GRADUATE SEMINAR ON APPLIED LOGIC (S4A6) SS 2023

Diophantine applications of o-minimality

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Time and Place. Fridays 12.15-14, N0.003

Organizational Meeting. Tuesday July 16th 10.15am N.003
If you want to give a talk, send me (Philipp Hieronymi) an email by 26.7.2024 indicating which talks you are interested in. If possible, list at least three topics.

Abstract. In this seminar we cover the basics of some of the recent applications of o-minimality. In particular, we study the Pila-Zannier strategy and use it to give a new proof of Manin-Mumford [10]. We then develop the theory of complex analysis in o-minimal structures in detail, following Peterzil and Starchenko [9]. These results are then used in Tsimerman's new proof of Ax' theorem [13]. We finish by studying the beginnings of the o-minimal GAGA theory following Bakker's notes [1].

In addition to giving a talk, each participant (who wants to get credit) is required to submit a 4 or more page summary of their topic prepared using latex.

Prerequisites. This seminar is designed for students who have taken *V4A7 - Advanced Mathematical Logic I* in Summer Semester 2024. Thus we assume knowledge of o-minimal structures, in particular cell decomposition and the Pila-Wilkie theorem. We try to keep the assumptions on background knowledge from arithmetic/algebraic geometry as minimal as possible.

Talks.

(1) **Pila-Zannier strategy and Mann's theorem** (1 talk)
Present a proof of Mann's theorem [11, Theorem 2.1] using the Pila-Wilkie strategy. Follow the argument in [14, Chapter 1].
References: Scanlon [11], Marker [3], Valle Thiele [14]

(2) **An introduction to abelian varieties** (1 talk)
Introduce the theory of abelian varieties. Follow [4, Sections 2-3] and/or [14, Section 2]. Do not attempt to give proofs, but rather cover all definitions and theorems necessary for the next talk.
References: Orr [4], Valle Thiele [14]

(3) **An o-minimal proof of Manin-Mumford** (1 talk)
Follow [12, Section 5.1] to give a proof of Manin-Mumford over a number field. Black box Ax's theorem (but state it in the appropriate generality) and Masser's theorem, but otherwise try to give full details.
References: Scanlon [12], Pila-Zannier [10]

(4) **Counting Lattice Points and O-Minimal Structures** (1-2 talks, optional)
Motivate and prove [2, Theorem 1.3]. Assume knowledge Section 3 of [2], and black box Section 2 of [2] if necessary. Focus on the use of o-minimality.
References: Barroero and Widmer [2]

(5) **O-minimal complex analysis of one variable functions** (2-3 talks)
Cover [9, Section 3.1] including the proofs of [9, Theorem 3.2] from [5]. Skip all material already covered in Adv. Math. Logic I.
References: Peterzil-Starchenko [9, 5]

(6) **O-minimal complex analysis of functions of several variables** (2-3 talks)
Cover [9, Section 3.2] including the proofs of [9, Theorem 3.5] from [6, Section 2]. Do not cover [6, Section 3].
References: Peterzil-Starchenko [9, 6]

(7) **O-minimal Chows** (2-3 talks)
State the classical Chow Theorem. Prove the o-minimal Chow over $\mathbb{R}$ following [1, Section 1.3]. Now prove [9, Theorems 4.5 and 5.3] in the general setting. This can be achieved by covering Sections 1-5 in [7].
References: Peterzil-Starchenko [7, 8, 9]

(8) **A proof of the classical Ax-Schanuel using O-minimality** (1-2 talks)
Explain the equivalence of the different formulations of Ax–Schanuel [13, Theorems 1.1-1.3], show how to deduce Ax-Schanuel-Weierstrass, and then give the proof from [13, Section 2].
References: Tsimerman [13]

(9) **Definable GAGA** (2 talks)
Introduce definable topological spaces [1, Section 2] and the definabilization functor [1, Definition 2.1.5]. Then cover basic definable analytic spaces [1, Section 2.2] and definable analytic spaces [1, Section 2.3], and introduce the analytification functor. State the general version of o-minimal Chow [1, Corollary 3.4.4] and prove it using a finite definable affine cover.
References: Bakker [1]

References

1. Ben Bakker, *Hodge theory and o-minimality*, `https://drive.google.com/file/d/1KNXtWekGk1mAyJki-xT6bnbr9ofjdZGs/view`, 2019, Notes from the Felix Klein lecture series in Bonn.
2. Fabrizio Barroero and Martin Widmer, *Counting lattice points and O-minimal structures*, Int. Math. Res. Not. IMRN (2014), no. 18, 4932–4957. MR 3264671
3. David Marker, *The Pila-Zannier proof of the Manin-Mumford Conjecture*, `http://homepages.math.uic.edu/~marker/orsay/mm.pdf`, 2010.
4. Martin Orr, *Introduction to abelian varieties and the Ax-Lindemann-Weierstrass theorem*, O-minimality and diophantine geometry, London Math. Soc. Lecture Note Ser., vol. 421, Cambridge Univ. Press, Cambridge, 2015, pp. 100–128. MR 3496443
5. Ya'acov Peterzil and Sergei Starchenko, *Expansions of algebraically closed fields in o-minimal structures*, Selecta Math. (N.S.) **7** (2001), no. 3, 409–445. MR 1868299
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7. ———, *Complex analytic geometry in a nonstandard setting*, Model theory with applications to algebra and analysis. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 349, Cambridge Univ. Press, Cambridge, 2008, pp. 117–165. MR 2441378
8. ———, *Complex analytic geometry and analytic-geometric categories*, J. Reine Angew. Math. **626** (2009), 39–74. MR 2492989

9. ______, *Tame complex analysis and o-minimality*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 58–81. MR 2827785
10. Jonathan Pila and Umberto Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **19** (2008), no. 2, 149–162. MR 2411018
11. Thomas Scanlon, *Counting special points: logic, Diophantine geometry, and transcendence theory*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 1, 51–71. MR 2869007
12. ______, *O-minimality as an approach to the André-Oort conjecture*, Around the Zilber-Pink conjecture/Autour de la conjecture de Zilber-Pink, Panor. Synthèses, vol. 52, Soc. Math. France, Paris, 2017, pp. 111–165. MR 3728313
13. Jacob Tsimerman, *Ax-Schanuel and o-minimality*, O-minimality and diophantine geometry, London Math. Soc. Lecture Note Ser., vol. 421, Cambridge Univ. Press, Cambridge, 2015, pp. 216–221. MR 3496448
14. Lucas Valle Thiele, *The Pila-Zannier method for abelian varieties*, `http://www.math.uni-bonn.de/people/phierony/ValleThiele.pdf`, 2022, Bachelor thesis.