WS 2024/25

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Exercises for **Topology I**

Sheet 9

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Topics for Bachelor's theses. On **Wednesday, December 18**, instead of a regular lecture we will have presentations of possible topics for Bachelor's theses in topology.

Exercise 1.

1. Construct for every short exact sequence $0 \to C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3 \to 0$ of chain complexes natural boundary maps $\partial\colon H_{n+1}(C_3) \to H_n(C_1)$ fitting into a long exact sequence

$$\cdots \to H_{n+1}(C_3) \xrightarrow{\partial} H_n(C_1) \xrightarrow{i_*} H_n(C_2) \xrightarrow{p_*} H_n(C_3) \xrightarrow{\partial} \cdots$$

Here *natural* means that for every commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & C_1' & \longrightarrow & C_2' & \longrightarrow & C_3' & \longrightarrow & 0 \end{array}$$

of chain complexes with exact rows and every $n \geq 0$ the square

$$\begin{array}{ccc} H_{n+1}(C_3) & \xrightarrow{\partial} & H_n(C_1) \\ {\scriptstyle f_{3*}}\downarrow & & \downarrow{\scriptstyle f_{1*}} \\ H_{n+1}(C_3') & \xrightarrow[\partial]{} & H_n(C_1') \end{array}$$

should commute.

2. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of abelian groups. Use the above to construct for every space X a long exact sequence

$$\cdots \to H_{n+1}(X, A_3) \to H_n(X, A_1) \to H_n(X, A_2) \to H_n(X, A_3) \to \cdots$$

such that these long exact sequences are natural in maps of topological spaces.

Remark. The maps $H_{n+1}(X, A_3) \to H_n(X, A_1)$ are called *Bockstein homomorphisms*, and are usually denoted by β. A particularly important special case is the short exact sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$, in which case the Bocksteins are degree shifting homomorphisms $H_{n+1}(X, \mathbb{Z}/p) \to H_n(X, \mathbb{Z}/p)$ of the $\mathbb{Z}/p$-homology of X.

please turn over

Exercise 2. Let X be a space and let $U, V \subseteq X$ be open with $X = U \cup V$.

1. Let $\mathcal{S}_0(X) \subseteq \mathcal{S}(X)$ be the simplicial subset of small simplices with respect to this cover, i.e. consisting of those $\nabla^n \to X$ that factor through U or V. Show that for any abelian group A the sequence of chain complexes

$$0 \longrightarrow C(U \cap V, A) \xrightarrow{\binom{i_{1*}}{i_{2*}}} C(U, A) \oplus C(V, A) \xrightarrow{(k_{1*}\ \ -k_{2*})} C(\mathcal{S}_0(X), A) \longrightarrow 0$$

is exact, where i_1, i_2, k_1, k_2 are the evident embeddings.

2. Conclude that there exists a long exact sequence

$$\cdots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(U \cap V, A) \xrightarrow{\binom{i_{1*}}{i_{2*}}} H_n(U, A) \oplus H_n(V, A) \xrightarrow{(k_{1*}\ \ -k_{2*})} H_n(X, A) \longrightarrow \cdots$$

natural in the following sense: if $X' = U' \cup V'$ is another topological space with an open cover, and $f\colon X \to X'$ is continuous such that $f(U) \subseteq U', f(V) \subseteq V'$, then the effects of f on the various homology groups define a map of long exact sequences.

Remark. This long exact equence is called the *Mayer–Vietoris sequence*.

Exercise 3. Use the Mayer–Vietoris sequence to compute the homology groups $H_k(S^n, A)$ for all $k, n \geq 0$ and every abelian group A.

$*$ **Exercise 4** (*10 bonus points*).

1. Let C be a levelwise free chain complex that is in addition exact. Show that the identity of C is chain homotopic to the zero map.

2. Prove the following algebraic version of Whitehead's Theorem: if $f\colon C \to D$ is a quasi-isomorphism of levelwise free chain complexes, then f is a chain homotopy equivalence.

   Hint. First construct for every chain map f a 'mapping cone' complex $C(f)$ with $C(f)_n = C_{n-1} \oplus D_n$ (and a clever choice of differential) and show that it is exact if f is a quasi-isomorphism.