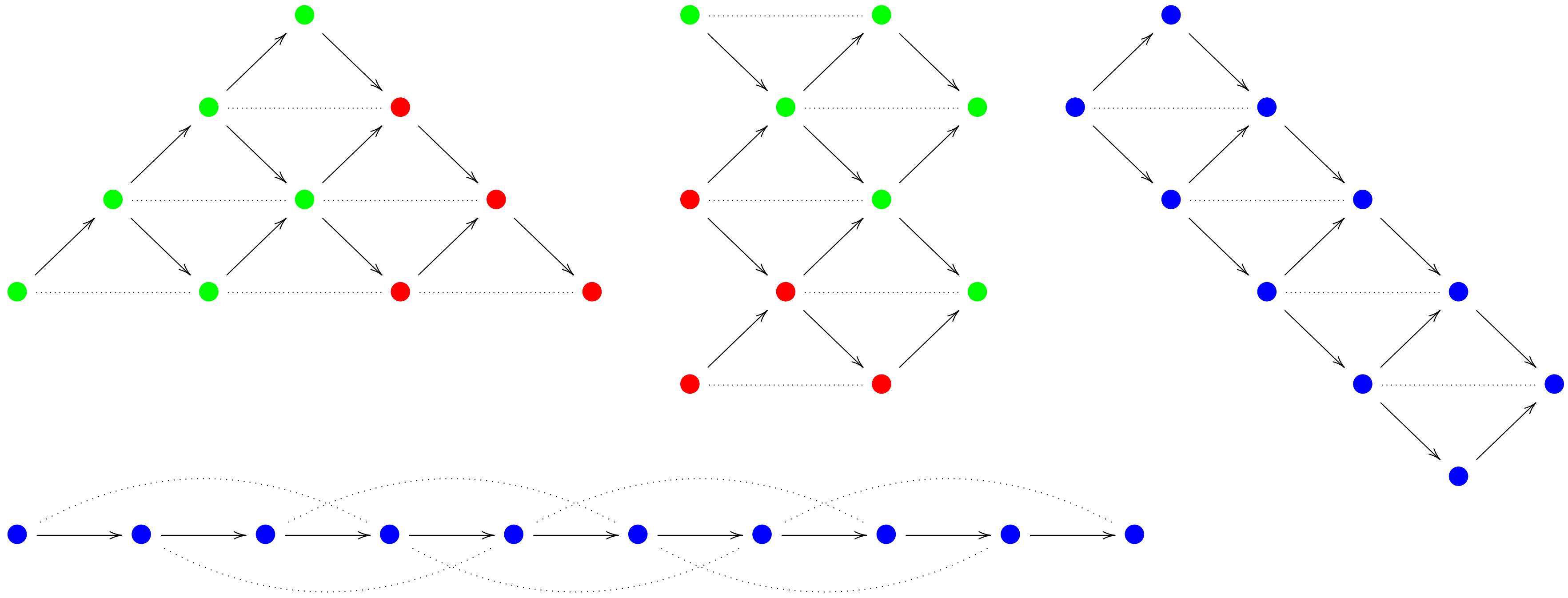
On Derived Equivalences of Triangles, Rectangles and Lines

Sefi Ladkani

Max-Planck-Institute for Mathematics, Bonn

http://guests.mpim-bonn.mpg.de/sefil/

What is the connection between . . .



$A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$

Context

- Derived *accessible algebras* [Lenzing - de la Peña 2008]
- Structured equivalence of *Euler forms* as an indicator of derived equivalence.
- Categories of singularities; *weighted projective lines* [Lenzing et al.]
- *Auslander algebras* and *initial modules* [Geiss-Leclerc-Schröer]
- *Cluster algebra* structures on . . .
  - Upper-triangular *unipotent matrices* [Geiss-Leclerc-Schröer]
  - *Grassmannians* [Scott 2006]

Lines

k – field, $\overrightarrow{A_n}$ – the quiver

$$\bullet_1 \xrightarrow{x} \bullet_2 \xrightarrow{x} \bullet_3 \xrightarrow{x} \cdots \xrightarrow{x} \bullet_n$$

The *path algebra* $k\overrightarrow{A_n}$ is the *incidence algebra* of the linear order on $\{1, 2, \dots, n\}$.

For $r \geq 2$, consider $A(n, r) = k\overrightarrow{A_n}/(x^r)$ –
the path algebra modulo the ideal generated by all the relations x^r.

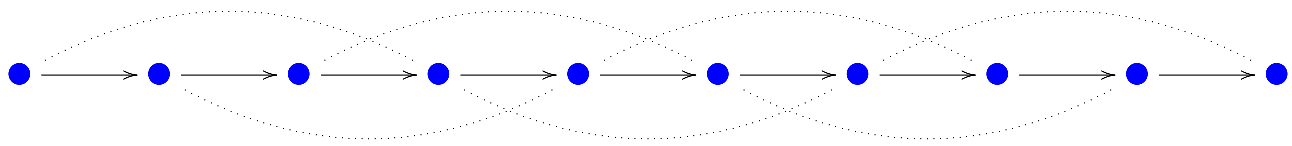
- $A(n, r)$ is of finite representation type,
- We are interested in its *derived equivalence* class, following [Lenzing - de la Peña, 2008].

The algebras $A(n,r)$

- $A(n,2) \sim k\overrightarrow{A_n}$.
- The derived equivalence class of $A(n,3)$ for $1 \le n \le 11$:

$$A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8, C(2,3,5), C(2,3,6), C(2,3,7)$$

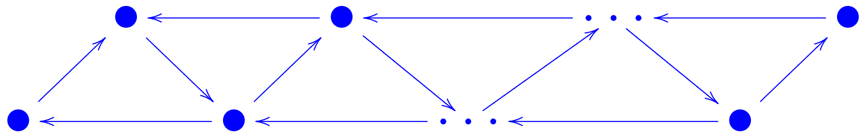
where $C(2,p,q)$ is the *canonical algebra* of weight type $(2,p,q)$ [Lenzing - de la Peña 2008].

 $\sim \quad C(2,3,6)$

- Characterization of the pairs (n,r) for which $A(n,r)$ is *piecewise hereditary* [Happel - U. Seidel].

The ADE Chain: $A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$

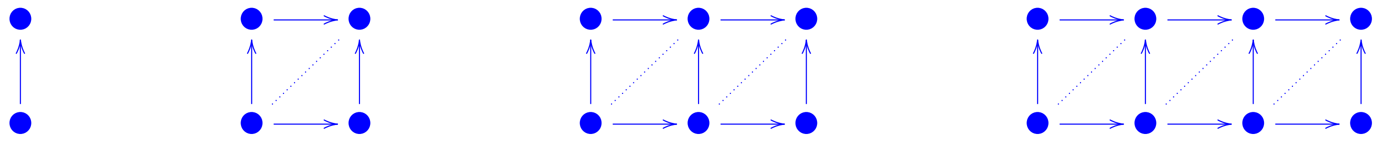
- The *cluster type* of . . .
  - the quiver



    with n vertices [Barot-Geiss-Zelevinsky 2006].
  - the coordinate rings of the *Grassmannians* [Scott 2006]

$$\mathrm{Gr}_{3,5}, \mathrm{Gr}_{3,6}, \mathrm{Gr}_{3,7}, \mathrm{Gr}_{3,8} \qquad (A_2, D_4, E_6, E_8)$$

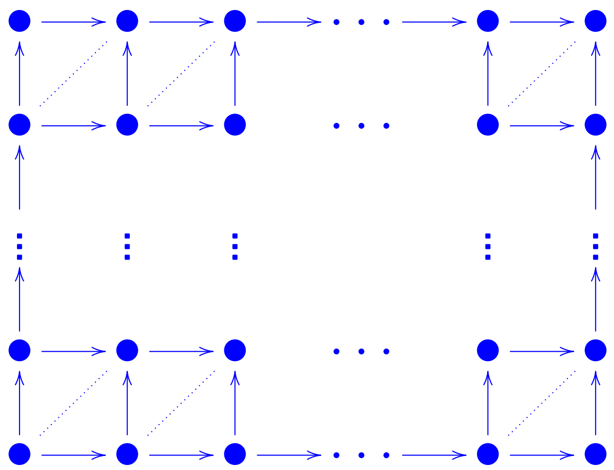
- The derived equivalence class of the incidence algebras



Rectangles

X, Y posets $\Rightarrow$ $X \times Y$ poset with $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

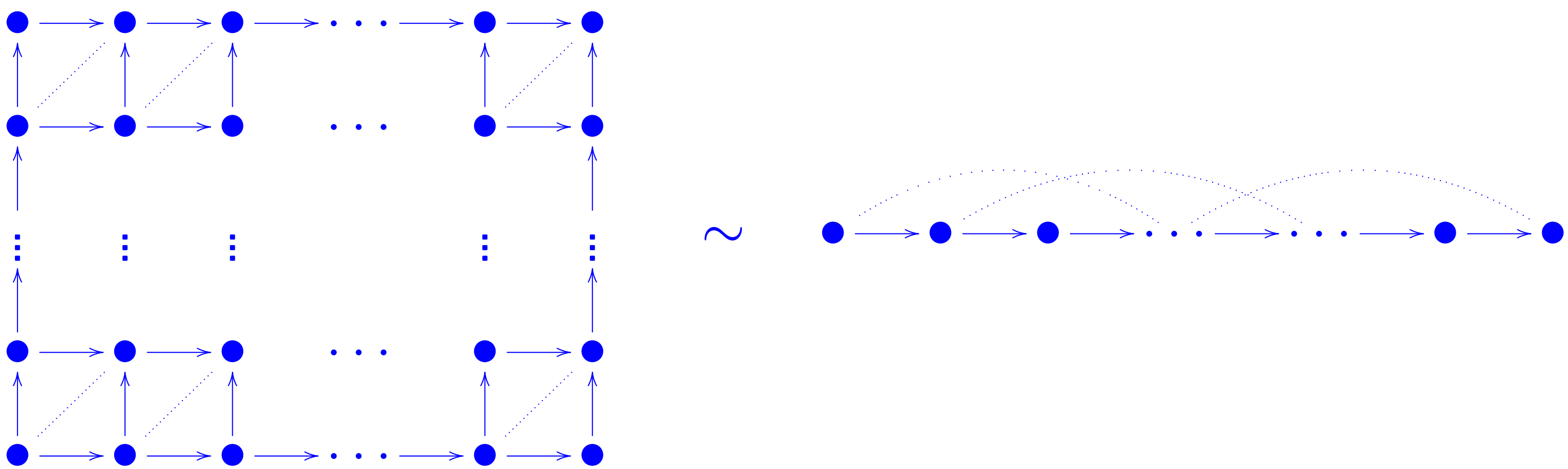
Let $n, m \geq 1$. Consider the incidence algebra of $\overrightarrow{A_n} \times \overrightarrow{A_m}$:



- Fully commutative quiver.
- Global dimension 2 (when $m, n \geq 2$).
- Periodic *Coxeter transformation*; even *fractionally Calabi-Yau* of dimension $\frac{n-1}{n+1} + \frac{m-1}{m+1}$.

Derived equivalence of rectangles and lines

Theorem 1. $k(\overrightarrow{A_n} \times \overrightarrow{A_m}) \sim A(m \cdot n, m+1)$.



Generalizes $A(n,2) \sim k\overrightarrow{A_n}$ and $A(2n,3) \sim k(\overrightarrow{A_n} \times \overrightarrow{A_2})$, hence $A(-,m)$ can be viewed as *higher ADE chains*.

Invariants of derived equivalence

Derived equivalent algebras (with finite global dimension)

$\Downarrow$

Equivalent Euler forms

with respect to bases of indecomposable projectives: *Cartan matrices*

$\Downarrow$

Similar Coxeter transformations

$\Downarrow$

Same Coxeter polynomial

Example – quivers with three vertices

Let $Q_{a,b}$ be the quiver $\bullet \overset{\vdots\, a}{\rightrightarrows} \bullet \overset{\vdots\, b}{\rightrightarrows} \bullet$,

with Cartan matrix and Coxeter polynomial

$$C_{a,b} = \begin{pmatrix} 1 & b & ba \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \qquad \chi_{a,b}(T) = T^3 + (3 - a^2 - b^2)T^2 + (3 - a^2 - b^2)T + 1.$$

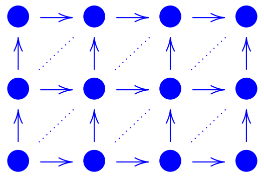
Then

$$\chi_{a,b} = \chi_{a',b'} \iff a^2 + b^2 = a'^2 + b'^2 \quad \text{(e.g. } \{1,8\} \text{ and } \{4,7\}\text{)}$$

but

$$C_{a,b} \sim C_{a',b'} \text{ (over } \mathbb{Z}!\text{)} \iff \{a,b\} = \{a',b'\} \iff kQ_{a,b} \sim kQ_{a',b'}.$$

Theorem 1 – Examining the Cartan matrices

 $= \overrightarrow{A_3} \times \overrightarrow{A_4}$

 $= A(12,4)$

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&0&1&1&0&1&1&0&1&1\\
0&0&1&0&0&1&0&0&1&0&0&1\\
0&0&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&0&1&1&0&1&1\\
0&0&0&0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\qquad
\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0\\
0&1&1&1&1&0&0&0&0&0&0&0\\
0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

$$A = \begin{pmatrix} 1&1&1\\ 0&1&1\\ 0&0&1 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1&0&0\\ 1&1&0\\ 1&1&1 \end{pmatrix}$$

A statement on matrices . . .

Proposition. Let A be a square invertible matrix over a commutative ring K. Then the bilinear forms represented by the matrices

$$C = \begin{pmatrix} A & A & \dots & A & A \\ 0 & A & A & \ddots & A \\ \vdots & 0 & A & \ddots & \vdots \\ \vdots & & \ddots & \ddots & A \\ 0 & \dots & \dots & 0 & A \end{pmatrix} \quad and \quad \begin{pmatrix} A & A^T & 0 & \dots & 0 \\ 0 & A & A^T & \ddots & \vdots \\ \vdots & 0 & A & \ddots & 0 \\ \vdots & & \ddots & \ddots & A^T \\ 0 & \dots & \dots & 0 & A \end{pmatrix} = C'$$

are *equivalent* over K.

A statement on matrices . . .

Proof. Let $S = -A^{-1}A^T$ and set

$$P = \begin{pmatrix} & & 0 & -S^{n-2} & S^{n-1} \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & S^{n-2} & 0 \\ 0 & -S & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ -I & S & 0 & & \\ I & 0 & & & \end{pmatrix}$$

Then $P^TCP = C'$, since

$$P^T \begin{pmatrix} A & A & \dots & A & A \\ 0 & A & A & \ddots & A \\ \vdots & 0 & A & \ddots & \vdots \\ \vdots & & \ddots & \ddots & A \\ 0 & \dots & \dots & 0 & A \end{pmatrix} P = P^T \begin{pmatrix} & & & 0 & AS^{n-1} \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ & 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & AS & 0 & & \\ A & 0 & & & \end{pmatrix} = \begin{pmatrix} A & A^T & 0 & \dots & 0 \\ 0 & A & A^T & \ddots & \vdots \\ \vdots & 0 & A & \ddots & 0 \\ \vdots & & \ddots & \ddots & A^T \\ 0 & \dots & \dots & 0 & A \end{pmatrix}.$$

. . . interpreted as derived equivalence

Λ − finite-dimensional algebra over k with $\operatorname{gl.dim}\Lambda < \infty$.

$D\Lambda = \operatorname{Hom}_k(\Lambda, k)$, with multiplication maps

$$\Lambda \otimes_\Lambda D\Lambda \to D\Lambda, \qquad D\Lambda \otimes_\Lambda \Lambda \to D\Lambda, \qquad D\Lambda \otimes D\Lambda \to 0.$$

Theorem 2.

$$\Lambda \otimes_k k\overrightarrow{A_n} = \begin{pmatrix} \Lambda & \Lambda & \dots & \Lambda & \Lambda \\ 0 & \Lambda & \Lambda & \ddots & \Lambda \\ \vdots & 0 & \Lambda & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \Lambda \\ 0 & \dots & \dots & 0 & \Lambda \end{pmatrix} \sim \begin{pmatrix} \Lambda & D\Lambda & 0 & \dots & 0 \\ 0 & \Lambda & D\Lambda & \ddots & \vdots \\ \vdots & 0 & \Lambda & \ddots & 0 \\ \vdots & & \ddots & \ddots & D\Lambda \\ 0 & \dots & \dots & 0 & \Lambda \end{pmatrix} = \Gamma$$

Corollary. Taking $\Lambda = k\overrightarrow{A_m}$ we get Theorem 1.

Proof of Theorem 2 – a tilting complex

$\Lambda \otimes_k k\overrightarrow{A_n}$ module: $M_1 \to M_2 \to \cdots \to M_n$ $M_i \in \mathrm{mod}\,\Lambda$

Let $\nu = - \overset{\mathbf{L}}{\otimes}_\Lambda D\Lambda$ be the *Serre functor*, $F = \nu[1]$,

$$
\begin{array}{llllllll}
T_0 : & \Lambda \to & 0 \to & 0 \to & \cdots \to & 0 \to & 0 \\
T_1 : & 0 \to & F\Lambda \to & 0 \to & \cdots \to & 0 \to & 0 \\
\vdots & & & & & & \\
T_{n-1} : & 0 \to & 0 \to & 0 \to & \cdots \to & 0 \to & F^{n-1}\Lambda
\end{array}
$$

Then $T = T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$ is a *tilting complex* with $\mathrm{End}_\Lambda T \simeq \Gamma$.

There are generalized versions for certain other auto-equivalences F.

Relevance

- Stable category of vector bundles on *weighted projective lines* [Kussin-Lenzing-Meltzer-de la Peña]

$$\underline{\mathrm{vect}}\,\mathbb{X}_{2,3,p} \simeq \mathcal{D}^b\,(A(2(p-1),3))$$

- Categories of (graded) singularities [loc. cit.]

$$x^2 + y^3 + z^p$$

- The cluster algebra structure on the coordinate ring of the *Grassmannian* $\mathrm{Gr}_{m+1,n+m+2}$ is related to $A_n \times A_m$ [Scott 2006].

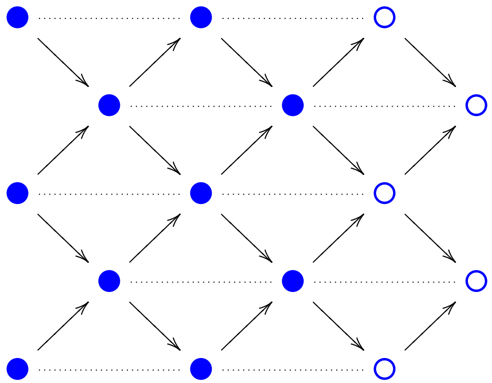
Some initial modules of path algebras

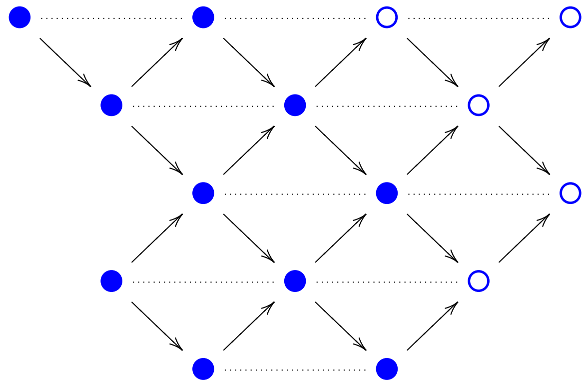
Q – an acyclic quiver, kQ – its path algebra

τ – the *Auslander-Reiten* translation

Consider the *initial modules* [Geiss-Leclerc-Schröer] of the form

$$kQ \oplus \tau^{-1}kQ \oplus \cdots \oplus \tau^{-r}kQ \qquad (r \geq 0)$$





Endomorphism rings of initial modules

Theorem 3. Let Q be an acyclic quiver and $r \geq 0$ such that

$$\tau^{-1}kQ, \tau^{-2}kQ, \dots, \tau^{-r}kQ$$

are all kQ-modules. Then

$$\text{End}_{kQ}\Big(kQ \oplus \tau^{-1}kQ \oplus \cdots \oplus \tau^{-r}kQ\Big) \sim kQ \otimes_k kA_{r+1}$$

Remark. No restrictions on r when Q is not Dynkin.

Theorem 3 – Strategy of proof

- Examine the *Euler forms* (this time, with respect to the basis of simples)

$$C = \begin{pmatrix} A & -A & 0 & \dots & 0 \\ 0 & A & -A & \ddots & \vdots \\ \vdots & 0 & A & \ddots & 0 \\ \vdots & & \ddots & \ddots & -A \\ 0 & \dots & \dots & 0 & A \end{pmatrix} \qquad \begin{pmatrix} A & A^T & 0 & \dots & 0 \\ 0 & A & A^T & \ddots & \vdots \\ \vdots & 0 & A & \ddots & 0 \\ \vdots & & \ddots & \ddots & A^T \\ 0 & \dots & \dots & 0 & A \end{pmatrix} = C'$$

- Observe structured equivalence $C' = P^T C P$ with

$$P = \mathrm{diag}(I, S, S^2, \dots, S^r), \qquad S = -A^{-1}A^T$$

- Construct appropriate tilting complex.
- Generalized version.

Auslander algebras

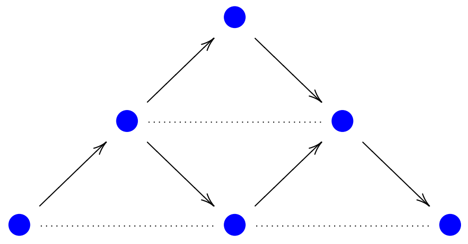
Λ – algebra of finite representation type.

The *Auslander algebra* of Λ is

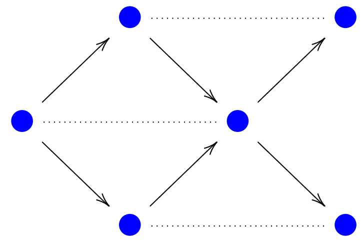
$$\text{Auslander}(\Lambda) = \text{End}_\Lambda\Big(\bigoplus_M M\Big)$$

where M runs over all indecomposable Λ-modules.

The Auslander algebras of derived equivalent algebras need *not* be derived equivalent:



$\sim kD_6$

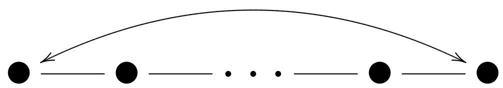


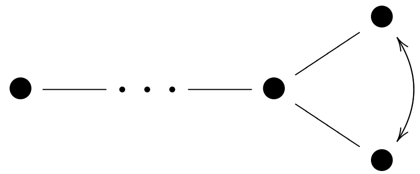
$\sim kE_6$

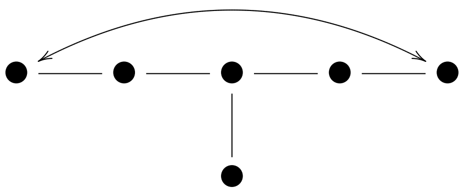
Auslander algebras of Dynkin quivers

A table of Dynkin quivers Q for which $\bigoplus M = \bigoplus_i \tau^{-i} kQ$.

Diagram	*Orientation*	*Derived type of Auslander algebra*
A_{2n}	none	
A_{2n+1}	symmetric	$A_{2n+1} \times A_{n+1}$
D_{2n}	any	$D_{2n} \times A_{2n-1}$
D_{2n+1}	symmetric	$D_{2n+1} \times A_{2n}$
E_6	symmetric	$E_6 \times A_6$
E_7	any	$E_7 \times A_9$
E_8	any	$E_8 \times A_{15}$

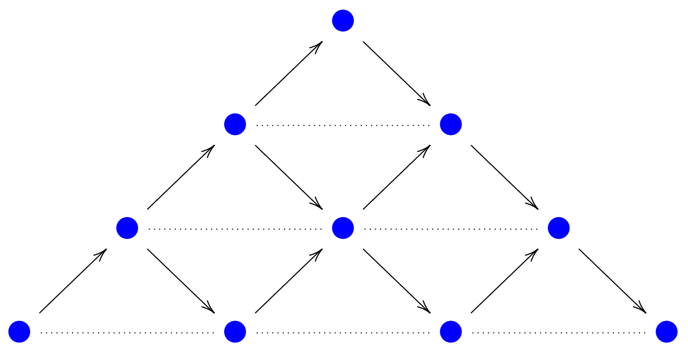






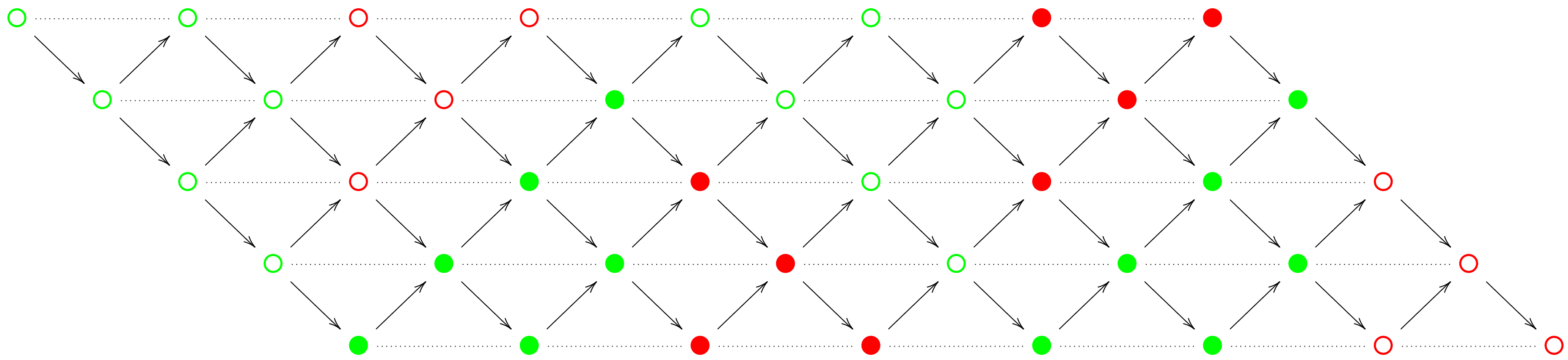
Triangles

Consider the Auslander algebras of $\overrightarrow{A_{2n}}$ (linear orientation):



Problem. Theorem 3 *cannot* be directly applied.

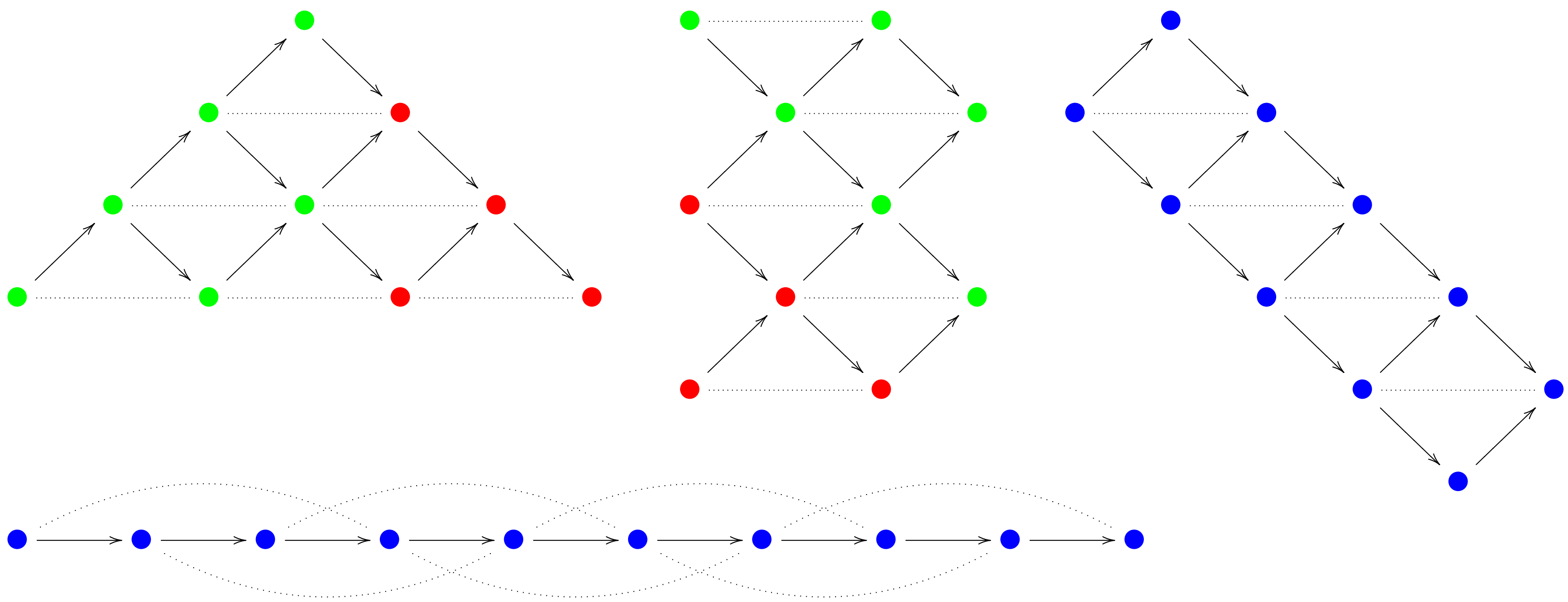
Derived equivalence through repetitive algebras



Corollary. Combining *Happel's Theorem* and Theorem 3,

$$\text{Auslander}(\overrightarrow{A_{2n}}) \sim \text{End}_{k\overleftrightarrow{A_{2n+1}}}\left(\bigoplus_{i=0}^{n-1} \tau^{-i} k\overleftrightarrow{A_{2n+1}}\right) \sim k(A_{2n+1} \times A_n)$$

. . . All these algebras are derived equivalent



$$\text{Auslander}(\overrightarrow{A_4}) \sim \text{End}_{k\overleftrightarrow{A_5}}\left(k\overleftrightarrow{A_5} \oplus \tau^{-1} k\overleftrightarrow{A_5}\right) \sim k(A_2 \times A_5) \sim A(10,3)$$