Ordinal Computability and Fine Structure

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GÖDEL's constructible hierarchy

- $L_0 = \emptyset$
- $L_{\alpha+1} = \{\{x \in L_\alpha \mid L_\alpha \vDash \varphi(x, \vec{y})\} \mid \varphi \in \mathrm{Fml}, \vec{y} \in L_\alpha\}$
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$
- $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$

Everything in L is *named* by finitely many ordinals:

$$\begin{aligned}\{x \in L_\alpha \mid L_\alpha \vDash \varphi(x, \vec{y}\,)\} &= \{x \in L_\alpha \mid L_\alpha \vDash \varphi(x, \{x \in L_\beta \mid L_\beta \vDash \psi(x, \vec{z}\,)\})\} \\ &= \{x \in L_\alpha \mid L_\alpha \vDash \varphi(x, \{x \in L_\beta \mid L_\beta \vDash \psi(x, \{...\})\})\} \\ &\sim (\alpha, \varphi, \beta, \psi, ...)\end{aligned}$$

This corresponds to *one* ordinal via GÖDEL pairing.

Computing bounded truth in L

$$L_{\alpha+1} \vDash x \in \{x \in L_\alpha \,|\, L_\alpha \vDash \varphi(x, \vec{y}\,)\}$$

$$\leftrightarrow \; L_\alpha \vDash \varphi(x, \vec{y}\,)$$

$$\leftrightarrow \; L_\alpha \vDash (\psi_0 \vee \psi_1)(x, \vec{y}\,)$$

$$\leftrightarrow \; L_\alpha \vDash \psi_0(x, \vec{y}\,) \text{ or } L_\alpha \vDash \psi_1(x, \vec{y}\,)$$

$$\leftrightarrow \; \ldots$$

$$L_\alpha \vDash \exists v \psi(v, \vec{y}\,)$$

$$\leftrightarrow \; \text{there is } x \in L_\alpha \text{ such that } L_\alpha \vDash \psi(x, \vec{y}\,)$$

$$\leftrightarrow \; \ldots$$

One can arrange, that the RHS formulas are smaller in an adequate well-order.

If we define a *constructible truth predicate* $F\colon \mathrm{Ord} \rightarrow 2$ by

$$F(\ulcorner L_\alpha \vDash \varphi(x, \vec{y}\,)\urcorner) = 1 \text{ iff } L_\alpha \vDash \varphi(x, \vec{y}\,)$$

then F has a recursive definition of the form:

$$F(\alpha) = \begin{cases} 1 \text{ iff } \exists \nu < \alpha\ H(\alpha, \nu, F(\nu)) = 1 \\ 0 \text{ else} \end{cases}$$

for some computable function H.

Computing $F(3)$ with a stack of ordinals:

time	stack contents	numerical code
1	$F(3)?$	$2^3 = 8$
2	$F(3)?, F(0)?$	$2^3 + 2^0 = 9$
3	$F(3)?, F(0)!(=0)$	
4	$F(3)?, F(1)?$	$2^3 + 2^1 = 10$
5	$F(3)?, F(1)?, F(0)?$	$2^3 + 2^1 + 2^0 = 11$
6	$F(3)?, F(1)?, F(0)!$	
7	$F(3)?, F(1)!$	
8	$F(3)!$	

Computing $F(\omega+2)$ with a stack of ordinals:

time	stack contents	ordinal code
1	$F(\omega+2)?$	$2^{\omega+2}$
2	$F(\omega+2)?, F(0)?$	$2^{\omega+2}+2^0$
3	$F(\omega+2)?, F(0)!(=0)$	
4	$F(\omega+2)?, F(1)?$	$2^{\omega+2}+2^1$
5	$F(\omega+2)?, F(1)?, F(0)?$	$2^{\omega+2}+2^1+2^0$
6	$F(\omega+2)?, F(1)?, F(0)!$	
7	$F(\omega+2)?, F(1)!$	
$\vdots$	$\vdots$	$\vdots$
	$F(\omega+2)?, F(n)?$	$2^{\omega+2}+2^n$
	$\vdots$	
	$F(\omega+2)?, F(n)!$	
	$\vdots$	
	$F(\omega+2)?, F(\omega)?$	$2^{\omega+2}+2^\omega=\lim_{n<\omega}(2^{\omega+2}+2^n)$
	$\vdots$	

```
  value:=2                            %% set value to undefined
MainLoop:
  nu:=last(stack)
  alpha:=llast(stack)
  if nu = alpha then
1:  do
    remove_last_element_of(stack)
    value:=0                          %% set value equal to 0
    goto SubLoop
    end
  else
2:  do
    stack:=stack + 1                  %% push the ordinal 0 onto the stack
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    goto MainLoop
    end
SubLoop:
  nu:=last(stack)
  alpha:=llast(stack)
  if alpha = UNDEFINED then STOP
  else
    do
    if H(alpha,nu,value)=1 then
3:    do
      remove_last_element_of(stack)
      value:=1
      goto SubLoop
      end
    else
4:    do
      stack:=stack + (3**y)*2      %% push y+1
      value:=2                     %% set value to undefined
      goto MainLoop
      end
    end
```

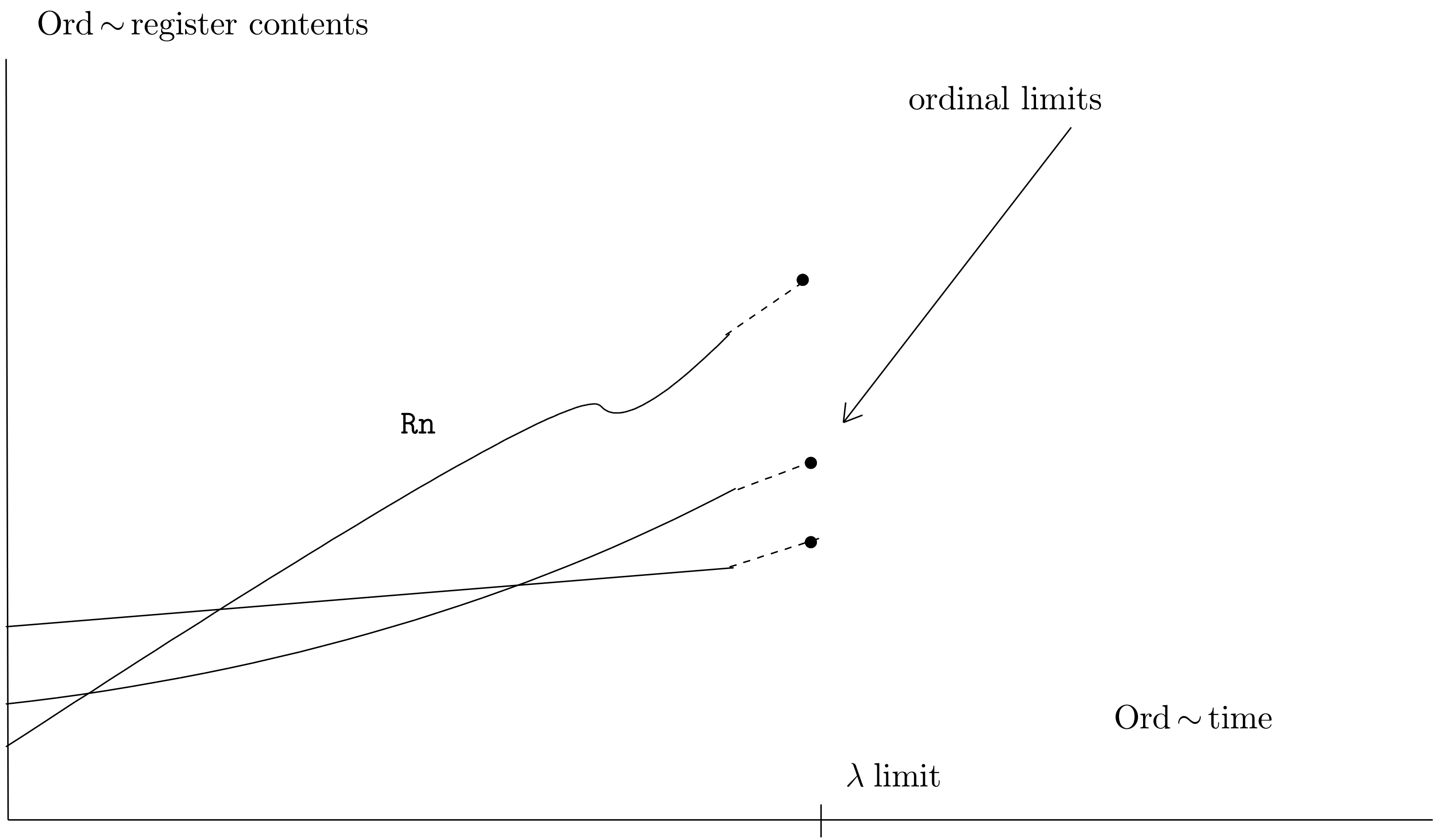
Ord ∼ register contents
ordinal limits
Rn
Ord ∼ time
λ limit

Consider a structure $M = (\mathrm{Ord}, <, M)$, $M\colon \mathrm{Ord}^{<\omega} \rightharpoonup \mathrm{Ord}$. For $\alpha \in \mathrm{Ord}$ let

$$M^{\alpha} = (\alpha, <, M \cap \alpha^{<\omega});$$

for $X \subseteq \alpha$ let $M^{\alpha}[X]$ be the substructure of M^{α} generated by X.

M is a SILVER *machine* if it satisfies the following axioms:

- (*Condensation*) For $\alpha \in \mathrm{Ord}$ and $X \subseteq \alpha$ there is a unique β such that $M^{\beta} \cong M^{\alpha}[X]$;
- (*Finiteness property*) For $\alpha \in \mathrm{Ord}$ there is a finite set $z \subseteq \alpha$ such that for all $X \subseteq \alpha + 1$

$$M^{\alpha+1}[X] \subseteq M^{\alpha}[(X \cap \alpha) \cup z] \cup \{\alpha\};$$

- (*Collapsing property*) If the limit ordinal β is singular in L then there is $\alpha < \beta$ and a finite set $p \subseteq \mathrm{Ord}$ such that $M[\alpha \cup p] \cap \beta$ is cofinal in β.

If there is a SILVER machine then combinatorial principles like $\Box$ and Morass hold.