Exercise Sheet 2

Discussed on 21.04.2021

Problem 1. (a) Given maps of rings $A \to B \to C$, there is a natural exact sequence

$$C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0.$$

(b) Let K/k be a finite extension of fields. Then K/k is separable if and only if $\Omega^1_{K/k} = 0$.

(c) Let k be a field of characteristic 0 and let K/k be a finitely generated field extension (i.e. K is the fraction field of a finitely generated k-algebra). Show that then $\dim_K \Omega^1_{K/k}$ is equal to the transcendence degree of K over k.

(d) Let k be a field of characteristic 0 and let X be a k-scheme of finite type. If X is reduced then there is an open and dense subset $U \subset X$ such that U is smooth over k.

Hint: Reduce to the case that X is affine and integral, then look at the behavior of $\Omega^1_{X/k}$ at the generic point of X.

(e) Let k be a field of characteristic 0 and G a group scheme over k which is locally of finite type. If G is reduced then G is smooth.

Remark: The above statement is true for all fields k if one replaces "reduced" by "geometrically reduced". In characteristic 0, every group scheme of finite type over k is automatically reduced (hence smooth).

Problem 2. (a) Let R, A and B be rings and $I \subset R$ an ideal with $I^2 = 0$. Assume we are given the outer commuting square of the following diagram:

$$\begin{array}{ccc} \operatorname{Spec} R/I & \longrightarrow & \operatorname{Spec} B \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} A \end{array}$$

Assume additionally that there is some map $\varphi\colon \operatorname{Spec} R \to \operatorname{Spec} B$ such that the diagram commutes; fix a choice φ_0. Then the map

$$\{\varphi\colon \operatorname{Spec} R \to \operatorname{Spec} B \text{ s.t. diagram commutes}\} \xrightarrow{\sim} \operatorname{Der}_A(B, I),$$
$$\varphi \mapsto \varphi^* - \varphi_0^*$$

is a bijection.

(b) Let k be a field and $J \subset k[T_1, \ldots, T_n]$ an ideal such that $Z := V(J) \subset \mathbb{A}^n_k$ is smooth over k. Assume that the exact sequence

$$0 \to J/J^2 \to \Omega^1_{\mathbb{A}^n_k/k} \to \Omega^1_{Z/k} \to 0$$

of sheaves on Z is splits. Then for all k-algebras R and ideals $I \subset R$ with $I^2 = 0$ the map $Z(R) \to Z(R/I)$ is surjective.

(c) Combine (a) and (b) to deduce: If k is a field and Z is a smooth scheme over k then for all k-algebras R and ideals $I \subset R$ with $I^2 = 0$ the map $Z(R) \to Z(R/I)$ is surjective.

Problem 3. Let X be a proper smooth curve over $\mathbb{C}$ and let X^{an} be the corresponding Riemann surface. Show that there is a natural map $(X, \mathcal{O}_X) \to (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ of locally ringed spaces with the following property: For every Riemann surface Y and all morphisms $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed spaces over $\mathbb{C}$, there is a unique morphism $Y \to X^{\mathrm{an}}$ of Riemann surfaces such that the following diagram commutes:

$$\begin{array}{ccc} (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) & \longrightarrow & (X, \mathcal{O}_X) \\ \uparrow & \nearrow & \\ (Y, \mathcal{O}_Y) & & \end{array}$$

Hint: Show first that for any locally ringed space $(Z, \mathcal{O}_Z)$ and any ring A, morphisms $(Z, \mathcal{O}_Z) \to \operatorname{Spec} A$ correspond to ring homomorphisms $A \to \mathcal{O}_Z(Z)$.