Free Rotational Motion of Rigid Bodies

What is to observe in the 3D-XplorMath exhibit
Solid Body (Euler's Polhode) ?

A brick – in the program of edge lengths $aa \geq bb \geq cc \geq 0$ – is a good example of a solid (also: rigid) body. The program illustrates the free rotational movement of such a brick (i.e. gravity is ignored): Select *Solid Body (Euler's Polhode)*, stop the alternation between two pictures by a mouse click and select *Do Poinsot Construction From Polhode* at the bottom of the Action Menu. The resulting animation shows a freely tumbling brick. By changing aa, bb, or cc one may watch other bricks tumbling.
There are three other input parameters, dd, ee, ff. These are initial conditions for the the tumbling motion. If one chooses $(dd, ee, ff) \approx (1, 0.1, 0.1)$ or $(dd, ee, ff) \approx (0.1, 0.1, 1)$ then there is not much tumbling. These motions are almost rotations around the longest axis (aa) of the brick, respectively the shortest axis (cc) . The fact that these rotation axes stay close to their initial position is expressed by saying: *the rotations around the longest and the shortest axis are stable.* Now look again at the default initial con-

ditions $(dd, ee, ff) \approx (0.1, 1, 0.1)$. One observes that the momentary axis of rotation moves almost to the direction opposite to the initial direction and then returns back. One says: *the rotation around the middle axis of the brick is unstable.* – By putting a tape around a book and trying to throw it so that it rotates around one of the three axes one can experimentally test these theoretical predictions.

The *explanation* of this behaviour has a mathematical part and a physical part. The physical part is contained in the initial picture, the mathematical part is the connection between the initial picture and the annimation. We explain the mathematical part in

Part I: From Angular Velocity to Rotational Motion

It is available in the *Topics* part of the *Documentation*. This mathematical part has no physical limitations, any of the space curves in the program can be used as *angular velocity curve* and in the Action Menu one can select animations that show the resulting motions.
The physical part requires in addition to *angular velocity* the physical notions *tensor of inertia* and *an-*

gular momentum. These are explained below. What can one say before this theory about the initial picture of the program? We see two space curves. The one on the sphere is the angular momentum as a function of time *in the coordinate system of the brick*. The other one is the angular velocity curve (called *Polhode*). Both are intersections of quadratic surfaces, represented by dots in the picture. The two curves are related by a fixed linear map – given by the tensor of inertia. To emphasize this linear map the quadratic surfaces alternate between the domain and the range of this map. Finally, these two curves together determine Euler's differential equation for either of them. For example the derivative of the angular momentum curve is the cross product of the corresponding position vectors of the angular momentum curve and the angular velocity curve, in formulas: $\vec{\ell'}(t) = \vec{\ell}(t) \times \vec{\omega}(t)$. The Action Menu entry *Show Repère Mobile and ODE* illustrates this connection. The dotted curves on the sphere are solutions for other initial conditions dd, ee, ff with the same value $dd^2 + ee^2 + ff^2$. The default morph varies bb between aa and cc, it illustrates how the *family* of polhodes depends on the shape of the brick.

And here is the theory:

Part II: Tensor of Inertia and Angular Momentum

The tensor of inertia is a map that transforms *angular velocity* into *angular momentum.*
Historical note: The word *tensor* is a generic word that describes objects from linear algebra that can be given by components (indices!) with respect to a base. The *tensor of inertia* is a *linear map* from the 3-dim vector space of angular velocities to the 3-dim vector space of angular momenta. What we need below is that for each solid body there exists an orthonormal frame $\{\vec{e}_x(t),\ \vec{e}_y(t),\ \vec{e}_z(t)\}$ in the rest space of the body (i.e. moving with the body) so that the tensor of inertia Θ is a diagonal map:

$$\begin{aligned}&\textit{angular momentum} = \Theta(\vec{\omega}(t)) = \\ &\omega_x(t)\cdot\Theta_x\vec{e}_x(t) + \omega_y(t)\cdot\Theta_y\vec{e}_y(t) + \omega_z(t)\cdot\Theta_z\vec{e}_z(t).\end{aligned}$$

$\Theta_x, \Theta_y, \Theta_z$ are called *principal moments of inertia.*

We now explain the tensor of inertia in some more detail. The result of the explanation will be the above formula for the angular momentum. We view a solid body as a collection of points of **mass** m_i and position vector $\vec{x}_i(t)$; the pairwise distances between these points are constant. The origin is the center of mass of these points, i.e. $\sum_i m_i \vec{x}_i(t) = \vec{0}$. For each mass point we have the following definitions, the corresponding notions for the solid body are obtained by summation:

linear momentum: $\vec{p}_i(t) := m_i \vec{x}_i\,'(t)$

angular momentum with respect to the origin:

$$\vec{\ell}_i(t) := \vec{x}_i(t) \times \vec{p}_i(t)$$

kinetic energy: $E_i(t) := \frac{1}{2} m_i \langle \vec{x}_i\,'(t), \vec{x}_i\,'(t) \rangle$.

The body is rigid, i.e. the distances between the points are constant, therefore there is an angular velocity function $\vec{\omega}(t)$ that relates the positions and velocities:

rotational motion: $\vec{x}_i\,'(t) = \vec{\omega}(t) \times \vec{x}_i(t)$.

angular momentum: $\vec{\ell}_i(t) = \vec{x}_i(t) \times (\vec{\omega}(t) \times \vec{x}_i(t))$.

$$=: \Theta_i(\vec{\omega}(t)).$$

This tensor of inertia is most easily understood if we use the relation between cross-product and matrix-product and insert it into the above definitions. We obtain the expressions for angular momentum and kinetic energy in terms of the tensor of inertia and the angular velocity as follows:

$$\vec{\omega} \times \vec{x} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

We obtain

$\vec{\ell}_i(t) =$

$$m_i \begin{pmatrix} 0 & z_i & -y_i \\ -z_i & 0 & x_i \\ y_i & -x_i & 0 \end{pmatrix} \begin{pmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$
$$= m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & y_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$= \Theta_i(\vec{\omega})$ (Note the symmetry of the matrix of Θ_i).

$$E_i(t) = \frac{1}{2} \langle \Theta_i(\vec{\omega}),\ \vec{\omega} \rangle.$$

The symmetry of $\Theta := \sum_i \Theta_i$ implies that we have an orthonormal eigen basis for Θ. The corresponding eigen values are the principal moments of inertia, $\Theta_x, \Theta_y, \Theta_z$.

Finally, we will derive Euler's equations, a first order ODE for $\vec{\omega}(t)$. Together with part I this determines the motion of a solid body that rotates without exterior forces. We will always take the eigen basis of Θ as the moving frame of part I.

Newton's laws imply that the total angular momentum is constant in situations that are more general than the force free rotation of a solid body. We omit this general theory and show only that the conservation of angular momentum is equivalent to Euler's equations.

$$\vec{\ell}(t) := \sum_i \vec{\ell}_i(t) = \Theta(\vec{\omega}(t)) = \sum_{\xi\in\{x,y,z\}} \omega_\xi(t)\Theta_\xi \vec{e}_\xi(t)$$

implies

$$\frac{d}{dt}\vec{\ell}(t) =$$
$$\sum_{\xi\in\{x,y,z\}} \omega_\xi(t)'\Theta_\xi \vec{e}_\xi(t) + \sum_{\xi\in\{x,y,z\}} \omega_\xi(t)\Theta_\xi \vec{e}_\xi{}'(t).$$

Insert $\vec{e}_\xi{}'(t) = \vec{\omega}(t) \times \vec{e}_\xi(t)$ to get

$$\sum_{\xi\in\{x,y,z\}} \omega_\xi(t)\Theta_\xi \vec{e}_\xi{}'(t) = \vec{\omega}(t) \times \vec{\ell}(t),$$

next compute the cross product in the base given by the moving frame:

$$\vec{\omega}(t) \times \vec{\ell}(t) = \sum_{\xi \in \{x,y,z\}} \left(\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} \ell_x \\ \ell_y \\ \ell_z \end{pmatrix} \right)_\xi \cdot \vec{e}_\xi(t),$$

finally compare coefficients to get Euler's equations:

$$\boxed{\begin{array}{l} \begin{pmatrix} \ell_x \\ \ell_y \\ \ell_z \end{pmatrix}' = \begin{pmatrix} \Theta_x\omega_x \\ \Theta_y\omega_y \\ \Theta_z\omega_z \end{pmatrix}' = -\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} \ell_x \\ \ell_y \\ \ell_z \end{pmatrix}, \\ \text{where the physics is contained in the relation} \\ \text{between } \omega \text{ and } \ell : \\ \ell_x = \Theta_x\omega_x,\ \ell_y = \Theta_y\omega_y,\ \ell_z = \Theta_z\omega_z. \end{array}}$$

Considered as differential equation for the ω-components these are Euler's equations. This ODE-system implies immediately that the two quadratic functions

$$|\vec{\ell}|^2 = \ell_x^2 + \ell_y^2 + \ell_z^2 = \Theta_x^2\omega_x^2 + \Theta_y^2\omega_y^2 + \Theta_z^2\omega_z^2 \quad \text{and}$$
$$2E = \ell_x\omega_x + \ell_y\omega_y + \ell_z\omega_z = \Theta_x\omega_x^2 + \Theta_y\omega_y^2 + \Theta_z\omega_z^2$$

are constant along solution curves. The solutions are therefore intersections of two ellipsoids. If one considers the ODE-system as differential equations for the ℓ-components then one of the ellipsoids is a sphere, the solutions $(\ell_x(t), \ell_y(t), \ell_z(t))$ are spherical curves. The choice of the ℓ-components as the functions to be determined therefore simplifies the visualization and also leads to a slightly simpler ODE-system, since the tensor of inertia enters only on the right side, linearly, into the equations.