j d_j$ with $b_k, c_j \in \mathbb{Z}$. By the above construction we have*

$$g \cdot x = \sum_{k=1}^{s} b_k \, (g \cdot e_k) + \sum_{j=1}^{t} c_j \, (g \cdot d_j)$$

for all $g \in G$.

We are now going to apply this method to define another $\mathrm{GL}(n, \mathbb{Z})$-module V_n. The module V_n will play an important role in the next section.

Lemma 6.5. *Let $V_n := \mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^n$ as an abelian group with standard generators $e_1, \dots, e_n$ and $d_1, \dots, d_n$. If we define on the generators*

$$\begin{array}{ll}
E_{ji} \cdot e_i = e_i + e_j, & E_{ji}^{-1} \cdot e_i = e_i - e_j, \\
E_{jk} \cdot e_i = e_i \quad (k \neq i), & E_{jk}^{-1} \cdot e_i = e_i \quad (k \neq i), \\
O_1 \cdot e_1 = d_1 - e_1, & \\
O_1 \cdot e_i = e_i + d_i \quad (i \neq 1), & \\
E_{ji} \cdot d_i = d_i + d_j, & E_{ji}^{-1} \cdot d_i = d_i + d_j, \\
E_{jk} \cdot d_i = d_i \quad (k \neq i), & E_{jk}^{-1} \cdot d_i = d_i \quad (k \neq i), \\
O_1 \cdot d_i = d_i \quad \text{for all } i &
\end{array}$$

we obtain a $\mathrm{GL}(n, \mathbb{Z})$-action on V_n.

Proof. We have to show that these definition extends to an action, i.e. we have to check the points a) - c) (see above).

First note, that by the above formulas the action defined on the generators $d_1, \dots, d_n$ of the group $(\mathbb{Z}/2\mathbb{Z})^n$ coincides with the standard $\mathrm{GL}(n, \mathbb{Z})$-action. Hence it is clear that the points a) - c) hold for $d_1, \dots, d_n$.

Furthermore note that the $\mathrm{SL}(n, \mathbb{Z})$-action defined on the generators $e_1, \dots, e_n$ of $\mathbb{Z}^n$ coincides with the standard $\mathrm{SL}(n, \mathbb{Z})$-action. Hence we just have to look on relations containing the generator O_1:

a) It suffices to show that $O_1 O_1 \cdot e_i = e_i$ for all i:

We have

- for $i \neq 1$

$$O_1 O_1 \cdot e_i = O_1 \cdot (e_i + d_i) = e_i + 2d_i = e_i,$$

- for $i = 1$

$$O_1 O_1 \cdot e_1 = O_1 \cdot (d_1 - e_1) = d_1 - d_1 + e_1 = e_1.$$

b) Here is nothing to show.

c) It suffices to show that $R \cdot e_i = e_i$, where R is one of the following relators

1.) $O_1 E_{ij} O_1 E_{ij}^{-1}$ if $i, j \neq 1$,
2.) $(O_1 E_{1j})^2$ if $j \neq 1$,
3.) $(O_1 E_{j1})^2$ if $j \neq 1$,
4.) O_1^2.

1.) We show that $O_1 E_{ij} O_1 E_{ij}^{-1} \cdot e_k = e_k$ if $i, j \neq 1$, for all k:

- for $k \neq 1, j$

$$O_1 E_{ij} O_1 E_{ij}^{-1} \cdot e_k = O_1 E_{ij} \cdot (e_k + d_k) = e_k + 2d_k = e_k,$$

- for $k = 1$

$$O_1 E_{ij} O_1 E_{ij}^{-1} \cdot e_1 = O_1 E_{ij} \cdot (d_1 - e_1) = d_1 - d_1 + e_1 = e_1,$$

- for $k = j$

$$\begin{aligned} O_1 E_{ij} O_1 E_{ij}^{-1} \cdot e_j &= O_1 E_{ij} O_1 \cdot (e_j - e_i) \\ &= O_1 E_{ij} \cdot (e_j + d_j - e_i + d_i) \\ &= O_1 \cdot (e_j + e_i + d_j + d_i - e_i + d_i) \\ &= O_1 \cdot (e_j + d_j) = e_j + 2d_j = e_j. \end{aligned}$$

2.) We show that $(O_1 E_{1j})^2 \cdot e_i = e_i$ for all i $(j \neq 1)$:
We have

- for $i \neq 1, j$

$$(O_1E_{1j})^2 \cdot e_i = O_1E_{1j} \cdot (e_i + d_i) = e_i + 2d_i = e_i,$$

- for $i = j$

$$\begin{aligned}(O_1E_{1j})^2 \cdot e_j &= O_1E_{1j}O_1 \cdot (e_j + e_1)\\ &= O_1E_{1j} \cdot (e_j + d_j - e_1 + d_1)\\ &= = O_1(e_j + e_1 + d_j + d_1 - e_1 + d_1)\\ &= O_1 \cdot (e_j + d_j) = e_j + 2d_j = e_j,\end{aligned}$$

- for $i = 1$

$$(O_1E_{1j})^2 \cdot e_1 = O_1E_{1j} \cdot (d_1 - e_1) = d_1 - d_1 + e_1 = e_1.$$

3.) We show that $(O_1E_{j1})^2 \cdot e_i = e_i$ for all i ($j \neq 1$):
We have

- for $i \neq 1$

$$(O_1E_{j1})^2 \cdot e_i = O_1E_{j1} \cdot (e_i + d_i) = e_i + 2d_i = e_i,$$

- for $i = 1$

$$\begin{aligned}(O_1E_{j1})^2 \cdot e_1 &= O_1E_{j1}O_1 \cdot (e_1 + e_j)\\ &= O_1E_{j1} \cdot (d_1 - e_1 + e_j + d_j)\\ &= O_1 \cdot (d_1 + d_j - e_1 - e_j + e_j + d_j)\\ &= O_1 \cdot (d_1 - e_1)\\ &= d_1 - d_1 + e_1 = e_1,\end{aligned}$$

□

Lemma 6.6. *Let V_n be the $\mathrm{GL}(n,\mathbb{Z})$-module defined above. Then we have for all i*

$$\begin{aligned}O_i \cdot e_i &= d_i - e_i,\\ O_j \cdot e_i &= e_i + d_i \quad (i \neq j),\\ O_j \cdot d_i &= d_i.\end{aligned}$$

Proof. Notice that

$$O_i = E_{1i}^{-1}E_{i1}^{2}O_1E_{1i}E_{i1}^{-2}$$

by Proposition 1.5. Hence we obtain

- for $j \neq 1, i$

$$E_{1i}^{-1}E_{i1}^{2}O_1E_{1i}E_{i1}^{-2} \cdot e_j = E_{1i}^{-1}E_{i1}^{2} \cdot (e_j + d_j) = e_j + d_j,$$

- for $j = 1$

$$\begin{aligned} & E_{1i}^{-1}E_{i1}^{2}O_1E_{1i}E_{i1}^{-2} \cdot e_1 = E_{1i}^{-1}E_{i1}^{2}O_1E_{1i} \cdot (e_1 - 2e_i) \\ = \ & E_{1i}^{-1}E_{i1}^{2}O_1 \cdot (-e_1 - 2e_i) = E_{1i}^{-1}E_{i1}^{2} \cdot (e_1 - d_1 - 2e_i) \\ = \ & E_{1i}^{-1} \cdot (e_1 - d_1) = e_1 - d_1, \end{aligned}$$

- for $j = i$

$$\begin{aligned} & E_{1i}^{-1}E_{i1}^{2}O_1E_{1i}E_{i1}^{-2} \cdot e_i = E_{1i}^{-1}E_{i1}^{2}O_1 \cdot (e_i + e_1) \\ = \ & E_{1i}^{-1}E_{i1}^{2} \cdot (e_i + d_i + d_1 - e_1) = E_{1i}^{-1} \cdot (d_i + d_1 - e_1 - e_i) \\ = \ & d_i - e_i. \end{aligned}$$

□

6.2 THE ABELIANIZED GROUP K_n^{ab}

By Chapter 4 the following sequence is exact

$$1 \to K_n \to \Gamma^+(C_2, \pi) \to \mathrm{GL}(n-1, \mathbb{Z}) \to 1. \tag{6.1}$$

Define $K_n^{\text{ab}} := K_n/[K_n, K_n]$, where $[K_n, K_n]$ is the commutator subgroup of K_n. By Theorem 4.14 K_n^{ab} is a finitely generated abelian group. Furthermore K_n^{ab} is a $\mathrm{GL}(n-1, \mathbb{Z})$-module by Proposition 2.17. To be more precise let $A \in \mathrm{GL}(n-1, \mathbb{Z})$ and $a \in \Gamma^+(C_2, \pi)$ with $\sigma_{-1}(a) = A$. Then the action of A on an element $[k] \in K_n^{\text{ab}}$ is given by

$$A \cdot [k] := [a \circ k \circ a^{-1}].$$

We are now interested in the structure of the $\mathrm{GL}(n-1, \mathbb{Z})$-module K_n^{ab}.

Lemma 6.7. *In $K_n \leq \Gamma^+(C_2, \pi)$ the following relations hold for $n \geq 3$ and $1 \leq i, j \leq n-1$ with $i \neq j$:*

$$(\psi_i^2)^2 = (\zeta_{ij}^a)^{-1} \circ \psi_j^2 \circ \zeta_{ij}^c \circ \psi_j^{-2}.$$

Proof. $(\psi_i^2)^2 = (\zeta_{ij}^a)^{-1} \circ \psi_j^2 \circ \zeta_{ij}^c \circ \psi_j^{-2}$:

$$\left\{ \begin{array}{lllllllll} y_i & \overset{\psi_j^{-2}}{\mapsto} & y_i & \overset{\zeta_{ij}^c}{\mapsto} & y_i y_j x^{-1} y_j x & \overset{\psi_j^2}{\mapsto} & y_i y_j x y_j x^3 & \overset{(\zeta_{ij}^a)^{-1}}{\mapsto} & y_i x^4 \\ y_j & \overset{\psi_j^{-2}}{\mapsto} & y_j x^{-2} & \overset{\zeta_{ij}^c}{\mapsto} & y_j x^{-2} & \overset{\psi_j^2}{\mapsto} & y_j & \overset{(\zeta_{ij}^a)^{-1}}{\mapsto} & y_j \end{array} \right\}.$$

So the resulting automorphism is $(\psi_i^2)^2$. □

Lemma 6.8. *In K_n^{ab} the following relations hold*

(a) *for $n \geq 2$:*

$$\begin{array}{ll} [\mathrm{kon}_{xi}] = [\delta_i] - [\varepsilon_i], & [\mathrm{kon}_{ix}^2] = 0, \\ [\varphi_i^2] = [\psi_i^2], & [\alpha_i] = [\varepsilon_i] + [\delta_i] - 2[\psi_i^2], \\ 2[\alpha_i] = 0, & [\alpha_i] = -[\beta_i]. \end{array}$$

(b) *for $n \geq 3$:*

$$\begin{array}{ll} 2[\psi_i^2] = 0, & [\alpha_i] = [\varepsilon_i] + [\delta_i], \\ [\kappa_{jk}] = 0, & [\mathrm{kon}_{ijx}^-] = [\alpha_i] + [\alpha_j], \\ [\zeta_{ij}^a] = [\zeta_{ij}^b] = [\zeta_{ij}^c] = [\zeta_{ij}^d] = 0, & [\mathrm{kon}_{ij}] = [\alpha_j]. \end{array}$$

(c) *for $n \geq 4$:*

$$[\kappa_{ijk}] = 0.$$

Proof. (a) Let $n \geq 2$. With the help of formulas in the proof of Proposition 4.9 and Lemma 4.11 we can conclude that

$$\begin{aligned} &[\mathrm{kon}_{xi}] = [\delta_i \circ \varepsilon_i^{-1}] = [\delta_i] - [\varepsilon_i], \\ &[\alpha_i] = [\psi_i^{-2} \circ \varepsilon_i \circ \psi_i^{-2} \circ \delta_i] = [\varepsilon_i] + [\delta_i] - 2[\psi_i^2], \\ &[\beta_i] = [\psi_i^2 \circ \delta_i^{-1} \circ \psi_i^2 \circ \varepsilon_i^{-1}] = -[\varepsilon_i] - [\delta_i] + 2[\psi_i^2], \\ &[\mathrm{kon}_{ix}^2] = [\alpha_i \circ \beta_i] = [\varepsilon_i] + [\delta_i] - 2[\psi_i^2] - [\varepsilon_i] - [\delta_i] + 2[\psi_i^2] = 0, \\ &[\varphi_i^2] = [\mathrm{kon}_{ix}^2 \circ \psi_i^2] = [\mathrm{kon}_{ix}^2] + [\psi_i^2] = [\psi_i^2]. \end{aligned}$$

Since $\alpha_i^2 = \mathrm{id}$ by Proposition 4.9, we have

$$0 = [\alpha_i^2] = 2[\alpha_i].$$

(b) Now let $n \geq 3$. Again we use the formulas in the proof of Lemma 4.11:

$$\begin{aligned}
&[\zeta_{ij}^c] = [\varepsilon_j^{-1} \circ \psi_i^{-2} \circ \varepsilon_j \circ \psi_i^2] = 0,\\
&[\zeta_{ij}^a] = [\mathrm{kon}_{ij}^{-1} \circ \psi_i^2 \circ \mathrm{kon}_{ij} \circ \zeta_{ij}^c \circ \psi_i^{-2}] = [\zeta_{ij}^c] = 0,\\
&[\zeta_{ij}^b] = [\alpha_i \circ (\zeta_{ij}^c)^{-1} \circ \alpha_i] = 2[\alpha_i] - [\zeta_{ij}^c] = 0,\\
&[\zeta_{ij}^d] = [\beta_i \circ (\zeta_{ij}^a)^{-1} \circ \beta_i] = 2[\beta_i] - [\zeta_{ij}^a] = 0.
\end{aligned}$$

Now Lemma 6.7 yields

$$2[\psi_i^2] = [(\psi_i^2)^2] = [(\zeta_{ij}^a)^{-1} \circ \psi_j^2 \circ \zeta_{ij}^c \circ \psi_j^{-2}] = 0.$$

And hence we get together with part (a)

$$[\alpha_i] = [\varepsilon_i] + [\delta_i] - 2[\psi_i^2] = [\varepsilon_i] + [\delta_i]. \tag{6.2}$$

Further

$$\begin{aligned}
&[\mathrm{kon}_{ijx}^-] = [\alpha_i \circ \beta_j] = [\alpha_i] + [\beta_j] \overset{\text{(a)}}{=} [\alpha_i] - [\alpha_j] = [\alpha_i] + [\alpha_j],\\
&[\kappa_{jk}] = [\varepsilon_j \circ \varepsilon_k \circ \varepsilon_j^{-1} \circ \varepsilon_k^{-1}] = 0,\\
&[\mathrm{kon}_{ij}] = [\beta_i \circ \varepsilon_j^{-1} \circ \beta_i \circ \delta_j^{-1}] = -[\varepsilon_j] - [\delta_j] \overset{(6.2)}{=} -[\alpha_j] = [\alpha_j].
\end{aligned}$$

(c) Finally let $n \geq 4$. We have to show that $[\kappa_{ijk}] = 0$:

$$\begin{aligned}
&[\kappa_{ijk}] = [\mathrm{kon}_{xj} \circ \varepsilon_j \circ (\mathrm{kon}_{ijx}^-)^{-1} \circ \varepsilon_j \circ \mathrm{kon}_{ijx}^- \circ \mathrm{kon}_{ik}^{-1} \circ \zeta_{ik}^d \circ (\mathrm{kon}_{ijx}^-)^{-1} \circ\\
&\varepsilon_j^{-1} \circ \mathrm{kon}_{ijx}^- \circ (\zeta_{ik}^d)^{-1} \circ \mathrm{kon}_{ik} \circ \varepsilon_j^{-1} \circ \mathrm{kon}_{xj}^{-1}] = 0.
\end{aligned}$$

□

By Corollary 4.15, the group K_n is generated by ε_i, α_i and ψ_i^2 for $i = 1, \ldots, n-1$. This leads us to the following generator set of K_n^{ab}.

Proposition 6.9. *Let $n \geq 2$. Then the group K_n^{ab} is generated by $[\varepsilon_i]$, $[\alpha_i]$ and $[\psi_i^2]$ for $i = 1, \ldots, n-1$.*

- *For $n \geq 2$ the order of $[\alpha_i]$ is either one or two.*

- *For $n \geq 3$ the order $[\psi_i^2]$ is either one or two.*

Proof. Corollary 4.15 and Lemma 6.8. □

In Section 6.3 we will give a finite presentation of the group K_2 and continue the discussion about K_2^{ab} there. In particular we will identify the isomorphism type of K_2^{ab}. For the rest of this section assume $n \geq 3$. We describe the $\mathrm{GL}(n-1,\mathbb{Z})$-action on K_n^{ab} with the help of Lemma 6.8 and the formulas in the proof of Lemma 4.12 (in the cases, which are not listed, the action is trivial):

- The action of $\mathrm{GL}(n-1,\mathbb{Z})$ on the $[\varepsilon_i]$'s:

$$\begin{aligned}
E_{ij}\cdot[\varepsilon_i] &= [\nu_{ji}\circ\varepsilon_i\circ\nu_{ji}^{-1}] = [\varepsilon_i]\\
E_{ji}\cdot[\varepsilon_i] &= [\nu_{ij}\circ\varepsilon_i\circ\nu_{ij}^{-1}] = [\varepsilon_i]+[\varepsilon_j]\\
E_{ij}^{-1}\cdot[\varepsilon_i] &= [\nu_{ji}^{-1}\circ\varepsilon_i\circ\nu_{ji}] = [\varepsilon_i]\\
E_{ji}^{-1}\cdot[\varepsilon_i] &= [\nu_{ij}^{-1}\circ\varepsilon_i\circ\nu_{ij}] = [\varepsilon_i\circ\varepsilon_j^{-1}] = [\varepsilon_i]-[\varepsilon_j]\\
O_i\cdot[\varepsilon_i] &= [\mathrm{kon}_{ix}\circ\varepsilon_i\circ\mathrm{kon}_{ix}^{-1}] = [\mathrm{kon}_{ix}^2\circ\delta_i\circ\mathrm{kon}_{ix}^{-2}] = [\delta_i]\\
&= [\alpha_i]-[\varepsilon_i]\\
O_j\cdot[\varepsilon_i] &= [\mathrm{kon}_{jx}\circ\varepsilon_i\circ\mathrm{kon}_{jx}^{-1}] = [\varepsilon_i\circ\mathrm{kon}_{ji}^{-1}] = 2[\varepsilon_i]+[\delta_i]\\
&= -[\delta_i] = [\varepsilon_i]-[\alpha_i]
\end{aligned}$$

- The action of $\mathrm{GL}(n-1,\mathbb{Z})$ on the $[\delta_i]$'s:

$$\begin{aligned}
E_{ij}\cdot[\delta_i] &= [\nu_{ji}\circ\delta_i\circ\nu_{ji}^{-1}] = [\delta_i]\\
E_{ji}\cdot[\delta_i] &= [\nu_{ij}\circ\delta_i\circ\nu_{ij}^{-1}] = [\delta_i]+[\delta_j]\\
E_{ij}^{-1}\cdot[\delta_i] &= [\nu_{ji}^{-1}\circ\delta_i\circ\nu_{ji}] = [\delta_i]\\
E_{ji}^{-1}\cdot[\delta_i] &= [\nu_{ij}^{-1}\circ\delta_i\circ\nu_{ij}] = [\delta_j^{-1}\circ\delta_i] = [\delta_i]-[\delta_j]\\
O_i\cdot[\delta_i] &= [\mathrm{kon}_{ix}\circ\delta_i\circ\mathrm{kon}_{ix}^{-1}] = [\varepsilon_i]\\
O_j\cdot[\delta_i] &= [\mathrm{kon}_{jx}\circ\delta_i\circ\mathrm{kon}_{jx}^{-1}] = [\delta_i\circ\alpha_j\circ\mathrm{kon}_{ji}^{-1}\circ\alpha_j]\\
&= 2[\delta_i]+[\varepsilon_i] = -[\varepsilon_i]
\end{aligned}$$

- The action of $\mathrm{GL}(n-1,\mathbb{Z})$ on the $[\alpha_i]$'s ($[\alpha_i] = [\varepsilon_i] + [\delta_i]$):

$$\begin{aligned}
E_{ij} \cdot [\alpha_i] &= [\varepsilon_i] + [\delta_i] = [\alpha_i] \\
E_{ji} \cdot [\alpha_i] &= [\varepsilon_i] + [\varepsilon_j] + [\delta_i] + [\delta_j] = [\alpha_i] + [\alpha_j] \\
E_{ij}^{-1} \cdot [\alpha_i] &= [\varepsilon_i] + [\delta_i] = [\alpha_i] \\
E_{ji}^{-1} \cdot [\alpha_i] &= [\varepsilon_i] - [\varepsilon_j] + [\delta_i] - [\delta_j] = [\alpha_i] - [\alpha_j] \\
O_i \cdot [\alpha_i] &= [\delta_i] + [\varepsilon_i] = [\alpha_i] \\
O_j \cdot [\alpha_i] &= -[\delta_i] - [\varepsilon_i] = -[\alpha_i] = [\alpha_i]
\end{aligned}$$

- The action of $\mathrm{GL}(n-1,\mathbb{Z})$ on the $[\psi_i^2]$'s:

$$\begin{aligned}
E_{ij} \cdot [\psi_i^2] &= [\nu_{ji} \circ \psi_i^2 \circ \nu_{ji}^{-1}] = [\psi_i^2 \circ \mathrm{kon}_{ji}^{-1} \circ \psi_j^{-2} \circ \mathrm{kon}_{ji}] \\
&= [\psi_i^2] + [\psi_j^2] \\
E_{ji} \cdot [\psi_i^2] &= [\nu_{ij} \circ \psi_i^2 \circ \nu_{ij}^{-1}] = [\mathrm{kon}_{ij}^{-1} \circ \psi_i^2 \circ \mathrm{kon}_{ij}] = [\psi_i^2] \\
E_{ij}^{-1} \cdot [\psi_i^2] &= [\nu_{ji}^{-1} \circ \psi_i^2 \circ \nu_{ji}] = [\psi_i^2] + [\psi_j^2] \\
E_{ji}^{-1} \cdot [\psi_i^2] &= [\nu_{ij}^{-1} \circ \psi_i^2 \circ \nu_{ij}] = [\psi_i^2] \\
O_i \cdot [\psi_i^2] &= [\mathrm{kon}_{ix} \circ \psi_i^2 \circ \mathrm{kon}_{ix}^{-1}] = [\psi_i^2].
\end{aligned}$$

Proposition 6.10. *Let $n \geq 3$. Further let*

$$V_{n-1} = \mathbb{Z}^{n-1} \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}$$

be the $\mathrm{GL}(n-1,\mathbb{Z})$*-module defined in Lemma* 6.5 *and*

$$M_{n-1} = (\mathbb{Z}/2\mathbb{Z})^{n-1}$$

be the $\mathrm{GL}(n-1,\mathbb{Z})$*-module with the dual standard action. Then there are surjective* $\mathrm{GL}(n-1,\mathbb{Z})$*-equivariant homomorphisms*

$$\Phi_n : V_{n-1} \oplus M_{n-1} \twoheadrightarrow K_n^{\mathrm{ab}}$$

for all $n \geq 3$.

Proof. Let $e_1, \ldots, e_{n-1}$ and $d_1, \ldots, d_{n-1}$ denote the standard generators of V_{n-1} and $f_1, \ldots, f_{n-1}$ the standard generators of M_{n-1}. Define $\Phi_n : V_n \oplus M \to K_n^{\mathrm{ab}}$ by

$$\begin{aligned}
e_i &\mapsto [\varepsilon_i], \\
d_i &\mapsto [\alpha_i], \\
f_i &\mapsto [\psi_i^2]
\end{aligned}$$

for $i = 1, \dots, n-1$. By the above formulas it is clear that Φ_n is a $\mathrm{GL}(n-1,\mathbb{Z})$-homomorphism. Since $[\varepsilon_i]$, $[\alpha_i]$ and $[\psi_i^2]$ generate K_n^{ab} by Proposition 6.9, the map Φ_n is surjective. $\square$

Definition 6.11. Define

$$\mathcal{A}_n := \langle \alpha_1, \dots, \alpha_{n-1} \rangle$$

to be the subgroup of K_n^{ab} generated by the α_i,

$$\mathcal{P}_n := \langle \psi_1^2, \dots, \psi_{n-1}^2 \rangle$$

to be the subgroup of K_n^{ab} generated by the ψ_i^2 and

$$\mathcal{E}_n := \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$$

to be the subgroup of K_n^{ab} generated by the ε_i. $\diamond$

Remark 6.12. Actually, we see by the above formulas that, $\mathcal{A}_n$ and $\mathcal{P}_n$ are $\mathrm{GL}(n-1,\mathbb{Z})$-submodules of K_n^{ab}. The subgroup $\mathcal{E}_n$ is only a $\mathrm{SL}(n-1,\mathbb{Z})$-submodule of K_n^{ab}.

Proposition 6.13. *Let $n \geq 3$. Further let $M = (\mathbb{Z}/2\mathbb{Z})^{n-1}$. Then we have*

- *The submodule $\mathcal{A}_n \leq K_n^{\mathrm{ab}}$ is either isomorphic to M with the standard action or to 0.*
- *The submodule $\mathcal{P}_n \leq K_n^{\mathrm{ab}}$ is either isomorphic to M with the dual standard action or to 0.*

Proof. Let $M = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ be the $\mathrm{GL}(n-1,\mathbb{Z})$-module with the standard action and let $e_1, \dots, e_{n-1}$ denote the standard generators of M. Define

$$f_n : M \twoheadrightarrow \mathcal{A}_n$$

to be the homomorphism, which sends e_i to $[\alpha_i]$. By the above formulas it is clear, that f_n is $\mathrm{GL}(n-1,\mathbb{Z})$ equivariant. Hence $\ker(f_n)$ is a submodule of M. Since M is irreducible as $\mathrm{GL}(n-1,\mathbb{Z})$-modules by Proposition 6.3, we obtain

$$\ker(f_n) = 0 \text{ or } \ker(f_n) = M.$$

So f_n is an isomorphism or f_n is the zero-map.
The proof for P_n is analogous. $\square$

Proposition 6.14. *Let $n \geq 3$. Further let $M = \mathbb{Z}^{n-1}$ together with the standard* $\mathrm{SL}(n-1,\mathbb{Z})$*-action. The subgroup $\mathcal{E}_n$ of K_n^{ab} is as a* $\mathrm{SL}(n-1,\mathbb{Z})$*-submodule either isomorphic to M or to some finite* $\mathrm{SL}(n-1,\mathbb{Z})$*-module.*

Proof. Let $e_1, \dots, e_{n-1}$ denote the standard basis of M. Define

$$f_n : M \twoheadrightarrow \mathcal{E}_n$$

to be the homomorphism, which sends e_i to $[\varepsilon_i]$. By the above formulas f_n is a $\mathrm{SL}(n-1,\mathbb{Z})$-homomorphism. Hence $\ker(f_n)$ is a submodule of M. By Proposition 6.1 we have either

$$\ker(f_n) = 0 \quad \text{or} \quad \ker(f_n) \text{ has finite index in } M.$$

In the case $\ker(f_n) = 0$, the homomorphism f_n is an isomorphism and $\mathcal{E}_n$ is isomorphic to M as $\mathrm{SL}(n-1,\mathbb{Z})$-module. In the case $\ker(f_n)$ has finite index in M, we obtain

$$\mathcal{E}_n \cong M/\ker(f_n),$$

which is a finite $\mathrm{SL}(n-1,\mathbb{Z})$-module. □

The results of Propsition 6.13 and Proposition 6.14 are all we know about the structure of K_n for $n \geq 3$. But our conjecture is that

- the submodule $\mathcal{A}_n \leq K_n^{\mathrm{ab}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ with the standard $\mathrm{GL}(n-1,\mathbb{Z})$-action,
- the submodule $\mathcal{P}_n \leq K_n^{\mathrm{ab}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ with the dual standard $\mathrm{GL}(n-1,\mathbb{Z})$-action,
- the $\mathrm{SL}(n-1,\mathbb{Z})$-submodule $\mathcal{E}_n$ of K_n^{ab} is isomorphic to $\mathbb{Z}^{n-1}$ with the standard $\mathrm{SL}(n-1,\mathbb{Z})$-action.

Moreover we conjecture the following.

Conjecture 6.15. *Let $n \geq 3$. The* $\mathrm{GL}(n-1,\mathbb{Z})$*-homomorphism*

$$\Phi_n : V_{n-1} \oplus M_{n-1} \twoheadrightarrow K_n^{\mathrm{ab}}$$

is an isomorphism for each n.

6.3 THE SPECIAL CASE $n = 2$

In the case $n = 2$ we get by Theorem 4.14 an exact sequence

$$1 \to K_2 \to \Gamma^+(C_2, \pi) \to \mathrm{GL}(1, \mathbb{Z}) \to 1.$$

Thus we see that the index of K_2 in $\Gamma^+(C_2, \pi)$ is two. Since by Lemma 4.1 the index of $\Gamma^+(C_2, \pi)$ in $\mathrm{Aut}(F_2)$ is six, we conclude that the index of K_2 in $\mathrm{Aut}(F_2)$ is twelve.

By Proposition 3.1, we know the following finite presentation of $\mathrm{Aut}(F_2)$.

$$\begin{aligned} \mathrm{Aut}(F_2) = \langle \pi_{12}, \sigma_1, \nu_{12} \quad | \quad & \pi_{12}^2 = 1,\ \sigma_1^2 = 1,\ (\sigma_1 \circ \pi_{12})^4 = 1, \\ & \sigma_1^{-1} \circ \nu_{12}^{-1} \circ \sigma_1^{-1} \circ \nu_{12}^{-1} \circ \sigma_1 \circ \nu_{12} \circ \sigma_1 \circ \nu_{12} = 1, \\ & (\nu_{12} \circ \pi_{12} \circ \sigma_1 \circ \pi_{12})^2 = 1,\ (\sigma_1 \circ \pi_{12} \circ \nu_{12})^3 = 1 \rangle. \end{aligned}$$

By applying the Reidemeister rewriting process (see for example [16] Chapter 2.3) we can calculate a finite presentation of K_2. We used the computer algebra system MAGMA to do this. For the program, see the Appendix in Chapter 8. Here is the result.

Proposition 6.16. *The group K_2 has the following finite presentation*

$$\begin{aligned} K_2 = \langle \varepsilon_1, \alpha_1, \psi_1^2 \quad | \quad & \alpha_1^2 = 1,\ [\alpha_1, \varepsilon_1] = 1 \\ & [\alpha_1, \psi_1^2] \circ [\alpha_1, \psi_1^{-2}] = 1,\ [\varepsilon_1^{-1}, \psi_1^2] \circ [\varepsilon_1, \psi_1^{-2}] = 1 \rangle. \end{aligned}$$

Starting from this presentation we can compute the isomorphism type of the abelianized group K_2^{ab}.

Corollary 6.17. *The abelianized group K_2^{ab} has the following finite presentation*

$$\begin{aligned} K_2^{\mathrm{ab}} = \langle \varepsilon_1, \alpha_1, \psi_1^2 \quad | \quad & \alpha_1^2 = 1,\ [\alpha_1, \varepsilon_1] = 1, \\ & [\alpha_1, \psi_1^2] = 1,\ [\varepsilon_1, \psi_1^2] = 1 \rangle. \end{aligned}$$

In particular, we have

$$K_2^{\mathrm{ab}} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}.$$

The $\mathrm{GL}(1, \mathbb{Z})$*-action is given by*

$$\begin{aligned} O_1 \cdot [\varepsilon_1] &= [\alpha_1] + 2[\psi_1^2] - [\varepsilon_1], \\ O_1 \cdot [\alpha_1] &= [\alpha_1], \\ O_1 \cdot [\psi_1^2] &= [\psi_1^2]. \end{aligned}$$

By Proposition 2.6, the abelian group $\gamma_1(K_2)/\gamma_2(K_2)$ is generated by

$$[\varepsilon_1, \alpha_1] \cdot \gamma_2(K_2), \qquad [\varepsilon_1, \psi_1^2] \cdot \gamma_2(K_2), \qquad [\alpha_1, \psi_1^2] \cdot \gamma_2(K_2).$$

But $[\varepsilon_1, \alpha_1] = [\alpha_1, \varepsilon_1]^{-1} = 1$ by Proposition 6.16 and so the group $\gamma_1(K_2)/\gamma_2(K_2)$ is generated by $[\varepsilon_1, \psi_1^2] \cdot \gamma_2(K_2)$ and $[\alpha_1, \psi_1^2] \cdot \gamma_2(K_2)$.

Lemma 6.18. *The simple commutators $[\varepsilon_1, \psi_1^2]$ and $[\alpha_1, \psi_1^2]$ have order one or two modulo $\gamma_2(K_2)$.*

Proof. We know from Proposition 6.16 that $[\varepsilon_1^{-1}, \psi_1^2] \circ [\varepsilon_1, \psi_1^{-2}] = 1$. It follows by Lemma 2.9 that

$$1 = [\varepsilon_1^{-1}, \psi_1^2] \circ [\varepsilon_1, \psi_1^{-2}] \equiv [\psi_1^2, \varepsilon_1] \circ [\psi_1^2, \varepsilon_1] \mod \gamma_2(K_2).$$

□

A short calculation with MAGMA (see Appendix) shows that the order of $[\varepsilon_1, \psi_1^2]$ and $[\alpha_1, \psi_1^2]$ is two modulo $\gamma_2(K_2)$. In fact we obtain the following proposition.

Proposition 6.19. *The group $\gamma_1(K_2)/\gamma_2(K_2)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. In particular the order of $\gamma_1(K_2)/\gamma_2(K_2)$ is finite.*

If we apply now Corollary 2.13, we see that all quotients $\gamma_i(K_2)/\gamma_{i+1}(K_2)$ are of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{2,i}}$ for $i \geq 1$. Moreover we obtain the following proposition.

Proposition 6.20. *Let $i \geq 1$. Then the group $\gamma_i(K_2)/\gamma_{i+1}(K_2)$ is a finite abelian group of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{2,i}}$ with*

$$0 \leq b_{2,i} \leq 3^{i-1} \cdot 2.$$

Proof. Apply Corollary 2.13 together with Proposition 6.17 and Proposition 6.19. □

We computed the numbers $b_{2,i}$ for $i = 1, \ldots, 9$ with the help of MAGMA (see Apendix). Here is the result.

$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$b_{2,5}$	$b_{2,6}$	$b_{2,7}$	$b_{2,8}$	$b_{2,9}$	$\ldots$
2	4	6	10	14	22	32	48	70	

Remark 6.21. After computing one value of the $b_{2,i}$'s, say b_{2,i_0},we can improve the estimation of the $b_{2,i}$'s ($i \geq i_0$) by Corollary 2.13. This is illustrated in the following table.

$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$b_{2,5}$	$b_{2,6}$	$b_{2,7}$	$b_{2,8}$	$b_{2,9}$	$\dots$
2	≤ 6	≤ 18	≤ 54	≤ 162	≤ 486	≤ 1458	≤ 4374	≤ 13122	
2	4	≤ 12	≤ 36	≤ 108	≤ 324	≤ 972	≤ 2916	≤ 8748	
2	4	6	≤ 18	≤ 54	≤ 162	≤ 486	≤ 1458	≤ 4374	
2	4	6	10	≤ 30	≤ 90	≤ 270	≤ 810	≤ 2430	
2	4	6	10	14	≤ 42	≤ 126	≤ 378	≤ 1134	
2	4	6	10	14	22	≤ 66	≤ 198	≤ 594	
2	4	6	10	14	22	32	≤ 96	≤ 288	
2	4	6	10	14	22	32	48	≤ 144	
2	4	6	10	14	22	32	48	70	

Finally, we give a conjecture about the numbers $b_{2,i}$, which is based on an observation about the known values of $b_{2,1} - b_{2,9}$.

Conjecture 6.22. *The number $b_{2,i}$ is given by the following formula*

$$b_{2,i} = \begin{cases} b_{2,i-1} + b_{2,i-3} & \text{for } i \text{ odd} \\ b_{2,i-1} + b_{2,i-3} + 2 & \text{for } i \text{ even.} \end{cases}$$

6.4 HIGHER QUOTIENTS OF THE LOWER CENTRAL SERIES

In this section we consider quotients of the lower central series of K_n for $n \geq 3$. We use the notation $\gamma_i^n := \gamma_i(K_n)$. Our second main theorem states the surprising fact that the the quotients $\gamma_i^n/\gamma_{i+1}^n$ ($i \geq 1$) are finite abelian groups of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{n,i}}$ with

$$0 \leq b_{n,i} \leq (3n-3)^{i-1} \cdot \left(3n^2 - 7n + 4\right).$$

By Corollay 4.15 we know that K_n is generated by ε_i, α_i and ψ_i^2 for $i = 1, \dots, n-1$. Hence by Proposition 2.6 the abelian group γ_1^n/γ_2^n is generated by the following elements

$$
\begin{array}{ll}
[\varepsilon_i, \varepsilon_j] \cdot \gamma_2^n \ (i < j), & [\varepsilon_i, \alpha_j] \cdot \gamma_2^n, \\
[\varepsilon_i, \psi_j^2] \cdot \gamma_2^n, & [\alpha_i, \alpha_j] \cdot \gamma_2^n \ (i < j), \\
[\alpha_i, \psi_j^2] \cdot \gamma_2^n, & [\psi_i^2, \psi_j^2] \cdot \gamma_2^n \ (i < j).
\end{array}
$$

Since $[\psi_i^2]$ and $[\alpha_i]$ have order one or two in K_n^{ab} (by Proposition 4.9 and Lemma 6.8), the elements $[\varepsilon_i, \alpha_j] \cdot \gamma_2^n$, $[\varepsilon_i, \psi_j^2] \cdot \gamma_2^n$, $[\alpha_i, \alpha_j] \cdot \gamma_2^n$, $[\alpha_i, \psi_j^2] \cdot \gamma_2^n$, $[\psi_i^2, \psi_j^2] \cdot \gamma_2^n$ have all finite order in γ_1^n / γ_2^n by Lemma 2.8. In fact they have order one or two.

Lemma 6.23. *Let $n \geq 2$. Then the following relations hold in K_n for $1 \leq i, j \leq n-1$:*

a) $[\varepsilon_i, \delta_j] = 1$, $[\varepsilon_i, \alpha_i] = 1$ *and* $[\psi_i^2, \psi_j^2] = 1$,

b) $[\alpha_i, \psi_i^2] = \mathrm{kon}_{ix}^2$ *and* $[\alpha_i, \psi_j^2] = \psi_j^{-4}$ $(i \neq j)$,

c) $[\alpha_i, \alpha_j] = \mathrm{kon}_{ix}^2 \circ \mathrm{kon}_{jx}^{-2}$ $(i \neq j)$.

Proof. a) $[\varepsilon_i, \delta_j] = 1$:

$$
\left\{ x \overset{\delta_j^{-1}}{\mapsto} y_j^{-1} x \overset{\varepsilon_i^{-1}}{\mapsto} y_j^{-1} x y_i^{-1} \overset{\delta_j}{\mapsto} x y_j^{-1} \overset{\varepsilon_i}{\mapsto} x \right\},
$$

$[\varepsilon_i, \alpha_i] = 1$:

$$
\left\{ \begin{array}{lllllllll}
x & \overset{\alpha_i}{\mapsto} & x^{-1} & \overset{\varepsilon_i^{-1}}{\mapsto} & y_i x^{-1} & \overset{\alpha_i}{\mapsto} & x y_i^{-1} & \overset{\varepsilon_i}{\mapsto} & x \\
y_i & \overset{\alpha_i}{\mapsto} & x y_i^{-1} x^{-1} & \overset{\varepsilon_i^{-1}}{\mapsto} & x y_i^{-1} x^{-1} & \overset{\alpha_i}{\mapsto} & y_i & \overset{\varepsilon_i}{\mapsto} & y_i
\end{array} \right\},
$$

$[\psi_i^2, \psi_j^2] = 1$:

$$
\left\{ \begin{array}{lllllllll}
y_i & \overset{\psi_j^{-2}}{\mapsto} & y_i & \overset{\psi_i^{-2}}{\mapsto} & y_i x^{-2} & \overset{\psi_j^2}{\mapsto} & y_i x^{-2} & \overset{\psi_i^2}{\mapsto} & y_i \\
y_j & \overset{\psi_j^{-2}}{\mapsto} & y_j x^{-2} & \overset{\psi_i^{-2}}{\mapsto} & y_j x^{-2} & \overset{\psi_j^2}{\mapsto} & y_j & \overset{\psi_i^2}{\mapsto} & y_j
\end{array} \right\},
$$

b) $[\alpha_i, \psi_i^2] = \mathrm{kon}_{ix}^2$:

$$
\left\{ \begin{array}{lllllllll}
x & \overset{\psi_i^{-2}}{\mapsto} & x & \overset{\alpha_i}{\mapsto} & x^{-1} & \overset{\psi_i^2}{\mapsto} & x^{-1} & \overset{\alpha_i}{\mapsto} & x \\
y_i & \overset{\psi_i^{-2}}{\mapsto} & y_i x^{-2} & \overset{\alpha_i}{\mapsto} & x y_i^{-1} x & \overset{\psi_i^2}{\mapsto} & x^{-1} y_i^{-1} x & \overset{\alpha_i}{\mapsto} & x^2 y_i x^{-2}
\end{array} \right\},
$$

$[\alpha_i, \psi_j^2] = \psi_j^{-4}$:

$$\left\{\begin{array}{lllllllll} x & \overset{\psi_j^{-2}}{\mapsto} & x & \overset{\alpha_i}{\mapsto} & x^{-1} & \overset{\psi_j^{2}}{\mapsto} & x^{-1} & \overset{\alpha_i}{\mapsto} & x \\ y_i & \overset{\psi_j^{-2}}{\mapsto} & y_i & \overset{\alpha_i}{\mapsto} & xy_i^{-1}x^{-1} & \overset{\psi_j^{2}}{\mapsto} & xy_i^{-1}x^{-1} & \overset{\alpha_i}{\mapsto} & y_i \\ y_j & \overset{\psi_j^{-2}}{\mapsto} & y_jx^{-2} & \overset{\alpha_i}{\mapsto} & y_jx^2 & \overset{\psi_j^{2}}{\mapsto} & y_jx^4 & \overset{\alpha_i}{\mapsto} & y_jx^{-4} \end{array}\right\},$$

c) $[\alpha_i, \alpha_j] = \mathrm{kon}_{ix}^2 \circ \mathrm{kon}_{jx}^{-2}$:

$$\left\{\begin{array}{lllllllll} x & \overset{\alpha_j}{\mapsto} & x^{-1} & \overset{\alpha_i}{\mapsto} & x & \overset{\alpha_j}{\mapsto} & x^{-1} & \overset{\alpha_i}{\mapsto} & x \\ y_i & \overset{\alpha_j}{\mapsto} & y_i & \overset{\alpha_i}{\mapsto} & xy_i^{-1}x^{-1} & \overset{\alpha_j}{\mapsto} & x^{-1}y_i^{-1}x & \overset{\alpha_i}{\mapsto} & x^2y_ix^{-2} \\ y_j & \overset{\alpha_j}{\mapsto} & xy_j^{-1}x^{-1} & \overset{\alpha_i}{\mapsto} & x^{-1}y_j^{-1}x & \overset{\alpha_j}{\mapsto} & x^2y_jx^{-2} & \overset{\alpha_i}{\mapsto} & x^{-2}y_jx^2 \end{array}\right\}.$$

□

Proposition 6.24. *For $n \geq 3$ the group γ_1^n/γ_2^n is generated by*

$$\begin{array}{ll} [\varepsilon_i, \alpha_j] \cdot \gamma_2^n \ (i \neq j), & [\varepsilon_i, \psi_j^2] \cdot \gamma_2^n, \\ [\alpha_i, \psi_j^2] \cdot \gamma_2^n. & \end{array}$$

where each of these generators has order one or two. In particular γ_1^n/γ_2^n is a finite abelian group of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{n,2}}$ with

$$0 \leq b_{n,2} \leq 3n^2 - 7n + 4.$$

Proof. By the above remark we know that the abelian group γ_1^n/γ_2^n is generated by the following elements

$$\begin{array}{ll} [\varepsilon_i, \varepsilon_j] \cdot \gamma_2^n \ (i < j), & [\varepsilon_i, \alpha_j] \cdot \gamma_2^n, \\ [\varepsilon_i, \psi_j^2] \cdot \gamma_2^n, & [\alpha_i, \alpha_j] \cdot \gamma_2^n \ (i < j), \\ [\alpha_i, \psi_j^2] \cdot \gamma_2^n, & [\psi_i^2, \psi_j^2] \cdot \gamma_2^n \ (i < j). \end{array}$$

By Lemma 6.23 we have

- $[\varepsilon_i, \alpha_i] \cdot \gamma_2^n = 0$ and $[\psi_i^2, \psi_j^2] \cdot \gamma_2^n = 0$,
- $[\alpha_i, \alpha_j] \cdot \gamma_2^n = [\alpha_i, \psi_i^2][\psi_j^2, \alpha_j] \cdot \gamma_2^n = [\alpha_i, \psi_i^2] \cdot \gamma_2^n - [\alpha_j, \psi_j^2] \cdot \gamma_2^n \ \ (i \neq j)$.

Thus it suffices to show that $[\varepsilon_i, \varepsilon_j]$ is a sum of $[\varepsilon_i, \alpha_j]$, $[\varepsilon_i, \psi_j^2]$ and $[\alpha_i, \psi_j^2]$ modulo γ_2^n.

$$\begin{aligned}
[\varepsilon_i, \varepsilon_j] &= \varepsilon_i \circ \varepsilon_j \circ \varepsilon_i^{-1} \circ \varepsilon_j^{-1} \overset{\text{L. 6.23 a)}}{=} \varepsilon_i \circ \varepsilon_j \circ \delta_j \circ \varepsilon_i^{-1} \circ \delta_j^{-1} \circ \varepsilon_j^{-1} \\
&= [\varepsilon_i, \varepsilon_j \circ \delta_j] \overset{\text{Prop. 4.9}}{=} [\varepsilon_i, \varepsilon_j \circ \psi_j^2 \circ \varepsilon_j^{-1} \circ \psi_j^2 \circ \alpha_j] \\
&\overset{\text{L. 2.2}}{\equiv} [\varepsilon_i, \varepsilon_j] \circ [\varepsilon_i, \psi_j^2] \circ [\varepsilon_i, \varepsilon_j^{-1}] \circ [\varepsilon_i, \psi_j^2] \circ [\varepsilon_i, \alpha_j] \\
&\overset{\text{L. 2.9}}{\equiv} [\varepsilon_i, \varepsilon_j] \circ [\varepsilon_i, \psi_j^2] \circ [\varepsilon_i, \varepsilon_j]^{-1} \circ [\varepsilon_i, \psi_j^2] \circ [\varepsilon_i, \alpha_j] \\
&\equiv [\varepsilon_i, \psi_j^2]^2 \circ [\varepsilon_i, \alpha_j] \mod \gamma_2^n.
\end{aligned}$$

This means

$$[\varepsilon_i, \varepsilon_j] \cdot \gamma_2^n = [\varepsilon_i, \alpha_j] \cdot \gamma_2^n.$$

For the estimation of the $b_{n,2}$, we count the numbers of generators:
We have $(n-1)(n-2)$ generators of the form $[\varepsilon_i, \alpha_j] \cdot \gamma_2^n$ $(i \neq j)$ and $2(n-1)^2$ generators of the form $[\varepsilon_i, \psi_j^2] \cdot \gamma_2^n$ or $[\alpha_i, \psi_j^2] \cdot \gamma_2^n$. It follows that there are at most

$$(n-1)(n-2) + 2(n-1)^2 = 3n^2 - 7n + 4$$

generators. Hence we obtain

$$0 \leq b_{n,2} \leq 3n^2 - 7n + 4.$$

□

Our second main theorem states the surprising fact that the quotients $\gamma_i(K_n)/\gamma_{i+1}(K_n)$ are finite groups for all $i \geq 1$.

Theorem 6.25. *Let $n \geq 2$ and $i \geq 1$. Then the group $\gamma_i(K_n)/\gamma_{i+1}(K_n)$ is a finite abelian group of the form $(\mathbb{Z}/2\mathbb{Z})^{b_{n,i}}$ with*

$$0 \leq b_{n,i} \leq (3n-3)^{i-1} \cdot \left(3n^2 - 7n + 4\right).$$

Proof. For the case $n = 2$ see Corollary 6.20. For $n \geq 3$ apply Corollary 2.13 to Proposition 6.24 and Proposition 6.9. □

By Proposition 4.9 the group K_n is not torsion-free. Thus by Proposition 2.16 there are two possibilities:

Proposition 6.26. *Let $n \geq 3$. Then we have either*

- *there is a natural number $i_0 \in \mathbb{N}_0$ such that $\gamma^n_{i_0}/\gamma^n_{i_0+1}$ is not torsion-free or*
- *K_n is not residually nilpotent, i.e. $\bigcap_{i=0}^{\infty} \gamma_i(K_n) \neq 1$.*

Chapter 7

Further results

In this chapter we present some results concerned with the relationship between the classical Torelli group $\mathrm{IA}(F_n)$ and the generalized Torelli group K_n.

7.1 $\mathrm{IA}(F_{n-1})$ as a subgroup of K_n

Let $n \geq 3$ and F_n be the free group generated by x, $y_1, \ldots, y_{n-1}$. Define

$$A_{n-1} := \left\{ \varphi \in \mathrm{Aut}(F_n) \;\middle|\; \begin{array}{l} \varphi(x) = x \\ \varphi(y_i) \in \langle y_1, \ldots, y_{n-1} \rangle \end{array} \right\} \leq \mathrm{Aut}(F_n).$$

Let $G_{n-1} \leq F_n$ be the subgroup generated by $y_1, \ldots, y_{n-1}$, which is a free group on the $n-1$ free generators $y_1, \ldots, y_{n-1}$. Define $\iota : \mathrm{Aut}(G_{n-1}) \to A_{n-1}$ to be the homomorphism which sends an automorphism $\varphi \in \mathrm{Aut}(G_{n-1})$ to the automorphism defined by

$$\{x \mapsto x, \quad y_1 \mapsto \varphi(y_1), \ldots, \quad y_{n-1} \mapsto \varphi(y_{n-1})\}.$$

Then the homomorphism ι is obviously an isomorphism. From now on we will identify A_{n-1} with $\mathrm{Aut}(G_{n-1})$.

The group $\mathrm{Aut}^+(G_{n-1})$ is generated by the automorphisms λ_{ij} and ν_{ij} for $1 \leq i, j \leq n-1$, $i \neq j$ and we see that $\mathrm{Aut}^+(G_{n-1}) \leq \Gamma^+(C_2, \pi)$ by Corollary 4.4. Let $\mathrm{IA}(G_{n-1})$ denote the classical Torelli group of $\mathrm{Aut}(G_{n-1})$. By Theorem 3.2 the group $\mathrm{IA}(G_{n-1})$ is generated by

$$\mathrm{kon}_{ij} : \{y_i \mapsto y_j y_i y_j^{-1}\} \quad \text{and} \quad \kappa_{ijk} : \{y_i \mapsto y_i y_j y_k y_j^{-1} y_k^{-1}\}.$$

By Lemma 4.11 we have $\mathrm{IA}(G_{n-1}) \leq K_n$.

Proposition 7.1. *Let $n \geq 3$. Then the following diagram commutes*

$$\begin{tikzcd}
K_n \arrow[r, hook] & \Gamma^+(C_2,\pi) \arrow[r, two heads, "\sigma_{-1}"] & \mathrm{GL}(n-1,\mathbb{Z}) \\
\mathrm{IA}(G_{n-1}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Aut}^+(G_{n-1}) \arrow[r, two heads, "\rho_1"] \arrow[u, hook] & \mathrm{SL}(n-1,\mathbb{Z}). \arrow[u, hook]
\end{tikzcd}$$

Proof. It suffices to show that the right square in the diagram commutes. We know from Chapter 3.1 that

$$\rho_1(\nu_{ij}) = E_{ji} \quad \text{and} \quad \rho_1(\lambda_{ij}) = E_{ji}.$$

Furthermore we have by Proposition 4.5

$$\begin{aligned}
\sigma_{-1}(\nu_{ij}) = E_{ji} \quad \text{and} \quad \sigma_{-1}(\lambda_{ij}) &\overset{\text{L. 4.4}}{=} \sigma_{-1}(\mathrm{kon}_{ix}^{-1} \circ \delta_j \circ \mathrm{kon}_{ix} \circ \delta_j \circ \nu_{ij}) \\
&= O_i^{-1} \cdot I_{n-1} \cdot O_i \cdot I_{n-1} \cdot E_{ji} = E_{ji}.
\end{aligned}$$

Since $\mathrm{Aut}^+(G_{n-1})$ is generated by ν_{ij} and λ_{ij} the Proposition follows. □

The inclusion $\iota : \mathrm{IA}(G_{n-1}) \hookrightarrow K_n$ induces a homomorphism

$$\iota : \mathrm{IA}(G_{n-1})^{\mathrm{ab}} \to K_n^{\mathrm{ab}}. \tag{7.1}$$

We compute the images of the generators of $\mathrm{IA}(G_{n-1})^{\mathrm{ab}}$ under ι:

$$\begin{aligned}
\iota[\mathrm{kon}_{ij}] &= [\mathrm{kon}_{ij}] \overset{\text{L. 6.8}}{=} [\alpha_j], \\
\iota[\kappa_{ijk}] &= [\kappa_{ijk}] \overset{\text{L. 6.8}}{=} 0.
\end{aligned}$$

Hence we proved the following proposition.

Proposition 7.2. *Let $n \geq 3$ and $\iota : \mathrm{IA}(G_{n-1})^{\mathrm{ab}} \to K_n^{\mathrm{ab}}$ be the map defined in* (7.1)*. Then we have*

$$\mathrm{Im}(\iota) = \mathcal{A}_n,$$

where $\mathcal{A}_n = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$ (see Chapter 6.2*). In particular $\mathrm{Im}(\iota)$ is either isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ or to* 0*.*

7.2 THE RELATION BETWEEN IA(F_n) AND K_n

In this section we assume that $n \geq 2$. Let F_n be the free group generated by $x, y_1, \ldots, y_{n-1}$ and $\operatorname{Inn}(F_n)$ be its group of inner automorphisms. For each $w \in F_n$ we have an induced inner automorphism $i(w)$, where

$$i(w)(y) = wyw^{-1}$$

for all $y \in F_n$. Since the center of F_n is trivial, the map $i : F_n \to \operatorname{Inn}(F_n)$ is an isomorphism, i.e.

$$\operatorname{Inn}(F_n) \cong F_n.$$

Hence we obtain the following generators for $\operatorname{Inn}(F_n)$.

Proposition 7.3. *The group* $\operatorname{Inn}(F_n)$ *is generated by the inner automorphisms* $i(x)$, $i(y_1), \ldots, i(y_{n-1})$.

Notice that for $n = 2$ we obtain

$$\mathrm{IA}(F_2) = \operatorname{Inn}(F_2) \cong F_2.$$

Hence $\mathrm{IA}(F_2)$ is a free group on two generators.

Lemma 7.4. *We have*

$$\operatorname{Inn}(F_n) \leq \mathrm{IA}(F_n).$$

Proof. We show that the generators of $\operatorname{Inn}(F_n)$ given by Proposition 7.3 are in $\mathrm{IA}(F_n)$:

$$\begin{aligned} i(x) &= \mathrm{kon}_{1x} \circ \ldots \circ \mathrm{kon}_{n-1,x} \in \mathrm{IA}(F_n) \\ i(y_j) &= \mathrm{kon}_{1j} \circ \ldots \circ \mathrm{kon}_{j-1,j} \circ \\ &\quad\ \mathrm{kon}_{j+1,j} \circ \ldots \circ \mathrm{kon}_{n-1,j} \circ \mathrm{kon}_{xj} \in \mathrm{IA}(F_n) \end{aligned}$$

for $1 \leq j \leq n-1$. □

Notice that by Lemma 4.3 we have

$$\operatorname{Inn}(F_n) \leq \mathrm{IA}(F_n) \leq \Gamma^+(C_2, \pi).$$

We are now going to analyze the representation σ_{-1} restricted to $\operatorname{Inn}(F_n)$. Let us first consider the images of the generators of $\operatorname{Inn}(F_n)$ under σ_{-1}.

Lemma 7.5. *We have*

$$\sigma_{-1}(i(x)) = -I_{n-1} \quad \text{and} \quad \sigma_{-1}(i(y_j)) = I_{n-1}$$

for $1 \leq j \leq n-1$. *In particular the image of* $\operatorname{Inn}(F_n)$ *under* σ_{-1} *is isomorphic to* C_2.

Proof.

$$\begin{aligned}
\sigma_{-1}(i(x)) &= \sigma_{-1}(\mathrm{kon}_{1x} \circ \ldots \circ \mathrm{kon}_{n-1,x}) \\
&\overset{\text{L. 4.5}}{=} O_1 \cdot O_2 \cdot \ldots \cdot O_{n-1} = -I_{n-1}, \\
\sigma_{-1}(i(y_j)) &= \sigma_{-1}(\mathrm{kon}_{1j} \circ \ldots \circ \mathrm{kon}_{j-1,j} \circ \\
&\qquad \mathrm{kon}_{j+1,j} \circ \ldots \circ \mathrm{kon}_{n-1,j} \circ \mathrm{kon}_{xj}) \\
&\overset{\text{L. 4.11}}{=} I_{n-1}.
\end{aligned}$$

□

If we identify $\operatorname{Inn}(F_n)$ with F_n itself and the image of $\operatorname{Inn}(F_n)$ under σ_{-1} with $C_2 = \langle g \mid g^2 = 1 \rangle$, we obtain by Lemma 7.5

$$\sigma_{-1} : F_n \to C_2.$$

This map is determined by

$$x \mapsto g, \quad y_1 \mapsto 1, \ \ldots \quad y_{n-1} \mapsto 1.$$

If we compare this map with $\pi : F_n \to C_2$ defined in Chapter 4.2, we see that they are identical. Thus we recover here the map $\pi : F_n \to C_2$, which was the starting point of the whole. Recapitulatory we have the following proposition.

Proposition 7.6. *Let* $n \geq 2$. *Then the following diagram commutes*

$$\begin{tikzcd}
1 \arrow[r] & R \arrow[r] \arrow[d, "\cong"', "i"] & F_n \arrow[r, "\pi"] \arrow[d, "\cong"', "i"] & C_2 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \operatorname{Inn}(F_n) \cap K_n \arrow[r] & \operatorname{Inn}(F_n) \arrow[r, "\sigma_{-1}"] & C_2 \arrow[r] & 1.
\end{tikzcd}$$

Notice that, since R is a free group on $2n-1$ generators, the group $\operatorname{Inn}(F_n) \cap K_n$ is also free on $2n-1$ generators.

Corollary 7.7. *Let $n \geq 2$. The group $\mathrm{IA}(F_n) \cap K_n$ contains a free group on $2n-1$ generators. In fact for $n = 2$ the group $\mathrm{IA}(F_2) \cap K_2$ is a free group on three generators.*

Proof. This follows immediately from the fact that

$$\mathrm{Inn}(F_n) \leq \mathrm{IA}(F_n)$$

for all $n \geq 3$ and

$$\mathrm{Inn}(F_2) = \mathrm{IA}(F_2).$$

□

Until here we considered the restriction of σ_{-1} to $\mathrm{Inn}(F_n) \leq \Gamma^+(C_2, \pi)$. We are now going to study the map σ_{-1} restricted to $\mathrm{IA}(F_n)$. Thus let us first calculate the images of the generators of $\mathrm{IA}(F_n)$ under σ_{-1}.

Lemma 7.8. *Let $n \geq 2$. We have*

$$\begin{array}{ll} \sigma_{-1}(\kappa_{ijk}) = 1, & \sigma_{-1}(\mathrm{kon}_{ij}) = 1, \\ \sigma_{-1}(\kappa_{ij}) = 1, & \sigma_{-1}(\mathrm{kon}_{ix}) = O_i, \\ \sigma_{-1}(\mathrm{kon}_{xi}) = 1, & \sigma_{-1}(\tau_{ij}) = E_{ji}^{-2}. \end{array}$$

Proof. The automorphisms κ_{ijk}, kon_{ij}, κ_{ij} and kon_{xi} are in the kernel of σ_{-1} by Lemma 4.11 and Theorem 4.14. We know from Proposition 4.5 that $\sigma_{-1}(\mathrm{kon}_{ix}) = O_i$. Hence it suffices to show that $\sigma_{-1}(\tau_{ij}) = E_{ji}^{-2}$:

$$\begin{aligned} \sigma_{-1}(\tau_{ij}) &\overset{\text{Cor. 4.4}}{=} \sigma_{-1}(\mathrm{kon}_{ix}^{-1} \circ \nu_{ij} \circ \mathrm{kon}_{ix} \circ \nu_{ij}^{-1}) \\ &\overset{\text{L. 4.5}}{=} O_i \cdot E_{ji} \cdot O_i \cdot E_{ji}^{-1} = E_{ji}^{-2}. \end{aligned}$$

□

It follows that the image of σ_{-1} restricted to $\mathrm{IA}(F_n)$ is generated by E_{ij}^2 for $1 \leq i, j \leq n-1$ ($i \neq j$) and O_i for $1 \leq i \leq n-1$. By Proposition 5.3 this image equals

$$\Gamma_{n-1}(2) = \{M \in \mathrm{GL}(n-1, \mathbb{Z}) \mid M \equiv I_{n-1} \mod 2\}.$$

Hence we obtain the following exact sequence

$$1 \to \mathrm{IA}(F_n) \cap K_n \to \mathrm{IA}(F_n) \xrightarrow{\sigma_{-1}} \Gamma_{n-1}(2) \to 1.$$

Notice that for $n = 2$ this sequence coincides with the sequence given in Proposition 7.6.

On the other hand we can restrict the map $\rho_1 : \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$ to $K_n \leq \mathrm{Aut}(F_n)$. Thus let us calculate the images of the generators of K_n under ρ_1.

Lemma 7.9. *Let $n \geq 2$. We have*

$$\rho_1(\varepsilon_i) = \rho_1(\delta_i) = E_{i+1,1} \quad \text{and} \quad \rho_1(\psi_i^2) = E_{1,i+1}^2$$

for $1 \leq i \leq n-1$.

Proof. This is clear by Chapter 3.1 □

It follows that the image of ρ_1 restricted to K_n is generated by E_{1j}^2 for $2 \leq j \leq n$ and E_{i1} for $2 \leq i \leq n$. By Proposition 5.5 this group equals

$$\widetilde{\Gamma_n^1}(2) = \left\{ A \in \mathrm{SL}(n, \mathbb{Z}) \mid A \equiv \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline * & & I_{n-1} & \end{array} \right) \mod 2 \right\}.$$

Thus we obtain an exact sequence

$$1 \to \mathrm{IA}(F_n) \cap K_n \to K_n \xrightarrow{\rho_1} \widetilde{\Gamma_n^1}(2) \to 1.$$

Chapter 8

Appendix

```
Aut<p,q,u>:=Group<p,q,u | p^2,q^2,(p*q)^4,
u*q*u*q*u^-1*q^-1*u^-1*q^-1,(p*q*p*u)^2,(u*p*q)^3>;

K2:=sub<Aut | u , p*u^2*p , p*u*q*u*p*q >;

H<a,b,c>:=Rewrite(Aut,K2);
```

```
Result:
> H;
Finitely presented group H on 3 generators
Generators as words in group Aut
a = p * u * p * u^-1 * q * p
b = q * p * u * q * u^-1 * p
c = p * u * q * u * p * q
Relations
c^2 = Id(H)
a * c * a^-1 * c = Id(H)
b^2 * c * b^-2 * c = Id(H)
c * a^-1 * b * a * b * a * c * b^-1 * a^-1 * b^-1 = Id(H)
```

```
K2<a,b,c>:=Group<a,b,c | a^2, b*a*b^-1*a^-1,
(c^-1*a)^2*(c*a)^2,(c^-1*b)^2*(c*b^-1)^2>;
N2:=NilpotentQuotient(K2,2);
gamma12<a,b>:=CommutatorSubgroup(N2);
```

```
Result:
> gamma12;
GrpGPC : gamma12 of order 2^2 on 2 PC-generators
PC-Relations:
a^2 = Id(gamma12),
b^2 = Id(gamma12)
```

Here is an alternative, which can also be used to compute higher quotients:

```
K2<a,b,c>:=Group<a,b,c | a^2, b*a*b^-1*a^-1,
(c^-1*a)^2*(c*a)^2,(c^-1*b)^2*(c*b^-1)^2>;
N1,pi1:=NilpotentQuotient(K2,1);
N2,pi2:=NilpotentQuotient(K2,2);
f := hom< N2->N1 | [ pi2(g)->pi1(g) :
g in Generators(K2)]>;
Kernel(f);
```

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