HOMOTOPY & COHOMOLOGY

GK 1150

GRK 1150, Mathematisches Institut, Universität Bonn, 53115 Bonn

Winter School

"From Field Theories to Elliptic Objects"

February, 28th till March, 4th 2006
Schloss Mickeln, Düsseldorf

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Talk No. 3

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Field theories

- physical pt. of view
- Σ-models
- functorial definition of QFTs

physics

A section of a suitable bundle over spacetime Y is called a field, i.e. $\phi \in \Gamma(Y, E)$.

ask for fields which extremize an action

$$S[\phi(y)] = \int_Y \mathcal{L}(\phi, \partial\phi)\, dvol(Y)$$

The Euler-Lagrange eq's are now PDE for the field ϕ.

Quantization : 2 options

(a) path integrals, try to make sense of

$$\int \exp i S[\phi]\, [d\phi(Y)]$$

(b) need time, $Y = X \times \mathbb{R}$

define momentum densities

$$\pi := \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}$$

promote fields and momenta to operators $\hat{\phi}, \hat{\pi}$
and impose canonical (anti-) commutation relations:
@ equal time

$$\left[\hat{\pi}(x,t), \hat{\phi}(x',t) \right]_{\pm} = \pm i \delta(x,x')$$

need Hilbert space where they act on,
sometimes there's an obvious choice, sometimes not.

Σ-models

<u>def</u> a <u>sigma model</u> is (the physical theory underlying) the study of maps between metric manifolds

$$\phi : (\Sigma^n, g) \longrightarrow (M, G) \quad \text{subject to the}$$

action $$S[\phi] = \int_{\Sigma^n} \tfrac{1}{2} g^{ab} \partial_a \phi^\mu \partial_b \phi^\nu G_{\mu\nu} |\det g|^{\frac{1}{2}} d^n\sigma$$

<u>rmks</u>

- this functional generalizes the action of the free particle in M.
- solutions to the resulting non-linear PDE are called harmonic / wave maps for riem. / lorentzian g
- if $n = \dim \Sigma = 2$, then the action is invariant under Weyl rescaling of the metric $g_{ab} \to e^{2f} g_{ab}$, ~~so it is a classical CFT~~. i.e. depends only on $[g]$.

- SUSY extensions exist

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functorial definition of QFTs

A QFT is a functor $E: \mathcal{B}^d \to \mathrm{Hilb}$
in d spacetime dimensions.

$\mathcal{B}^d$ d-dim. bordism category:

objects: closed orient $d-1$-dim. mfs Y

mor: or. pr. diffeos

and bordisms $\Sigma: Y_1 \to Y_2$, s. th. $\partial\Sigma = \overline{Y_1} \sqcup Y_2$

$\Sigma \sim \Sigma'$ if diffeomorphic relative bdry.

Hilb

obj.: separable Hilbert spaces / $\mathbb{C}$

mor: bounded linear operators

rmk $\sqcup, \emptyset$ and $\otimes, \mathbb{C}$ make these into monoidal cats

rmk/def involutions $\overline{}: \mathcal{B}^d \circlearrowleft$, $\mathrm{Hilb} \circlearrowleft$

which act on the objects and morphisms by reversing the orientations / the complex structure

$$\overline{\mathcal{B}^d(Y_1, Y_2)} = \mathcal{B}^d(\overline{Y_1}, \overline{Y_2})$$

$$f: H_1 \to H_2 \quad \rightsquigarrow \quad \overline{f}: \overline{H_1} \to \overline{H_2}$$

anti-involution: $*: \mathcal{B}^d \circlearrowleft$, $\mathrm{Hilb} \circlearrowleft$

id on the objects, reversing orientation / cx structure on the morphisms

$\Sigma : Y_1 \to Y_2$, $\overline{\Sigma}^{*} : Y_2 \to Y_1$, $f^{*} : H_2 \to H_1$

this reflects unitarity in QM.

(ex $f = e^{-i\hat{H}t}$ $f^{*} = e^{+i\hat{H}t}$)

adjunction formulae:

natural transf:

$$\mathcal{B}^d(\phi, Y_1 \sqcup Y_2) \longrightarrow \mathcal{B}^d(\overline{Y_1}, Y_2)$$

$$\mathrm{Hilb}(\mathbb{C}, H_1 \otimes H_2) \longrightarrow \mathrm{Hilb}(\overline{H_1}, H_2).$$

(in general not surjective)

add more geometric data; e.g.

(1) a riemannian metric on $\mathcal{B}^1$, to get a functor

$E : \mathcal{E}\mathcal{B}^1 \to \mathrm{Hilb}$ "euclidean field theory"

(2) a conformal class $[g]$ on $\mathcal{B}^2$, to get a conformal field theory $C : \mathcal{C}\mathcal{B}^2 \to \mathrm{Hilb}$.

example

Fix a mf M, closed, oriented, riemannian

$\mathcal{H} = L^2(M)$

define an EFT by the following assignments:

$E(pt) = \mathcal{H}$, $E(I_T) = e^{-T\Delta}$, $E(S^1_{T/2\pi}) = \mathrm{Tr}\, e^{-T\Delta/2}$

example the harmonic oscillator

$H : T^*\mathbb{R} \to \mathbb{R}$

$H(p,q) = \frac{1}{2}(p^2+q^2)$

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{q}^2 = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}q^2$$

$\mathcal{H} = L^2(\mathbb{R})$, $\hat{p} = -i\frac{\partial}{\partial x}$, $\hat{q} = q$.

def $\hat{a} := \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$, $\hat{a}^+ := \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})$

$$[\hat{a}, \hat{a}^+]_- = \frac{1}{2}\left[x + \frac{\partial}{\partial x}, x - \frac{\partial}{\partial x}\right]_- = 1$$

exercise

- reformulate $\hat{H}$ in terms of these operators & derive spectrum $\hat{H}$

- construct the wavefunction of the vacuum by requiring $\hat{a}\psi = 0$

$$= \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\psi$$

$\psi_0 = \text{const} \cdot \exp -\frac{1}{2}x^2$ const $= \pi^{-1/4}$ ensures normalization

$$\hat{H} = (a^+a + \tfrac{1}{2})$$

$$\text{spec}\,\hat{H} = \left\{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots\right\}$$

$a^+\psi_0$ is eigenfunction with eigenvalue $\frac{1}{2}+1$

$a^\dagger$ creates modes out of the vacuum

a annihilates modes

fermionic version:

$$\hat{H} = \hat{\psi}^* \hat{\psi} - \frac{1}{2} \qquad [\hat{\psi}^*, \hat{\psi}]_+ = 1$$

$$\operatorname{spec} \hat{H} = \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$$

$$\downarrow \qquad \uparrow$$

$$\hat{\psi} \downarrow = 0$$

$$\hat{\psi}^* \downarrow = \uparrow$$

$$\hat{\psi}^* \uparrow = 0$$

$$\hat{\psi} \uparrow = \downarrow$$