Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 9

Due Date: 30.06.2014

Exercise 1:

(i) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor that admits an exact additive left adjoint functor $G : \mathcal{B} \to \mathcal{A}$. Show that F preserves injective objects.

(ii) Let X be a topological space. Show that the category $\mathrm{PSh}(X, \mathbb{Z})$ of presheaves of abelian groups has enough injectives.
Hint: Let $U \subset X$ be open. Use that

$$A \longmapsto \left(\underline{A}_U : V \mapsto \begin{cases} A & \textit{if } U \subset V \\ 0 & \textit{otherwise} \end{cases} \right)$$

defines a functor from abelian groups to $\mathrm{PSh}(X, \mathbb{Z})$ *adjoint to the functor* $\Gamma(U, -)$.

Exercise 2:

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories and let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be left exact additive functors.

(i) Let M be an object of $\mathcal{B}$ and let $0 \to M \to K^\bullet$ be an exact complex such that K^i is acyclic for F for all i. Show that there is a canonical isomorphism

$$H^i(F(K^\bullet)) \to R^i F(M).$$

Hint: Proceed by induction on i.

(ii) Assume that G is exact and G maps injective objects to F-acyclic objects. Show that

$$R^i(FG)(M) \cong R^i F(GM).$$

(iii) Assume that F is exact. Show that

$$R^i(FG)(M) \cong F(R^i G(M)).$$

Exercise 3:
Let X be a (locally ringed) topological space. A sheaf $\mathscr{F}$ on X is called *flabby* if the restriction maps

$$\Gamma(U, \mathscr{F}) \longrightarrow \Gamma(V, \mathscr{F})$$

are surjective for all $V \subset U \subset X$. Let

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

be a short exact sequence of abelian sheaves (or of $\mathcal{O}_X$-modules).

(i) Assume that $\mathscr{F}$ is flabby. Show that

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow \Gamma(X, \mathscr{H}) \longrightarrow 0$$

is exact.
Hint: Let $h \in \Gamma(X, \mathscr{H})$. Apply Zorn's lemma to the set

$$\{(U, g) \mid U \subset X \textit{ open}, \; g \in \Gamma(U, \mathscr{G}) \textit{ such that } g \mapsto h|_U\}.$$

(ii) Assume that $\mathscr{F}$ is flabby. Show that $\mathscr{G}$ is flabby if and only if $\mathscr{H}$ is flabby.

Exercise 4:
Let X be a locally ringed space.

(i) Show that an injective $\mathcal{O}_X$-module is flabby.
Hint: For an open embedding $j : U \hookrightarrow X$ and an abelian sheaf $\mathscr{F}$ on U consider the sheaf $j_!\mathscr{F}$ which is the sheafification of the presheaf

$$V \longmapsto \begin{cases} \mathscr{F}(V) & \textit{if } V \subset U \\ 0 & \textit{otherwise.} \end{cases}$$

Show that $j_!$ is adjoint to j^* and that

$$\mathscr{F}(X) = \mathrm{Hom}_X(\mathcal{O}_X, \mathscr{F}) \longrightarrow \mathrm{Hom}_X(j_!\mathcal{O}_u, \mathscr{F}) = \mathrm{Hom}_U(\mathcal{O}_U, \mathscr{F}|_U) = \mathscr{F}(U)$$

agrees with the restriction map.

(ii) Show that flabby sheaves are acyclic for $\Gamma(X, -)$.

(iii) Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Show that the cohomology of $\mathscr{F}$ as an abelian sheaf coincides with the cohomology of $\mathscr{F}$ as an $\mathcal{O}_X$-module.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`