Fock Spaces

Part 1

Julia Singer
Universität Bonn
singer@math.uni-bonn.de

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Overview

- ▷ Motivation from physics: Several identical particles
- ▷ Describing a system of fermions
- ▷ Creation and annihilation operators
- ▷ The Clifford algebra action
- ▷ Lagrangians and algebraic Fock spaces
- ▷ The Segal-Shale equivalence criterion
- ▷ Orientations
- ▷ Functoriality

Motivation from physics: Several identical particles

Assumption: the state of one particle is given by an element in a Hilbert space $\mathcal{H}$.

Aim: Description of systems consisting of several identical, non-interacting particles.

Fermions are a class of particles (e.g. including electrons ☺, protons ☻) with the following properties:

- ▷ Particles of the same sort are indistinguishable.
- ▷ No two particles can be in the same state.

Describing a system of fermions

- ▷ Description of a system of k particles: specify an element of $\mathcal{H}$ for each particle.
- ▷ States can be entangled $\Rightarrow$ more accurate description is given by $\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$.
- ▷ By the properties above, $v \otimes v = 0$ for all $v \in \mathcal{H}$, hence we get $\Lambda^k \mathcal{H}$.
- ▷ Interpretation of $v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \mathcal{H}$ as k particles occupying the states $v_1, \ldots, v_k$.
- ▷ For an unknown number of identical fermions: elements of the exterior algebra $\Lambda \mathcal{H}$.

This vector space will become the (algebraic) **Fock space**.

Creation and annihilation

On $\Lambda\mathcal{H}$, we have the operations of creation and annihilation of particles:

- ▷ For each $v \in \mathcal{H}$: **Creation operator** (also denoted by v), given by

$$v(\xi) = v \wedge \xi$$

for $\xi \in \Lambda H$.
„Create a particle in the state v“

- ▷ **Annihilation operator** D_v for each $v \in \mathcal{H}$ (here: norm 1), given by

$$D_v(v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_n) = w_1 \wedge w_2 \wedge \cdots \wedge w_n$$

if all $w_i \perp v$.

The unit $1 \in \Lambda\mathcal{H}$ is called the vacuum state.

Creation and annihilation algebras

Algebra generated by **creation** operators: Exterior algebra $\Lambda\mathcal{H}$, acting on the Fock space by left multiplication.

Definition of the **annihilation** operator D_w for $w \in \mathcal{H}$ on the exterior algebra:

- ▷ $v \in \mathcal{H} = \Lambda^1\mathcal{H} \;\Rightarrow\; D_w(v) = \langle w, v\rangle \cdot 1 \in \Lambda^0\mathcal{H}$.
- ▷ $\xi, \eta \in \Lambda\mathcal{H} \;\Rightarrow\; D_w(\xi \wedge \eta) = D_w(\xi) \wedge \eta + (-1)^{|\xi|}\xi \wedge D_w(\eta)$.
  (graded derivation)

In the complex case: conjugate $\mathbb{C}$-action

Relations between annihilation operators: $D_v D_v = 0$ and $D_w D_v + D_v D_w = 0$

The annihilation operators also generate an exterior algebra $(\Lambda\overline{\mathcal{H}})^{op}$.
(„$D_v D_w$ annihilates $w \wedge v$“)

The Clifford algebra action on $\Lambda\mathcal{H}$

Combining the creation and annihilation operators into one algebra acting on $\Lambda\mathcal{H}$

Denote by

- ▷ $\overline{\mathcal{H}}$ the vector space of annihilation operators D_w (from now on also written as $\overline{w}$)
- ▷ $\alpha : \mathcal{H} \oplus \overline{\mathcal{H}} \to \mathcal{H} \oplus \overline{\mathcal{H}}$ the isometric involution given on $\mathcal{H}$ by $v \mapsto \overline{v}$.

Let b be the symmetric bilinear form on $V = \mathcal{H} \oplus \overline{\mathcal{H}}$ given by

$$b(x, y) = \langle \alpha(x), y \rangle.$$

Proposition The creation and annihilation operators define an action of the Clifford algebra $\mathcal{C}\ell(V)$ (with respect to the bilinear form b) on the Fock space $\Lambda\mathcal{H}$.

Proof of the Proposition

$\mathcal{H}$ and $\overline{\mathcal{H}}$ are isotropic with respect to $b \Rightarrow$ they generate the exterior algebras $\Lambda\mathcal{H}$ and $\Lambda\overline{\mathcal{H}}^{op}$.

Relations between creation and annihilation operators

Let $v, w \in \mathcal{H}$. Write any element of $\Lambda\mathcal{H}$ as a sum of elements of the form $x \wedge w_1 \wedge \cdots \wedge w_k$ with $w_i \perp v$.

$\Rightarrow$ enough to check relations on these. Let $\xi = w_1 \wedge \cdots \wedge w_k$.

Then we get

$$\begin{aligned} & (D_v w + w D_v)(x \wedge \xi) \\ = \quad & D_v(w \wedge x \wedge \xi) + w \wedge \langle v, x\rangle \xi \\ = \quad & D_v(-x \wedge w \wedge \xi) + \langle v, x\rangle w \wedge \xi \\ = \quad & -\langle v, x\rangle w \wedge \xi + x D_v(w) \wedge \xi + \langle v, x\rangle w \wedge \xi \\ = \quad & \langle v, w\rangle x \wedge \xi. \end{aligned}$$

This implies the relation

$$\begin{aligned} \overline{v} w + w \overline{v} \quad &= \quad D_v w + w D_v \\ = \langle v, w\rangle \quad &= \quad \langle \alpha(\overline{v}), w\rangle \; = \; b(\overline{v}, w). \end{aligned}$$

Lagrangians and algebraic Fock spaces

Definition Let V be a Hilbert space with isometric involution α ($\alpha(v) = \overline{v}$, $\mathbb{C}$-antilinear in the complex case), $b(v, w) = \langle \alpha(v), w\rangle$.
A **Lagrangian** of V is a closed subspace L which is isotropic with respect to b and for which $L \oplus \overline{L} = V$.

Example For $V = \mathcal{H} \oplus \overline{\mathcal{H}}$, the subspace $\mathcal{H}$ is Lagrangian.

Constructing algebraic Fock spaces from these data:

Definition For a Hilbert space V with involution α and a choice of Lagrangian L, the associated algebraic Fock space is the $\mathcal{C}\ell(V)$-module $F_{alg}(L) = \Lambda L$.

Graded modules

- ▷ $\mathcal{C}\ell(V)$ is $\mathbb{F}_2$-graded
- ▷ Decompose $\Lambda(L)$ as

$$\Lambda L = \bigoplus_{n \text{ even}} \Lambda^n L \oplus \bigoplus_{n \text{ odd}} \Lambda^n L$$

- ▷ $\mathbb{F}_2$-grading on the Fock space

Compatibility $\Rightarrow F_{alg}(L)$ becomes a graded module over $\mathcal{C}\ell(V)$.

Properties of Clifford algebras

- ▷ For a Hilbert space V with isometric involution α, denote by $-V$ the same space equipped with the involution $-\alpha$.
- ▷ Convention: if no involution is specified, always assume $\alpha = id$.

Natural isomorphisms:

- ▷ $\mathcal{C}\ell(V \oplus W) = \mathcal{C}\ell(V) \otimes \mathcal{C}\ell(W)$ (graded tensor product)
- ▷ $\mathcal{C}\ell(-V) = \mathcal{C}\ell(V)^{op}$

A $\mathcal{C}\ell(V \oplus (-W))$-module structure can also be viewed as a $\mathcal{C}\ell(V) - \mathcal{C}\ell(W)$-bimodule structure.

Inner product and completion

For a Hilbert space V, define an inner product on $\Lambda^k V$ by

$$\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle).$$

Remark Creation and annihilation are adjoint:

$$w \in \mathcal{H},\ \eta, \xi \in F_{alg}(L) \Rightarrow \langle D_w(\xi), \eta \rangle = \langle \xi, w \wedge \eta \rangle.$$

The fermionic Fock space is the completion of the algebraic Fock space with respect to this inner product.

The Segal-Shale equivalence criterion

Changing the Lagrangian

Let V be a Hilbert space with orthonormal basis $\{e_i\}$, and $\phi : V \to W$ an operator into a second Hilbert space W.

Recall: ϕ is called **Hilbert-Schmidt** if

$$\sum_i \|\phi(e_i)\|^2 < \infty.$$

Theorem (Segal-Shale) Let L and L' be two Lagrangians of V. The corresponding Fock spaces $F(L)$ and $F(L')$ are isomorphic if and only if the composition $L' \to V \to \overline{L}$ is a Hilbert-Schmidt operator.
The grading is preserved if and only if $\dim(\overline{L} \cap L')$ is even.

The finite-dimensional case

Proposition Let V be an n-dimensional real inner product space. Then there is a bijection between the set of linear isometries $f : \mathbb{R}^n \to V$ and the set $\mathcal{L}$ of Lagrangian subspaces of $V \oplus (-\mathbb{R}^n)$, given by $f \mapsto \Gamma_f$ (the graph of f).

Proof (isotropic) Let $v, w \in \mathbb{R}^n$; then

$$\begin{aligned} b(v + f(v), w + f(w)) &= \langle -v + f(v), w + f(w) \rangle \\ &= \langle -v, w \rangle + \langle f(v), f(w) \rangle \;=\; 0. \end{aligned}$$

With the usual topology on both sets, the bijection becomes a homeomorphism.

Orientations

- ▷ Linear isometries $f : \mathbb{R}^n \to V$ correspond to elements of $O(n)$; connected components of $O(n)$ to orientations of V.
- ▷ Application of π_0 to the homeomorphism above $\Rightarrow$

  $$\text{Orientations of } V \;\cong\; \pi_0(\mathcal{L}).$$
- ▷ **Fact**: In the finite-dimensional case, any irreducible module over $\mathcal{C}\ell(V \oplus (-\mathbb{R}^n))$ is isomorphic to some Fock space.
- ▷ Segal-Shale for finite-dimensional spaces: Any operator is Hilbert-Schmidt, hence all irreducible modules are isomorphic.
- ▷ Isomorphism classes of *graded* irreducible modules over $\mathcal{C}\ell(V \oplus (-\mathbb{R}^n))$ correspond to orientations of V.

Functoriality

In which sense are the constructions $V \mapsto \mathcal{C}\ell(V)$ and $L \mapsto F_{alg}(L)$ functorial?

Let $\mathcal{C}$ be the category with

- ▷ objects (V, α) (Hilbert spaces with isometric involution)
- ▷ morphisms $\mathcal{C}(V, V')$ given by the set of Lagrangians of $V' \oplus (-V)$.

Composition of morphisms $L_1 \subset V_2 \oplus (-V_1)$ and $L_2 \subset V_3 \oplus (-V_2)$: Let

$$L_3 = (L_2 \oplus L_1) \cap U^{\perp} / (L_2 \oplus L_1) \cap U \subset (V_3 \oplus -V_2 \oplus V_2 \oplus -V_1) \cap U^{\perp}/U,$$

where $U = \{0, v_2, v_2, 0\}$ and $U^{\perp}$ is the annihilator with respect to b.

For trivial involutions:

$$\begin{aligned} L_3 = \{v_3 + v_1 \in V_3 \oplus (-V_1) \mid \exists v_2 \in V_2 \quad &: \quad v_3 + v_2 \in L_2 \\ &\wedge \quad v_2 + v_1 \in L_1\}. \end{aligned}$$

Let $\mathcal{D}$ be the category of graded algebras with morphism sets $\mathcal{D}(A, B)$ given by pointed graded $B - A$-bimodules.

Composition of morphisms $(M, m_0) \in \mathcal{D}(A, B),\ (N, n_0) \in \mathcal{D}(B, C)$: given by the pointed $C - A$-bimodule $(M \otimes_B N, m_0 \otimes n_0)$.

Associate

- ▷ $(V, \alpha) \;\mapsto\; \mathcal{C}\ell(V)$ on objects,
- ▷ $(L \subset V' \oplus (-V)) \;\mapsto\; F_{alg}(L)$ on morphisms.

In special cases, this gives a functor $\mathcal{C} \to \mathcal{D}$:

Proposition If $L_1 \in \mathcal{C}(V_1, V_2)$ and $L_2 \in \mathcal{C}(V_2, V_3)$ and if the von Neumann-algebra generated by $\mathcal{C}\ell(V_2)$ in $B(F(L))$ is of type $\mathtt{I}$, i.e. the algebra of bounded operators on some Hilbert space, then we have

$$F_{alg}(L_1) \otimes_{\mathcal{C}\ell(V_2)} F_{alg}(L_2) \cong F_{alg}(L_3),$$

under the assumption that $L_i \cap V_j = 0$

Generalised Lagrangians

Let V be a Hilbert space with involution.

Definition A generalised Lagrangian of V is a homomorphism $L : W \to V$ with

- ▷ $\dim \ker(L) < \infty$
- ▷ such that the closure $\overline{L}_W$ of $im(L)$ is a Lagrangian of V.

Associated algebraic Fock space:

$$F_{alg}(L) = \Lambda^{top}(\ker L)^* \otimes \Lambda(\overline{L}_W).$$

- ▷ $(\ker L)^*$ dual space; $top = \dim(\ker L)$
- ▷ $\mathcal{C}\ell(V)$-action on the second factor.

Some points to remember

- ▷ Lagrangian subspace of (V, α): $L \oplus \alpha(L) = V$ and $b|_{V \times V} = 0$
- ▷ Algebraic Fock space associated to a Lagrangian L:
  ΛL with action of the Clifford algebra $\mathcal{C}\ell(V)$, induced by creation and annihilation operations
- ▷ Fock space: given by completing ΛL
- ▷ Isomorphism classes of graded $\mathcal{C}\ell(V) - \mathcal{C}\ell_n$-bimodules correspond to orientations of V.
- ▷ In special cases: functoriality

Problem session

Exercise 1 (One possible state). Suppose that $\mathcal{H} = \mathbb{R}$ is the Hilbert space describing a system of one particle and only one possible state s. Then the corresponding Fock space $\mathcal{H} = \Lambda^0\mathcal{H} \oplus \Lambda^1\mathcal{H}$ is two-dimensional with basis $\{1, s\}$. By writing the creation and annihilation operators as matrices with respect to this basis, it is easy to see that they generate the whole endomorphism algebra $\mathrm{End}_{\mathbb{R}}(\Lambda\mathcal{H})$.

Exercise 2. This holds more generally: Let $\mathcal{H}$ be an n-dimensional Hilbert space. Then $\mathcal{C}\ell(\mathcal{H} \oplus \overline{\mathcal{H}}) \cong \mathrm{End}(\Lambda\mathcal{H})$.

Solution. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathcal{H}$. Denote the annihilation operators corresponding to e_i by D_{e_i} and choose the set of all elements $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ with $0 < r \leq n$ and $i_1 < i_2 < \cdots < i_r$ as a basis for the Fock space.

Comparison of the dimensions yields 2^{2n} for both algebras, hence it is enough to show that for each two basis elements $x = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ and $y = e_{j_1} \wedge \cdots \wedge e_{j_s}$, we can find an operator mapping x to y and all other basis elements to 0. An operator satisfying these conditions (up to a sign) is given by the composition

$$e_{j_1} \circ \cdots \circ e_{j_s} \circ D_{e_1} \circ \cdots \circ D_{e_n} \circ e_{k_1} \circ \cdots \circ e_{k_{n-r}},$$

where $e_{k_1}, \ldots, e_{k_{n-r}}$ denote the elements not occurring in x. $\square$