Seminar on compact Lie groups and homogeneous spaces

The first objective of the seminar is to convey a feeling for the special properties of Lie groups. We are going to concentrate mostly on the following selection of properties of compact, connected Lie groups.

Our first highlight is the theorem of Peter and Weyl, which states that any compact Lie group can be embedded into some unitary group and is, in particular, isomorphic to some matrix group.

Our next goal will be the existence and uniqueness of maximal tori, which allows, for example, to reduce several questions about compact connected Lie groups to understanding the abelian case and the action of the (finite) Weyl group. As an application, we are going to be able to compute the cohomology ring of $BU(n)$ and the representation ring of $SU(n)$.

The next block of the seminar deals with the related concept of homogeneous spaces. These can always be described as quotients of a Lie group by a subgroup. After developing some techniques from representation theory, we're going to take a look at almost complex and complex structures on homogeneous spaces following Borel and Hirzebruch. Here, we make use of the fact that the adjoint representation associated to a homogeneous space encodes enough information to determine its characteristic classes. Afterwards, we turn our attention to a paper by Singhof where he considers the question which homogeneous spaces admit an (almost) parallelisable structure. His approach translates a nice K-theoretic description to a problem in representation theory which can be attacked by methods we have developed so far.

The representation theory of Lie groups allows us to make some surprising statements about the topology of Lie groups: We are going to show that π_2 of a compact connected Lie group vanishes; one can also describe the fundamental group with these methods. Similar considerations yield a structure theorem for compact connected Lie groups describing them in terms of easier pieces, the simple Lie groups.

The last goal of the seminar is to investigate the elements represented by Lie groups with the left-invariant framing in stable homotopy groups of spheres (these are identified with framed bordism via the Pontrjagin-Thom isomorphism). A basic ingredient for this part of the seminar is the geometric concept of the transfer map of a fibre bundle.

First, we are going to deal with another application of the transfer, namely with the proof of the Adams conjecture which allowed Adams to describe the image of J. Then we are ready to describe the filtration of Lie groups in the Adams-Novikov spectral sequence. We are also going to sketch why the order of the represented element in the stable stem is a divisor of 72, in particular, this implies that Lie groups don't give any interesting elements at higher primes.

If you are interested in giving a talk, contact us, i.e. write a mail indicating 2-3 talks you're willing to give to **ozornova@math.uni-bonn.de**. Please do so before **September, 20th**.

Finally we want to encourage all those who give talks to always feel free to discuss their topics - including all technical questions or calculations - with us.

13.10.2011

Talk 1. Basic Lie group theory (70 min) As most first talks, this one is also dedicated to be introductory. Apart from recalling the basic definitions, the major goal is to give a feeling for those properties of Lie groups which make them stand out in the class of differentiable manifolds. Some of these properties are:

- A homomorphism of connected Lie groups is determined by the differential at the unit.([BtD85], I.3.4)
- A connected abelian Lie group G is diffeomorphic to $T^k \times \mathbb{R}^l$ for some k and l.([BtD85], I.3.6)
- Any continuous group homomorphism between Lie groups is differentiable.([BtD85], I.3.12)
- Every connected Lie group is generated by any neighbourhood of the unit element.

Make sure you introduce integral curves and the exponential map in this context and try to convey a good feeling for what they do. Also introduce the adjoint representation and use the 1-parameter families to indicate that its derivative is given by the Lie bracket.
Reference: Main reference are Chapters I.1-I.3 in [BtD85]. Also helpful could be [Bau09] and [MT91].

Talk 2. The theorem of Peter and Weyl (80min) As the title suggests the objective of this talk is to prove the Theorem of Peter and Weyl stating - in the embedding version - that every compact Lie group admits an embedding into some unitary group $U(n)$. Start by introducing invariant integration. Then prove the functional analysis version of the theorem by actually doing some functional analysis. As an extended corollary you'll get that every compact Lie group admits a faithful finite dimensional representation which then allows you to deduce the embedding version.
References: [Ada69] and Chapter III in [BtD85].

27.10.2011

Talk 3. Homogeneous spaces, Stiefel manifolds and their cohomology Start by recalling the notion of a homogeneous space and explain why it is justified to think of it as a quotient of a Lie group by a subgroup([BtD85], I.4.6). Introduce Stiefel manifolds and calculate their cohomology by means of the Leray-Serre spectral sequence as either done in [Bor53] or [MT91].
As a second example calculate the cohomology of $U(n)/U(1) \times ... \times U(1)$ following Theorem III.4.6 in [MT91]. Show that $U(1) \times ... \times U(1)$ is a maximal torus of $U(n)$. Later on, we will use this cohomology to determine the cohomology of $BU(n)$.
Reference: [Bor53], [BtD85] Chapter I.4, and [MT91] Chapters III.3 and III.4.

Talk 4. Representation theory of abelian groups In the later talks we will see that the representations of Lie groups are in some sense determined by the representation of their maximal abelian subgroups. Therefore we need to establish a

good understanding for representations of abelian groups which is the goal of this talk. To warm up prove basic facts as Schur's Lemma ([BtD85], II.1.10) and the fact that irreducible complex representations of an abelian Lie group are one-dimensional ([BtD85], II.1.13). Since understanding representations means understanding their irreducible summands and since a representation is determined (up to isomorphism) by its characters, it suffices to determine the characters of irreducible representations of abelian groups. On the way, explain the concept of a weight (space).
References:[BtD85], Chapter II 1,4 and 8.

10.11.2011

Talk 5. Maximal tori Introduce the notion of a maximal torus and show its existence and uniqueness up to conjugation. For the proof you need to recall Lefschetz Fixed Point Theorem as e.g. in [Bre93]. Deduce some corollaries of the uniqueness, in particular, that a maximal torus is the maximal abelian subgroup of a Lie group. Also determine the Weyl group of $U(n)$ and its action on T which factor through to an action on $H^*(T)$ and $H^*(BT)$, BT being the classifying space of the maximal torus. Finish your talk by determining the cohomology of $BU(n)$.
References: For the first part use [Bre72], Chapter 0; for corollaries, see [BtD85] III.2. As soon as the example session starts, go to [MT91] Chapter III.5.

Talk 6. Representation ring and root system of $SU(n)$ and other examples Define the ring of representations $R(G)$ and the action of the Weyl group W on the ring $R(T)$ where T is a fixed maximal torus. Prove that the map $R(G) \to R(T)^W$ is injective. Use this statement to find a generating set for $R(SU(n))$. Follow the idea of the example at the very end of Chapter IV.3 in [BtD85].
Define the root of a Lie group (see [BtD85] V.1.3). Determine the roots of $SU(n)$ (see V.6 in [BtD85]) as an easy but important example. Then turn to a more difficult one, namely introduce G_2 and determine its roots following Theorem 5.5 of [Ada96]. Do this in some detail because this is the only example of an exceptional Lie group we will see in the whole seminar. If time permits also give the very nice description of G_2 as automorphism group of the Cayley numbers as in Theorem 15.16 of [Ada96].

24.11.2011

Talk 7. Characteristic classes and Roots Your task is to introduce the techniques of working with root systems we will need later. You should, in particular, recall the contents of §§ 1-4 of [BH58]. As a second reference Chapters V.1-V.4 of [BtD85] might be useful. Then you have to establish a connection to characteristic classes as described in §§9-10 of [BH58]. For cohomology arguments, it seems easier to use [MT91] instead of [Bor53]. Make sure that you prove Theorem 10.8 of [BH58] since this is one of the main ingredients for the next talk. Use the classification of compact connected Lie groups and the fact that $H^2(G) = 0$ for semisimple G without proof; we're going to prove it later in Talks 10 and 11. Still, you should try not to get lost in details.

Talk 8. Complex structures on homogeneous spaces In this talk, we will see a first application of the theory of the previous talk. We can now characterise homogeneous spaces which admit a homogeneous (almost) complex structure; we can also give the number of such structures([BH58], 13.4 and 13.8). Furthermore, these considerations provide us with an example of two diffeomorphic manifolds carrying non-equivalent complex structures. For the exposition, you should follow §§12-13 of [BH58]. Try to take care of the assumptions. For the integrability condition, you might want to have a look into [KN69] Chapter IX.2 instead of [Frö55] which is the reference mentioned in [BH58].

8.12.2011

Talk 9. Parallelisable homogeneous spaces or "work of a neighbour"
Within [BoH] Borel and Hirzebruch state that the quotient of a a Lie group by a toral subgroup is parallelisable, in their language that reads: "the tangent bundle has trivial S-class". Instead dividing out this very special subgroups we can also ask whether an arbitrary homogeneous space is parallelisable. A paper by Singhof gives a partial answer to that. Deduce his sufficient criterion for a homogeneous space to be (stably) parallelisable. If time permits consider an example for a homogeneous space which is stably parallelisable but not parallelisable. Then try to convey a feeling for the main theorem and give as many details of the proof as possible without getting lost in technicalities. Therefore concentrate on the examples before and after the core of the main proof, e.g. Proposition 2.7, 2.8 or 2.9 and Proposition 5.11 and §6 of [Sin82].

Talk 10. The structure of compact Lie groups Your task is to prove a classical structure theorem for compact connected Lie groups, stating that those have a finite cover isomorphic to a direct product of some simple Lie groups and a torus. You might want to follow [BtD85], V.7, to show that $\pi_2 G = 0$. Then state the description of π_1 in terms of roots. If you have time, you could sketch the proof. In order to prove the structure theorem switch to [MT91], V.5 (in particular Corollary 5.31). Note that a deep theorem by Ado (II.5.11 in [MT91]) enters into the proof. If you have time left, you could also mention the classification of simple Lie groups by Dynkin diagrams (like V.5 of [BtD85]).

22.12.2011

Talk 11. The Adams Conjecture Start with recalling λ-rings and formulating the Adams conjecture. You should motivate it by sketching how it allows to decribe the image of the J-homomorphism; have a look into [Ada63]-[Ada66]. We will also need the Adams conjecture in Talk 14. Then follow [BG75] for the proof. You should assume the existence of stable transfer map with properties (3.2)-(3.4) and (4.3) of [BG75]; it will be defined in the next talk. Try to give most of the ideas of the proof; in any case, explain Theorem 5.7 of [BG75]. As a corollary, show also that H^2 of a semisimple compact Lie group is 0.

19.1.2012

Talk 12. The transfer map Start by introducing the transfer map on the level of spaces giving rise to a well-defined map in the stable category. Sketch why properties (3.2)-(3.4) of [BG75] hold. Then explain what the induced map on a generalised cohomology theory does. Give a geometric description of the induced map in framed bordism. After you defined the transfer explain the other maps and spaces in the diagram

$$\begin{tikzcd} S(\mathbb{C}P^{\infty+}) \arrow[r, "\tau"] & QS^0 \\ U \arrow[u, "R"] \arrow[ur, "\widetilde{J}"] & \end{tikzcd}$$

and prove that it commutes up to sign. Even if he does not explicitly prove this version of the commutative diagram, a look into [Sto79] might be helpful because there you find more details as in [BS74].

Talk 13. Lie groups as candidates for the stable stem Your main goal is to show that an element of stable stems given by a Lie group has at least its rank as Adams-Novikov filtration. For this, recall the Adams-Novikov spectral sequence and its filtration. Then follow [Kna78] to show the main result. It would be nice to see some examples: show how e.g. $SU(2)$ gives rise to the generator of π_3^s. You also should show Proposition 2.6 linking the elements represented by $SU(n)$ and $U(n)$. Then you may choose either to prove the generalisation of the main theorem given in Theorem 3.1 of [Kna78] or to do a more sophisticated example, e.g. $SU(3)$. If you have really much time, do both.

2.2.2012

Talk 14. The order of an element of π_*^s induced by a Lie group Your task is to present a paper by Ossa [Oss82] that determines strict constraints on the order of an element of the stable stem which is induced by a compact Lie group with the left-invariant framing. While presenting his prove concentrate on the more geometric parts, as for example Lemma 2.2 and the connection to representation rings which is established in Lemma 2.3. Prove his statements on λ-rings in reasonable detail and apply them to $SU(n)$, thereby proving the main theorem for $SU(n)$.

References

[Ada63] J. F. Adams. On the groups $J(X)$. I. *Topology*, 2:181–195, 1963.

[Ada65a] J. F. Adams. On the groups $J(X)$. II. *Topology*, 3:137–171, 1965.

[Ada65b] J. F. Adams. On the groups $J(X)$. III. *Topology*, 3:193–222, 1965.

[Ada66] J. F. Adams. On the groups $J(X)$. IV. *Topology*, 5:21–71, 1966.

[Ada69] J.F. Adams. *Lectures on Lie groups.* University of Chicago Press, 1969.

[Ada96] J. F. Adams. *Lectures on exceptional Lie groups.* Chicago Lectures in Mathematics. University of Chicago Press, 1996.

[Bau09] H. Baum. *Eichfeldtheorie, Eine Einführung in die Differentialgeometrie auf Faserbündeln*. Springer-Verlag, 2009.

[BG75] J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. *Topology*, 14:1–12, 1975.

[BH58] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces, I. *American Journal of Mathematics*, 80(2):458–538, 1958.

[BoH] Characteristic classes and homogeneous spaces, [iii.

[Bor53] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts. *The Annals of Mathematics*, 57(1):115–207, 1953.

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.

[Bre93] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[BS74] J.C. Becker and R.E. Schultz. Equivariant function spaces and stable homotopy theory I. *Commentarii Mathematici Helvetici*, 49:133–169, 1974.

[BtD85] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 12. Springer-Verlag, 1985.

[Frö55] A. Frölicher. Zur Differentialgeometrie der komplexen Strukturen. *Mathematische Annalen*, 129(1):50–95, 1955.

[KN69] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. II.* Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[Kna78] K.H. Knapp. Rank and Adams filtration of a Lie group. *Topology*, 17:41–52, 1978.

[MT91] M.M. Mimura and H. Toda. *Topology of Lie groups, I and II.* American Mathematical Society, 1991.

[Oss82] E. Ossa. Lie groups as framed manifolds. *Topology*, 21(3):315–323, 1982.

[Sin82] Wilhelm Singhof. Parallelizability of homogeneous spaces, I. *Mathematische Annalen*, 260:101–116, 1982.

[Sto79] S. Stolz. Beziehungen zwischen Transfer und J-homomorphismus. *Diplomarbeit*, 1979.