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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 5

Exercise 22. (Annihilator, 2 pts)
For an A-module M one defines its *annihilator* $\mathrm{Ann}(M) \subset A$ as the ideal of all elements $a \in A$ such that $am = 0$ in M for all $m \in M$. Now consider a multiplicative set $S \subset A$, assume that M is finite, and prove the following assertions:

(i) $S^{-1}\mathrm{Ann}(M) = \mathrm{Ann}(S^{-1}M)$.

(ii) $S^{-1}M = 0$ if and only if $\mathrm{Ann}(M) \cap S \neq \emptyset$.

Exercise 23. (Nakayama lemma, 3 points)
Let A be a ring and $\mathfrak{a} \subset A$ an ideal. Prove the following consequences of the Nakayama lemma:

(i) Let $N \to M$ be a homomorphism of A-modules such that the induced homomorphism $N/\mathfrak{a}N \to M/\mathfrak{a}M$ is surjective. If M is a finite A-module and $\mathfrak{a} \subset \mathfrak{R}$, then $N \to M$ is surjective.

(ii) Let $N \to M$ be a homomorphism of A-modules such that the induced homomorphism $N/\mathfrak{a}N \to M/\mathfrak{a}M$ is surjective. If M is a finite A-module, then there exists an element of the form $b = 1 + a$ with $a \in \mathfrak{a}$, such that the induced homomorphism $N_b \to M_b$ of A_b-modules is surjective.

(iii) Assume that $m_1, \ldots, m_n \in M$ generate $M/\mathfrak{a}M$. If M is finite, then there exists an element $b = 1 + a$ with $a \in \mathfrak{a}$, such that $m_1, \ldots, m_n$ generate the A_b-module M_b.

Exercise 24. (Non-zero divisors as multiplicative set, 3 points)
Let $S \subset A$ be the multiplicative set of all elements in A that are not zero divisors.

(i) Show that the natural map $A \to S^{-1}A$ is injective and that S is maximal with this property

(ii) Show that every element in $S^{-1}A$ is either a zero-divisor or a unit.

(ii) Assume every element in A is a zero-divisor or a unit. Show that then the natural ring homomorphism $A \to S^{-1}A$ is already an isomorphism.

Exercise 25. (Flat scalar extensions, 4 points)
Consider the following ring homomorphisms $A \to B$ and decide in which cases B is A-flat:
(i) $\mathbb{Z} \to \mathbb{F}_p$; (ii) $\mathbb{Z} \to \mathbb{Q}$: (iii) $A \to A[x]$; (iv) $\mathbb{Z} \to \mathbb{Q}[x,y]/(y^2 - x)$.

Exercise 26. (Localization, 4 points)
(i) Let k be a field and $A := k[x_1, x_2]/(x_2^2)$. Show that $S := \{f(x_1) + x_2 \cdot g(x_1) \mid f(x_1) \neq 0\}$ is a multiplicative set and prove that there exists an isomorphism of rings

$$S^{-1}A \cong k(x_1)[x_2]/(x_2^2).$$

(ii) Let A and B be rings. Consider the multiplicative set $S = \{(1,1), (1,0)\} \subset A \times B$. Show that there exists an isomorphism of rings

$$S^{-1}(A \times B) \cong A.$$

(iii) Let $S \subset A$ be a multiplicative set and M an A-module. Show that the natural map $M \to S^{-1}M$ is bijective if and only if for every element $s \in S$ multiplication $M \xrightarrow{\cdot s} M$ is bijective.

Solutions to be handed in before Monday May 11, 4pm.