Doctoral Student Seminar, Winter Term 2010/11

Geometric Group Theory

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The main idea behind geometric group theory is as follows: Study a group by looking at properties of some related metric space. A prominent example for this is the classifying space BG, which allows a geometric / topological approach to the group (co)homology $H_*(G)$.

The seminar consists of four more or less independent topics:

1. We will get to know ends of metric spaces. They reflect topological properties of complements of arbitrary large compacts in the space. Assigning to a group G its Cayley graph Γ, we can decide whether G is virtually $\mathbb{Z}$ or not by looking at the ends of Γ.
2. Here we will explicitely construct "small" classifying spaces for right-angled Artin groups. For this, we will consider the property of being CAT(0), which can be seen as a generalization of non-positive curvature to arbitrary metric spaces.
3. A construction for classifying spaces "up to dimension n" is given for so-called Bestvina–Brady groups, implying some finiteness conditions on their resolutions. This will be done using piecewise affine "Morse functions" on certain CW complexes.
4. The last part concerns Outer Space. It allows to deduce interesting results about the automorphism group of a free group, such as estimates for the virtual cohomology dimension as well as a nice proof that $\mathrm{Aut}(F_n)$ is generated by "elementary" automorphisms.

PART I: Ends

Remark. This first part can be seen as a warm-up for the seminar. The talks are not too difficult and one can very clearly see the idea behind geometric group theory: For a group G we define a topological space $\mathrm{Ends}(G)$ and analyze G by studying $\mathrm{Ends}(G)$.

Oct. 21, 2010

Talk 1: **Introduction to Ends / Beginning of the End.**

The aim of this talk is to define the space of ends of a f.g. (finitely generated) group. First, define the space of ends $\mathrm{Ends}(X)$ for a metric space X. Define what quasi-isometries are, briefly sketch the concept of proper and geodesic spaces and show that quasi-isometries between such spaces induce homeomorphisms between their spaces of ends. This can be found in [3], p.144/45. Define the Cayley graph for a group presentation and give the main ideas of Theorem 1.5 in [10]. Conclude.

TALK 2: **The Interplay between Goups and their Ends.**

Prove Theorem 8.32 (1)-(4) in [3]. If time permits, show that the space of ends of a tree is homeomorphic to the Cantor set ([3], Exercise 8.31).

State (without proof) the Theorem of Milnor–Svarc (see [10], §2) and show how this implies that a f.g. group is quasi-isometric to its subgroups of finite index.

PART II: Artin Groups and Salvetti Complexes

Remark. The aim of this part is to see that Salvetti complexes are finite dimensional classifying spaces for right-angled Artin groups. The leitmotif of this part is as follows: Salvetti complexes are locally CAT(0), hence their universal covers are (globally) CAT(0) and therefore contractible, implying that Salvetti complexes are $K(\pi, 1)$'s.

Nov. 4, 2010

TALK 3: **Right-angled Artin Groups.**

Carefully introduce braid groups to motivate the definition of Artin groups. Define the right-angled Artin group associated to a graph. Give examples, e.g. $\mathbb{Z}^2$ and F_2. Define Salvetti complexes $\mathcal{S}$ and mention that $\mathcal{S}_{\mathbb{Z}^2} \cong S^1 \times S^1$ and $\mathcal{S}_{F_2} \cong S^1 \vee S^1$. This material is covered in [5], p.2-10. Finally you should briefly mention that we want to prove that Salvetti complexes are classifying spaces for right-angled Artin groups.

TALK 4: **CAT(0) and locally CAT(0) spaces.**

Introduce geodesic spaces and uniquely geodesic spaces as in [3], p.4. Take your time to introduce the notion of CAT(0) and locally CAT(0). You should mention that CAT(0) can be thought of as generalizing non-positive curvature to arbitrary metric spaces. (You may want to draw pictures.) We do *not* need the notion of CAT(κ) for $\kappa \neq 0$. State (and prove) Corollary II.1.5 in [3] without mentioning Proposition II.1.4. (This should be a rather easy exercise.) To give an(other) intuitive meaning to CAT(0), state that a Riemannian manifold is locally CAT(0) if and only if its sectional curvature is ≤ 0 ([3], Corollary II.1A.6). If time permits, indicate the main ideas of the proof.

Nov. 18, 2010

TALK 5: **From local to global.**

State the Cartan–Hadamard Theorem ([3], II.4.1) which states that a simply connected, locally CAT(0) space is (globally) CAT(0). Since the remainder of this talk only consists of one simple example, you should devote most of your time to explaining the main ideas of the proof of this theorem. Example: Every simply connected Riemannian manifold of sectional curvature ≤ 0 is contractible. In particular, the universal cover of any hyperbolic surfaces is contractible, namely $\mathbb{H}^2$.

Talk 6: **Proof of the Main Theorem.**

Define flag complexes (e.g. [3], II.5.15) and cubical complexes (e.g. [9], p.122). Define links in cubical complexes (if you don't know what links are, see e.g. [5], p.8 to get an idea or ask the organizers).

Try to explain the proof of the Proposition in [9], §4.2.C, p. 122, as detailed as possible. (The statement can also be found in [3], Theorem II.5.20). Conclude that Salvetti complexes are classifying spaces for right-angled Artin groups.

PART III: Finiteness Properties of Groups

Remark. This part continues the study of right-angled Artin groups and Salvetti complexes, even though with a slightly different flavor.

Dec. 2, 2010

Talk 7: **Affine Morse Theory.**

Define affine CW complexes and give a survey on (affine) Morse theory on affine CW complexes. For this you will have to introduce ascending and descending links. The reference for the whole talk is [8], §8.3.A-B, p.187-190.

Talk 8: **Finiteness Properties of Groups.**

Your talk should cover [8], §8.3.C-D, p.190-192. If time permits, define FH_n and FL_n and provide some implications between finiteness properties ([2], §3).

Dec. 16, 2010

Talk 9: **The Theorem of Bestvina–Brady.**

State and proof the Theorem of Bestvina–Brady ([8], Theorem 8.3.12, p.192ff).

PART IV: Outer Space

Remark. The material of this part has been covered in Karen Vogtmann's Felix Klein Lectures 2010. If you have attanded them, we strongly recommend to also employ your personal notes. However, having attended these lectures is not a necessary condition for giving one of the following talks!

Jan. 20, 2011

Talk 10: **Constructing Outer Space.**

Define graphs (= 1-dimensional CW complexes), metric graphs and self-homotopy equivalences of graphs. Give a detailed and enlightening construction of Outer Space. The main reference for this talk is [7]. To get a general idea, [11], p.5-7, and [12] are also recommendable.

Talk 11: **Properties of Outer Space.**

Indicate why Outer Space is contractible ([7], Theorem, p.93). Show that the action of $\mathrm{Out}(F_n)$ on Outer Space is properly discontinuous and has finite stabelizers. (This follows from two facts: (1) the action is simplicial and (2) the self-isometry group of a finite metric graph is finite.) Finally, define the spine inside

Outer Spine and conclude that the virtual cohomology dimension of $\mathrm{Out}(F_n)$ is (at most) $2n-3$, see e.g. [11], p.8. (Equality then follows from the fact that $\mathrm{Out}(F_n)$ contains a free abelian subgroup of rank $2n-3$.)

Feb. 3, 2011

Talk 12: **Elementary Automorphisms generate $\mathrm{Aut}(F_n)$.**

The aim of this talk is to show that $\mathrm{Aut}(F_n)$ is generated by so-called *elementary automorphisms*. A proof of a (slightly) stronger result can be found in [1].

Talk 13: **Voting for next term's topic.**

Contribute your suggestion on next term's seminar and vote for your favorite(s).

References

[1] **H. Armstrong, B. Forrest, K. Vogtmann**, *A Presentation for* $\mathrm{Aut}(F_n)$, http://www.math.cornell.edu/∼vogtmann/papers/Presentation/

[2] **M. Bestvina, N. Brady**, *Morse Theory and Finiteness Properties of Groups*, Inventiones Mathematicae, Volume 129, **3**, 445–470, (1997).

[3] **M. R. Bridson, A. Haefliger**, *Metric Spaces of non-positive Curvature*, Grundlehren der mathematischen Wissenschaften, Volume 319, Springer, (1999).

[4] **K. S. Brown**, *The Geometry of Rewriting Systems: A Proof of the Anick–Groves–Squier Theorem*, Proceedings of the workshop on algorithms, word problems, and classification in combinatorial group theory, **23**, MSRI Publications, Springer, 121–136, (1992).

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[6] **D. E. Cohen**, *String Rewriting and Homology of Monoids*, Math. Struct. in Comp. Science, **11**, 207–240, Cambridge University Press, (2000).

[7] **M. Culler, K. Vogtmann**, *Moduli of Graphs and Automorphisms of Free Groups*, Invent. Math., Volume 1, **84**, 91–119, (1986).

[8] **R. Geoghegan**, *Toplogical Methods in Group Theory*, Graduate Text in Mathematics, **243**, Springer, (2008).

[9] **M. Gromov**, *Hyperbolic Groups*, Essays in Group Theory, 75–263, Math. Sci. Res. Inst. Publ., **8**, Springer, (1987).

[10] **C. Leininger, A. Reed**, *Notes on Geometric Group Theory*, http://www.math.uiuc.edu/_clein/

[11] **K. Vogtmann**, *Automorphisms of free groups and Outer space*, http://www.math.cornell.edu/∼vogtmann/papers/Autosurvey/

[12] **K. Vogtmann**, *What is Outer Space?*, http://www.math.cornell.edu/∼vogtmann/papers/WhatIs/KVWhatIs.pdf