Algebraic Geometry I

Exercise Sheet 4

Due Date:14.11.2013

Exercise 1:

Let $Z = V_+(\mathfrak{a}) \subset \mathbb{P}^n$ be a projective variety, where $\mathfrak{a} \subset k[T_0, \ldots, T_n]$ is a homogeneous prime ideal and let $S = k[T_0, \ldots, T_n]/\mathfrak{a}$ denote the corresponding homogenous coordinate ring.

(i) Show that the projection $p : \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ sending $(x_0, \ldots, x_n)$ to $(x_0 : \ldots : x_n)$ is a morphism of prevarieties.

(ii) Show that the *affine cone* $C(Z) = p^{-1}(Z) \cup \{0\} \subset \mathbb{A}^{n+1}$ of Z is a closed subvariety with $C(Z) = V(\mathfrak{a}) \subset \mathbb{A}^{n+1}$.

(iii) Let $f \in S$ be a non-constant homogenous element. Show that $p^{-1}(D_+(f)) = D(f) \subset C(Z)$.

(iv) Show that

$$\mathcal{O}_Z(D_+(f)) = \{f \in \mathcal{O}_{C(Z)}(D(f)) \mid f(\lambda x) = f(x) \text{ for all } \lambda \in k^\times \text{ and } x \in D(f)\} = S_{(f)}.$$

(Hint: We have already seen this for $f = T_i$. Deduce that it holds true for $f = T_i g$ for some homogenous g. Then deduce the general case by the gluing property.)

Exercise 2:

Let $f : X \to Y$ be a morphism of affine varieties.

(i) Show that f is *dominant* (i.e. $f(X)$ is dense in Y) if and only if $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is injective.

(ii) Assume that f is dominant and that $A = \mathcal{O}_X(X)$ is (via $f^\sharp$) integral over $B = \mathcal{O}_Y(Y)$. Show that f is surjective.

(iii) Assume that f is dominant. Show that that there exists an open subset $U \subset Y$ such that $U \subset f(X)$.

(Hint: Let $K = K(Y)$ be the fraction field of B. By Noether normalization there exist $T_1, \ldots, T_r \in A$ such that there is an injection $K[T_1, \ldots, T_r] \hookrightarrow A \otimes_B K$ making $A \otimes_B K$ into a finite $K[T_1, \ldots, T_r]$-algebra. Show that there exists some $g \in B$ such that every element of A_g is integral over $B_g[T_1, \ldots, T_r]$. Then use (ii) to conclude.)

Exercise 3:

(i) Show that $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$.

(Hint: consider the intersection of the $\mathcal{O}_{\mathbb{P}^n}(U_i)$ in the function field of $\mathbb{P}^n$, where the U_i are the standard affine spaces embedded into $\mathbb{P}^n$)

(ii) Let X be a prevariety and let Y be an affine variety. Show that $f \mapsto f^\sharp$ induces a bijection

$$\{f : X \to Y \text{ morphism of prevarieties}\} \cong \mathrm{Hom}_{k-\mathrm{alg}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)).$$

(iii) Let Y be an affine variety. Show that every map $\mathbb{P}^n \to Y$ is constant.

(iv) Let X be a prevariety such that every map $X \to \mathbb{P}^1$ has closed image. Let Y be an affine variety. Show that every map $X \to Y$ is constant.

We will see later that every projective variety has this property.

Exercise 4:

Let R be a ring and let I be a partially ordered index set. Let $(M_i, f_{ij})_{i,j\in I}$, $(M'_i, f'_{ij})_{i,j\in I}$ and $(M''_i, f''_{ij})_{i,j\in I}$ be inductive systems of R-modules.

(i) Show that $\varinjlim_I M_i$ exists in the category of R-modules.

(ii) Let $\phi_i : M'_i \to M_i$ and $\psi_i : M_i \to M''_i$ be R-module homomorphisms such that $f_{ij}\circ\phi_i = \phi_j\circ f'_{ij}$ and $f''_{ij} \circ \psi_i = \psi_j \circ f_{ij}$ for all $i \leq j$. Show that there are uniquely determined maps

$$\phi : M' = \varinjlim_I M'_i \longrightarrow M = \varinjlim_I M_i$$
$$\psi : M = \varinjlim_I M_i \longrightarrow M'' = \varinjlim_I M''_i$$

such that $f_i \circ \phi_i = \phi \circ f'_i$ and $f''_i \circ \psi_i = \psi \circ f_i$ for all $i \in I$, where $f'_i : M'_i \to M'$ (resp. $f'_i : M_i \to M$, resp. $f''_i : M''_i \to M''$) are the structure maps making M' (resp. M, resp. M'') into the colimit of $(M'_i, f'_{ij})_{i,j\in I}$ (resp. $(M_i, f_{ij})_{i,j\in I}$, resp. $(M''_i, f''_{ij})_{i,j\in I}$).

(ii) Show that $\varinjlim_I$ is right exact, i.e. that

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact, if the sequences

$$M'_i \longrightarrow M_i \longrightarrow M''_i \longrightarrow 0$$

are exact for all i.

(iv) Assume that in addition I is filtered. Show that $\varinjlim_I$ is exact, i.e. that

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact, if the sequences

$$0 \longrightarrow M'_i \longrightarrow M_i \longrightarrow M''_i \longrightarrow 0$$

are exact for all i.