WS 2024/25

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Exercises for **Topology I**
Sheet 5

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Exercise 1. Let $X = \mathbb{N}$ with the discrete topology and let $Y = \{n^{-1} : n \in \mathbb{N}_{>0}\} \cup \{0\}$ with the subspace topology of $\mathbb{R}$. We have a continuous map $f\colon X \to Y$ sending 0 to 0 and $n > 0$ to n^{-1}. Show:

1. The map f induces a bijection on π_0 and isomorphisms $\pi_k(X, x) \cong \pi_k(Y, f(x))$ for all $x \in X$ and $k \geq 1$.
2. Nevertheless, f is *not* a homotopy equivalence.

Exercise 2. Prove the following converse of Whitehead's Theorem: any homotopy equivalence $f\colon X \to Y$ of arbitrary topological spaces induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and isomorphisms $\pi_k(X, x) \cong \pi_k(Y, f(x))$ for all $x \in X$, $k \geq 1$.

Warning. Note that f is not necessarily a *based* homotopy equivalence $(X, x) \to (Y, f(x))$, i.e. a chosen homotopy inverse might not induce a map $\pi_k(Y, f(x)) \to \pi_k(X, x)$.

∗ **Exercise 3** (*10 bonus points*)**.** Let X, Y be CW-complexes, $x \in X_0, y \in Y_0$, and let $f\colon X \to Y$ be a homotopy equivalence such that $f(x) = y$. Show that f defines a *based* homotopy equivalence $(X, x) \to (Y, y)$, i.e. there exists a based map $g\colon (Y, y) \to (X, x)$ together with base point preserving homotopies $gf \sim \mathrm{id}_X$, $fg \sim \mathrm{id}_Y$. Can one drop the assumption that x, y be 0-cells?

Exercise 4. Let X be any CW-complex, let Y be an n-dimensional CW-complex, $n \geq 0$, and let $f\colon X \to Y$ be a continuous map inducing a bijection on π_0 as well as isomorphisms $\pi_k(X, x) \cong \pi_k(Y, f(x))$ for all $x \in X$ and $1 \leq k \leq n$.

1. Show that f admits a section up to homotopy, i.e. there exists a continuous map $g\colon Y \to X$ together with a homotopy $fg \sim \mathrm{id}_Y$.

   Hint. First reduce to cellular f and then note that in this case the inclusion $Y \hookrightarrow M(f)$ into the mapping cylinder factors through the n-skeleton $M(f)_n$.
2. Show that if X is of dimension $\leq n$, then f is already a homotopy equivalence.

Definition. Let $f\colon X \to Y$ be a continuous map of spaces. We define the (unreduced) *mapping cone* $C(f)$ as the quotient $M(f)/i(X)$ of the mapping cylinder by the image of the inclusion $i\colon X \hookrightarrow M(f)$. The (unreduced) *cone* of a space X is defined as $CX \coloneqq C(\mathrm{id}_X)$.

Exercise 5.
1. Let X be any topological space. Show that the cone $C(X)$ is contractible.
2. Let $f\colon X \to Y$ be a cellular map of CW-complexes. Equip $C(f)$ with the structure of a CW-complex such that Y is a subcomplex.
3. Show: if X is a CW-complex and $f\colon Y \hookrightarrow X$ is the inclusion of a subcomplex, then the collapse map $C(f) \twoheadrightarrow X/Y$, sending $[x]$ to $[x]$ for $x \in X$ and $[y, t]$ to the class $[y]$ for $y \in Y, t \in [0, 1]$ is welldefined and a homotopy equivalence.
4. Conclude: for any CW-complex X with a subcomplex Y, any basepoint $y \in Y_0$, and any based space Z we have an exact sequence of pointed sets
   $$[X/Y, Z]_* \xrightarrow{-\circ p} [X, Z]_* \xrightarrow{-\circ i} [Y, Z]_*$$
   induced by the projection $p\colon X \twoheadrightarrow X/Y$ and the inclusion $i\colon Y \hookrightarrow X$.

Please hand in your solutions by **November 18, 10am** via eCampus.