About Implicit Curves in the Plane*

Compare Implicit Surfaces in Space

There are three principal methods for describing curves in the plane:

a) As parametrized curves $c(t) = (x(t), y(t))$ with $x, y : (t_0, t_1) \mapsto \mathbb{R}$. For example the unit circle can be given as
$$x(t) = R \cdot \cos(t), \quad y(t) = R \cdot \sin(t), \quad t \in [0, 2\pi].$$

b) As the graph $y = F(x)$ of a function $F : [x_0, x_1] \mapsto \mathbb{R}$. For example the upper unit semi-circle can be given as the graph of the function $F(x) = \sqrt{1 - x^2}$ for $x \in (-1, +1)$.

c) Implicitly as a level set $\{f = c\}$ of a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$. For example the unit circle is the level $\{f = 1\}$ of the function $f(x, y) = x^2 + y^2$.

Implicit Curves in 3DXM:

Cassini Ovals $f(x, y) = ((x-aa)^2+y^2)((x+aa)^2+y^2)-bb^4$
Tacnodal Quartic $f(x, y) = y^3 + y^2 - x^4$
Teissier singular Sextic $f(x, y) = (y^2 - x^3)^2 - x^5 \cdot y$
Userdefined Implicit Curves: *available*
Parametrized Curves with Level Functions:
Cuspidal Cubic $f(x, y) = 27aa \cdot y^2 - 4(x + bb)x^2$
Nodal Cubic $f(x, y) = y^2 - (1 - x)x^2$

Clearly, method b) can easily be written as a special case

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

of methods a) or c) by using for a) the "graph parametrization" $x(t) = t,\ y(t) = F(t)$, and by using for c) the trivial level function $f(x, y) = y - F(x)$ and $\{f = 0\}$.

However, implicit curves $\{f(x, y) = c\}$ really give a different and somewhat richer class of objects than are given by explicit parametrization. For example, level sets may have several components; also, one is more interested in the singularities of level sets. In differential geometry one usually assumes that parametrized curves are without singularities, while in algebraic geometry the singularities of the level sets of polynomials are a major subfield of interest. The *Tacnodal Quartic* and the *Teissier Sextic* are examples in 3DXM.

Up to release 10.6 there are only a small number of implicit curves preprogrammed into 3DXM. What actually gets drawn are the solutions of the equation $f(x, y) = ff$ with $x_{min} \leq x \leq x_{max}, y_{min} \leq y \leq y_{max}$ (where these limits can be set in the *Settings Menu*, dialog entry *Set t,u,v,ranges...* and, as always, *ff* can be set in the dialog entry *Set Parameters, Modify Object.*
Note that user-defined implicit curves can be entered.

The default morphs of the implicit curves vary the parameter ff, so that, what one sees is a family of level curves of f. It may be helpful to think of the function $f(x, y)$ as giving the "height above sea-level" at the point (x, y), in which case the levels $\{f(x, y) = ff\}$ are just the level lines one is used to from topographic maps. If one chooses from

the *Animation Menu* the entry *Color Morph*, the program will draw such a topographic map with each level line a different color.
Some parametrized curves are provided with level functions. For these the *Animation Menu* has the entry *Morph Level Lines*. In the Cassini case this morph looks better with morphing parameter $bb = ff^{1/4}$.

Note that, while a parametrized curve depends on parameters only if the author chooses to embed it in some family, implicit curves always come naturally as 1-parameter family of curves. These families have been used to study singularities of curves via limits of nonsingular curves.

Tangents, Normals and Curvature

The gradient of the (height) function f is a vectorfield along and normal to the level lines. Therefore we have, even without parametrizing the curve, normals and tangents:

$$n = \frac{\mathrm{grad}\, f}{|\mathrm{grad}\, f|}, \quad t = (-n_y, n_x).$$

Assuming we had a parametrized curve with unit normal and tangent fields n, t then the formula $\dot{n}(s) = \kappa(s) \cdot \dot{c}(s)$ holds whether or not s is arc length parameter. This implies for our vector fields

$$\kappa = \langle \nabla_t n,\ t \rangle = \mathrm{hesse} f(t,t)/|\mathrm{grad}\, f|.$$

R.S.P.