Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 13

Due Date: 30.01.2014

Exercise 1:

Let k be a field and $n \geq 1$ and let $S = k[T_0, \dots, T_n]$ and $\mathbb{P}^n_k = \operatorname{Proj} S$. For $d \in \mathbb{Z}$ we write $S_d \subset S$ for the elements that are homogenous of degree d.

(i) Show that $\Gamma(\mathbb{P}^n_k, \mathcal{O}(d)) = S_d$.

(ii) Assume that $d \geq 0$ and write $N = \dim_k S_d - 1 = \binom{n+d}{d} - 1$. Show that the choice of a k-basis of S_d induces a surjection $\mathcal{O}_{\mathbb{P}^n_k}^{N+1} \rightarrow \mathcal{O}(d)$.

(iii) Show that the map $\mathbb{P}^n_k \rightarrow \mathbb{P}^N_k$ defined by the surjection from (ii) is a closed embedding which agrees with the d-fold Veronese embedding if we have chosen the standard basis of S_d (i.e. the basis given by the homogenous monomials).

(iv) Give a functorial description of the Veronese embedding.

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Exercise 2:

Let $X = \operatorname{Spec} A$ be an affine scheme and $Z \subset X$ be a closed subscheme defined by the quasi-coherent sheaf of ideals $\mathscr{I} = \tilde{I} \subset \mathcal{O}_X$. Let us write $\mathrm{Bl}_Z X = \operatorname{Proj}(\bigoplus_{d \geq 0} I^d)$ for the blow up of Z in X and $f : Y = \mathrm{Bl}_Z X \rightarrow X$ for the projection to X.

(i) Assume that $\mathscr{I}$ is locally free of rank 1. Show that f is an isomorphism.

(ii) Show that $f^{-1}(\mathscr{I})\mathcal{O}_Y$ is isomorphic to $\mathcal{O}_Y(1)$ and hence $f^{-1}(\mathscr{I})\mathcal{O}_Y$ is locally free of rank 1.

(iii) Let $g : Y' \rightarrow X$ be a morphism such that $g^{-1}(\mathscr{I})\mathcal{O}_{Y'}$ is locally free of rank 1. Show that there is a unique morphism $g' : Y' \rightarrow Y$ making the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\ & & X \end{array}$$

commutative.
(Hint: reduce to the case $Y' = \operatorname{Spec} B$ affine. Then use the functoriality of Proj to define a map $\operatorname{Proj}(\bigoplus_{d \geq 0} J^d) \rightarrow \operatorname{Proj}(\bigoplus_{d \geq 0} I^d)$, where $J = \Gamma(Y', g^{-1}(\mathscr{I})\mathcal{O}_{Y'})$ and use (i)).

Exercise 3:

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring such that S is generated by finitely many elements in S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$ and $U = X \backslash V(S_+) \subset \operatorname{Spec} S$. Finally let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a graded S-module.

(i) Let $f \in S_1$. Show that the canonical maps $\operatorname{Spec} S_f \to \operatorname{Spec} S_{(f)}$ glue to give a canonical map $\pi : U \to \operatorname{Proj} S$.

(ii) Let $f \in S_1$. Show that the canonical map $M_{(f)} \otimes_{S_{(f)}} S_f \to M_f$ induced by the inclusion $M_{(f)} \to M_f$ is an isomorphism.

(iii) Let $\mathscr{F}$ denote the quasi-coherent sheaf on $\operatorname{Spec} S$ such that by $\Gamma(\operatorname{Spec} S, \mathscr{F}) = M$ and let $\mathscr{G}$ denote the quasi-coherent sheaf on $\operatorname{Proj} S$ defined by the graded S-module M. Show that there is a canonical isomorphism $\pi^* \mathscr{G} \cong \mathscr{F}|_U$.

Exercise 4:

Let k be a field and let $X = V(T_1 T_2 - T_3^2) \subset \mathbb{A}_k^3$ and $Z = \{(0,0,0)\} \subset X$ viewed as a closed subscheme with the reduced scheme struture. Further let $\tilde{X} = \mathrm{Bl}_Z X$ denote the blow up of X in Z.

(i) Show that there is a morphism $f : X \backslash Z \to \mathbb{P}_k^1$ that is given by $(t_1, t_2, t_3) \mapsto (t_1 : t_3) = (t_3 : t_2)$ on k-valued points.

(ii) Show that f extends to a morphism $\tilde{f} : \tilde{X} \to \mathbb{P}_k^1$.

(iii) Show that $\tilde{f}^{-1}(U) \cong \mathbb{A}_k^1 \times U$ for all affine open subsets $U \subset \mathbb{P}_k^1$.
(In fact $\tilde{f} : \tilde{X} \to \mathbb{P}_k^1$ is the geometric vector bundle associated to $\mathcal{O}(2)$ on $\mathbb{P}_k^1$.)

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`