Graduate Student Seminar in Sommer Term 2011: Chromatic Homotopy Theory

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The stable homotopy groups of spheres are one of the central objects of algebraic topology. While it is relatively easy to compute the first few of them, the search for infinite families of elements in them requires more conceptual methods. These will be the topic of our seminar.

Particularly well adapted to this topic is the Adams-Novikov spectral sequence (ANSS). Originally, this was based on complex cobordism MU, but for our purposes it will be more convenient to base it on Brown-Peterson homology BP. Then the ANSS is a spectral sequence which converges to the stable homotopy groups of spheres, localized at a prime p. We will restrict to odd primes. Here, the 1-line of the ANSS represents exactly the elements in the (geometrically important) image of $J\colon \pi_*SO \to \pi_*^s(S^0)$. Furthermore, they come (up to multiples by a natural number) all to existence in the following way: There is a stable self-map $f\colon \Sigma^{2(p-1)}M(p) \to M(p)$ of the mod p Moore spectrum which induces an isomorphism in K-theory. Therefore, also its t-th iteration $f^t\colon \Sigma^{t\cdot 2(p-1)}M(p) \to M(p)$ is not zero. One gets now an infinite family of stable homotopy elements α_t as the composition

$$S^{t\cdot 2(p-1)-1} \to \Sigma^{t\cdot 2(p-1)-1}M(p) \to \Sigma^{-1}M(p) \to S^0.$$

Now we can ask ourselves: Can we generalize this to the 2- and the 3-line of the ANSS to get further infinite families β_t and γ_t? Indeed, there are complexes $V(2)$ (for $p > 3$) and $V(3)$ (for $p > 5$) with suitable self-maps, which produce stable homotopy elements in a manner similar to the one above. But how to show that these are non-zero?

Here again the ANSS comes into play: there is an algebraic procedure creating elements in the E^2-term of the ANSS which represent the β_t and γ_t. To show that these are non-zero, we have to study the E^2-term of the ANSS closer.

The main tool for this is the chromatic spectral sequence (CSS) converging to the E^2-term of the ANSS. The main players in the CSS are certain modules M_n^0 and everything in the E^2-term of the CSS is closely related to their cohomology. While the E^2-term of the ANSS is the cohomology of a Hopf algebroid over a big ring, by the Morava change-of-rings theorem the cohomology of the M_n^0 is the comology of a Hopf algebra over a field (the Morava stabilizer algebra), which is much simpler to compute. In addition, by another theorem of Morava it can be identified with the cohomology of a certain profinite group related to formal group laws, which allows to apply conceptual tools to prove general structure theorems.

At the end, we will compute with these methods the first two lines of the ANSS and show the non-triviality of the β_t and γ_t (or at least sketch the proofs, where the computations become too involved).

The seminar can be roughly divided into 3 parts.

- The first two talks introduce the basics, where everything (except talk 3) depends on.
- Talks 3-5 form the "geometric" part of the seminar, where the infinite families α_t, β_t and γ_t are constructed and their relationship to the ANSS is discussed. Talk 3 is really a classical topic since here some of the most important work of Adams is discussed.

- The talks 6-11 are devoted to the ANSS E_2-term and are nearly totally independent from the talks 3-5. Talk 6 introduces the chromatic spectral sequence, talks 7,8 and 10 are devoted to the Morava stabilizer algebra and group and its cohomology and talks 9 and 11 will actually apply this to compute the first two lines of the E_2-term of the ANSS and show the non-triviality of the β_t and γ_t. In talk 12, I will sketch in short words some more recent developements and recapitulate what we have done in the seminar.

I have no affirmation for the dates yet, but I think they should be correct.

List of Talks

Talk 1 - Brown-Peterson Homology - 14.4.
Define the Thom spectrum MU and describe its homotopy groups, homology groups ([Wil82], p.3-6) and its self-maps ([Wil82], 1.53). The main task of this talk is then to prove the existence of BP as a direct summand of $MU_{(p)}$ ([Wil82], 3.1) and discuss the generators v_i of its homotopy ([Wil82], 3.15). Along the way, you have to say at least something about formal group laws ([Wil82], section 2).

Talk 2 - The Adams-Novikov Spectral Sequence - 14.4.
The aim of this talk is to present the construction of the Adams-Novikov spectral sequence based on BP. There are several sources which present the construction of the general Adams spectral sequence, for example [Ada74], Chapter III.15, [Rav86], Chapter 2.2, or [Swi75], Chapter 19. To give a description of the E^2-term of the Adams spectral sequence, you need to define Hopf algebroids and their basic homological algebra, which is done in some way in all the sources, but the most extensive description can be found in [Rav86], Appendix A. You might also want to look at notes of Irakli, who gave once a very carefully prepared talk about the Adams spectral sequence. You have probably no time to do all the proofs, but this is not a great problem since these will not be important later; nevertheless you should at least carefully describe the construction.

There are some things you could do to give us a concrete feeling for the the special case of BP. First, you could state the Hopf algebroid structure of BP; second, you could draw a chart of the ANSS for BP in a certain range for an odd prime (or let the projector do this for you), e.g. $p = 3$, $t-s \leq 45$ ([Rav86], p.13); third, you could prove Proposition 4.4.2 of [Rav86].

Talk 3 - The α-family - 28.4.
In this talk, we will construct the first infinite family in the stable homotopy groups of spheres. The main theorem is [Ada66], 1.7, whose precise statement and proof are explained in §12 of the same paper. One of Adams's main tools is the e-invariant, which is explained in the same paper. You need little more than the definition of the e-invariant, Proposition 3.2 about the relationship between degree and e-invariant and §8 about the relationship between Hopf invariant and e-invariant. If time, you may also say something about the J-homomorphism.

Talk 4 - Toda-Smith complexes and the Adams-Novikov Spectral Sequences - 28.4.
Begin by defining the Toda-Smith complexes $V(n)$ and the corresponding stable homotopy elements α_t, β_t and γ_t (see [MRW77], bottom of p. 478, and [Nav10], p.1-2). Then define the greek letter elements in the E^2-term of the ANSS as, eg, in [Nav10], p.2, and discuss how they relate to the stable homotopy elements obtained from the Toda-Smith complexes using [Rav86], 2.3.4, (the original source is [JMWZ75]). If time, discuss how the 1-line of the Adams-Novikov spectral is related to the Image of J at odd primes ([Rav86], 5.3: p. 168-170).

Talk 5 - The Construction of $V(2)$ and $V(3)$ and the Steenrod Algebra (80 minutes) - 12.5.
Introduce the (dual) Steenrod algebra (at odd primes) and prove the (easier) back direction of Theorem 1.2 of [Nav10], characterizing Toda-Smith complexes by their ordinary (co)homology as (co)modules over the (dual) Steenrod algebra. Discuss also the remark after Theorem 1.2. Then prove (as much as you can of the) existence of $V(2)$ and $V(3)$ as in Theorem 1.1 of [Tod71].

Talk 6 - The Chromatic Spectral Sequence (70 minutes) - 12.5.
The aim of the talk is to introduce the chromatic spectral sequence. This is introduced in [MRW77], 3.A. The chromatic spectral sequence has a tight connection to the (generalized) greek letter elements, which is explained in 3.B. The key lemma is 3.7, which uses [Lan75], 2.5. (where the relevant part for us is explained at the beginning of section 3; you might also have a look at [Eis95], 3.3., for the necessary parts of commutative algebra), and [MR77], 3.2. You might also want to have a look at the beginning of section 4.3 of [Rav86].

Talk 7 - The Morava Change of Rings Theorem - 26.5.
The aim of this talk is to prove the change of ring isomorphism which is given by Theorem 6.1.1 in [Rav86]. Note that the Morava stabilizer algebra $\Sigma(n)$ is called $K(n)_*K(n)$ in [MRW77]. The proof begins on the bottom of p. 190. The main work is to prove the general change of ring isomorphism for Hopf algebroids A1.3.12.

Talk 8 - Structure and Cohomology of the Morava Stabilizer Algebra - 26.5.
The change of rings isomorphism of the last talk stresses the importance of the Morava stabilizer algebra. First, you need to study the filtration of the Morava stabilizer algebra induced by the one given before Theorem 4.3.24 in [Rav86]. Its associated graded is the universal enveloping algebra of a restricted Lie algebra (see p.68), which can be given explicitely (Theorem 6.3.3) - as always ignore $p = 2$. Using as input Lie algebra cohomology, the May spectral sequence can be used to compute the cohomology of the Morava stabilizer algebra. Give the simplification of its E^2-term as in Theorem 6.3.5 for $n < p - 1$.

Talk 9 - The 1-line of the ANSS and the β-familiy - 30.6.
Begin by proving Proposition 3.18a and b of [MRW77] - this is based on 6.3.21 in [Rav86]. The main aim of this talk is to compute the 1-line of the ANSS for $p > 2$ and show the nontriviality of the β-family. This is done in [MRW77], section 4 up to 4.9.

Talk 10 - The Morava Stabilizer Group - 30.6.
Introduce the group S_n and discuss its relationship to the Morava stabilizer algebra (Theorem 6.2.3 of [Rav86]). Then prove the rest of Proposition 3.18 of [MRW77] based on [Rav86], 6.3.12. At least state 6.2.10, better discuss its conceptual reasons as on p. 196-197.

Talk 11 - The 2-line of the ANSS and the γ-family (80 minutes) - 14.7.
The aim of this talk is to sketch the computation of the 2-line of the ANSS. First prove Theorem 5.3 of [MRW77] (without proving Proposition 5.4 in detail). Then give the proof strategy for Theorem 6.1 (p. 496). The computation is completed in §7 up to 7.5. At last, deduce corollary 7.8, where the non-triviality of the γ-family is proven.

Short Talk 12 - Recapitulation, Comments and Perspectives (Lennart Meier) - 14.7.

References

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