Seminar on 'Stability conditions and Stokes factors'[1]

Content. In a recent paper Bridgeland and Toledano Laredo explain the analogy between certain formulae arising in the study of irregular connections and in Joyce's work on counting functions of stable objects in abelian categories, thus leading to a more conceptual understanding of the latter. In particular, a certain differential equation studied by Joyce turns out to be equivalent to a similar one that comes up when an irregular connection $\nabla = d - (\frac{Z}{t^2} + \frac{f}{t})dt$ on $\mathbf{C}^*$ is deformed. Roughly, the Z is deformed in some open set U and the differential equation is a condition on the f, so that the connections glue to a flat connection on $U \times \mathbf{C}^*$ (isomonodromic deformations). The seminar will give us the opportunity to learn two rather different things: Ringel–Hall algebras and irregular connections. We will not assume any familiarity with either of the two.

The techniques, in particular the manipulation of the involved power series, should be useful whenever objects are counted. So besides the abstract theory, learning how to handle the various generating series is the main aim of the seminar.

The motivation for Joyce's work comes from Donaldson–Thomas theory (or the more recent version of Pandharipande–Thomas), which counts curves in Calabi–Yau threefolds. However the algebraic geometry (or even the definitions) of these invariants will not be discussed, at least not this term. It would be interesting to have a look also at the recent paper of Kontsevich and Soibelman (early copies are available on request), but we presumably won't have the time for it either.

The main reference for the seminar is the paper

[BT] T. Bridgeland, V. Toledano Laredo *Stability conditions and Stokes factors.* Preprint arXiv:0801.3974,

but we will also go through parts of articles by Joyce, Reineke, Schiffmann and others.

Organization. The seminar will take place every second Tuesday 2-6 pm starting October 21, 2008 with a meeting in Bonn (SR A, Beringstr. 4). All talks will be announced individually in the weekly program.

The dates for the meetings until Christmas are: October 21, November 4, November 11, November 25, December 2, December 16. The meeting on December 2 will take place in Mainz. The locations (Bonn or Mainz) for the other meetings have still to be decided.

For further information or if you are interested in giving a talk in the seminar please contact Daniel Huybrechts or Sven Meinhardt (huybrech or sven at math.uni-bonn.de). (DH and SM will give the talks at the first meeting, at least if no one else feels an urgent desire to do so.)

Below we sketch a provisional plan for the seminar which we should feel free to change. In particular, some talks might need more time than we think. The citations below refer to the references given in [BT].

[1]Sponsored by the SFB/TR 45. Bonn–Essen–Mainz.

I. Introduction, Hall algebras over finite fields and Reineke inversion.

This is the warm-up meeting. After some motivation and explanations of the setup, we will start with Hall algebras over finite fields. Proofs are easier than in characteristic zero.

I.1. Introduction: Stability conditions on abelian categories (stability function, Harder–Narasimhan property, examples). What we want to count and how.

I.2. Ringel–Hall algebras over finite fields: Sections 1.1.-1.5. in Schiffmann's lecture notes [35].

The Hall algebra $\mathbf{H}_{\mathcal{A}}$ for a finitary abelian categories $\mathcal{A}$ is generated by the isomorphism classes of objects in $\mathcal{A}$. Alternatively, it is the set of functions on $\mathrm{Ob}(\mathcal{A})/\cong$ with finite support. The Hall algebra comes with a natural product $[M]\cdot[N]$ expressed in terms of all possible extensions of M by N. The first result is: This product is associative (Prop. 1.1 in [35]).

The Hall algebra $\mathbf{H}(\mathcal{A})$ also comes with a natural coproduct Δ (Green's coproduct). Corresponding to the two definitions of the Hall algebra, there are two possible ways of defining the coproduct. The more geometric one is in terms of functions on $\mathrm{Ob}(\mathcal{A})$. That this does define a coproduct is Proposition 1.4 [35]. That it is compatible with the product structure is Theorem 1.9 [35]. (The proof of the latter is a bit longer, but seems elementary.)

I.3. As a more concrete example we will explain Section 4 and 5 of Reineke's article [29]. (We leave out Theorem 4.5 for the moment.) The new input is the existence of a Harder–Narasimhan filtration. The Hall algebra contains the composition algebra that is generated by characteristic functions χ_d of the natural strata R_d. Proposition 4.8 [29] expresses the characteristic function χ_d^{ss} of the semi-stable part R_d^{ss} in terms of χ_d and products of characteristic functions $\chi_{d_i}^{ss}$ of strata occurring in the HN filtrations in R_d. (This can be viewed as a recursion formula.) All these functions are put into one generating function $X(T)=\sum \chi_d * T^d$ and Proposition 4.8 can be rephrased in terms of $X(T)$ (see Proposition 4.12 [29]). The main result is then Theorem 5.1 which resolves the recursion in Proposition 4.8. This should be treated on a very formal level, for we will encounter the same formula in different settings later. Have a look also at Sections 9.1-9.4 in [BT].

II. Ringel–Hall algebras in characteristic zero.

Based on parts of Section 3 in [BT].

II.1. Section 3.1, 3.2 (leave the Ringel–Hall Lie algebra to the next talk), 3.7 and 3.8. We work in the abelian category $\mathcal{A}=\mathrm{Mod}(R)$ of finite-dimensional modules over a finite-dimensional algebra R. The space of d-dimensional representations Rep_d is an affine variety with a natural GL_d-action. The Ringel–Hall algebra $\mathcal{H}(\mathcal{A})$ in this setting consists of all constructible equivariant functions on Rep. Define product and coproduct in this new setting and explain the analogy to the Ringel–Hall algebra over finite fields. Counting points is replaced by taking Euler characteristics using cohomology with compact support. The analogue of the composition algebra $\mathcal{C}(\mathcal{A})\subset\mathcal{H}(\mathcal{A})$ is introduced. Explain why the coproduct is only defined in $\mathcal{C}(\mathcal{A})$. Prove Theorem 3.1 and Theorem 3.3 in analogy to last time. (The notation in Joyce's paper is heavy, but maybe this can be simplified in our situation and with what we have learnt already in mind.) Define the functions δ_γ in 3.7 and explain why Reineke's inversion formula works and yields Theorem 3.11.

II.2. Introduce the Ringel–Hall algebra and its extended version. Prove that with these definitions they are Lie algebras. Show that the composition algebra is the universal enveloping algebra of the Ringel–Hall algebra (Proposition 3.7). Maybe one could first look at the corresponding statement for Hall algebras over finite fields (see Reineke's article) and then explain what has to be modified in characteristic zero. We might need professional help here.

We suggest to skip Section 3.4 and 3.5, where certain completed versions of the Ringel–Hall algebra and the composition algebra are introduced. Later we should make clear where this is needed.

A brief outlook, following Sections 1.4 and 1.5, sketching how all this leads to irregular connections would be good. Those will be dicussed in the following meeting.

III. Irregular connections and Stokes.

Based on parts of Section 2 in [BT]

III.1. Recall the Riemann–Hilbert correspondence between flat connections on the trivial bundle on $\mathbf{C}^*$ and the monodromy $\pi_1(\mathbf{C}^*) \to \mathrm{GL}(n, \mathbf{C})$. Explain Hilbert's 21st problem which asks under which conditions a given monodromy can be realized by a flat connection with a pole of order one: $\nabla = d - \frac{f}{t}dt$. Discuss isomonodromic deformations of flat connections of this type and the equations that arise. (What is the best reference here? Maybe a good point to start is the article by C. Sabbah *The work of Bolibruch on isomonodromic deformations*. See his personal webpage.)

Generalize to the case of irregular connections (pole order two), see Section 2.3. The connection has now the form $\nabla = d - (\frac{Z}{t^2} + \frac{f}{t})dt$ with constant f and Z satisfying additional conditions. Restrict to the case $\mathrm{GL}(n)$ and leave all Lie algebra considerations for later. Prove Theorem 2.2 (see Section 6), which asserts the existence and uniqueness of flat sections. (The paper *On the generalized Riemann–Hilbert problem with irregular singularities* by Bolibruch, Malek, and Mitschi could be helpful.)

III.2. Prove Proposition 2.4 (see Section 5) and introduce Stokes factors, multipliers, etc. Conclude with an explanation of the relation to isomonodromic deformations in this case (see Section 2.15).

IV. Analogy: Stokes and Hall

The first part is based on Sections 2.7, 2.8, 2.9 in [BT]. For the second it is the beginning of Section 4.

IV.1. This part discusses the Lie algebra incarnations of the Stokes factor: ϵ_α, δ_α and κ_α. Section 2.7 introduces a certain completion $\widehat{U\mathfrak{g}}$ of the universal enveloping algebra. (Is this a standard procedure? Is there another reference for this part?) Taking the logarithm of the Stokes factors yields the Stokes map mapping f to $\sum \epsilon_\alpha$ (which implicitly depends on Z). Introduce $N_\ell \subset G$ and its Lie algebra $\mathfrak{n}_\ell$ (depending on a Stokes ray ℓ) and explain Lemma 2.9 (in particular the infinite sums that occur). Proposition 2.10 is the analogue of Reineke's formula. Explain that the analogy is valid. At this point we could add a comparison of the Stokes setting with the Ringel–Hall setting. What is missing in the Ringel–Hall setting is the connection $\nabla = d - (\frac{Z}{t^2} + \frac{f}{t})dt$. The stability function yields the Z, but f is a priori not present. The next step is to express the f in the Stokes world in terms of the ϵ and δ.

IV.2. Start with Section 2.10 and state Theorem 2.14 which expresses the δ in terms of f by a formula of the form $\delta = \sum \sum M_n(Z(\alpha_i)) \prod f_{\alpha_i}$. The coefficients M_n are iterated integrals and we have to learn what this is. Go to Section 4. This part should be in collaboration with V.1.

V. Explicit formulae comparing ϵ and δ with f.

Based on 2.10 and 2.11 and the relevant parts in Section 9 and 10 in [BT]

V.1 Continuation of IV.2. Prove Theorem 2.14.

V.2 This part should prove the analogous formula in Theorem 2.19 which expresses the ϵ in terms of f by an equation of the type $\epsilon = \sum \sum L_n(Z(\alpha_i)) \prod f_{\alpha_i}$. Roughly the L_n are the logarithm of the M_n. We need to cover parts of Section 9 (starting with 9.5) and 10 but should restrict to those that are strictly needed for Theorem 2.19. (Lemma 2.18 could be stated without proof.) If time permits discuss Theorem 2.20, which eventually expresses f in terms of the ϵ, should be included. It is this formula that completes the analogy between Joyce's Ringel–Hall setting and the more classical Stokes picture.

VI. The final pieces.

VI.1. The aim is to prove Theorem 2.21 and in particular that the function f in the Ringel–Hall setting is a holomorphic function on the space of stability conditions. This is Corollary 10.4. It seems that Joyce's functions F_n in Theorem 3.13, which are not explained in [BT], should simply be replaced by the J_n.

If time permits, one could also discuss the explicit formula relating f to κ. This is Theorem 2.23.

VI.2 This talk should summarize the discussion and explain Theorem 3.14 and Theorem 3.18.