

ON JACOBIAN ALGEBRAS FROM CLOSED SURFACES

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ABSTRACT. We show that the quivers with potentials associated to ideal triangulations of marked surfaces with empty boundary are not rigid, and their completed Jacobian algebras are finite-dimensional and symmetric.

1. INTRODUCTION

In [10] Labardini-Fragoso associated a quiver with potential to any ideal triangulation of a surface with marked points in such a way that flips of triangulations correspond to mutations of the associated quivers with potentials, thus providing a link between the work of Fomin, Shapiro and Thurston [6] on cluster algebras arising from marked surfaces and the theory of quivers with potentials initiated by Derksen, Weyman and Zelevinsky [5].

When the surface has non-empty boundary, the potential associated to any ideal triangulation is rigid and its Jacobian algebra is finite-dimensional [10]. However, when the surface has empty boundary, it was conjectured that the potential associated to any ideal triangulation is not rigid [10, Conjecture 34]. The question whether its Jacobian algebra is finite-dimensional or not has been open for some time, see [8, Problem 8.1], [9, Question 6.4] and the survey [2, Remark 3.17]. The only cases where finite-dimensionality has been established so far are the once-punctured torus [8, Example 8.2] and recently the spheres with arbitrary number of punctures [11].

Our main result, stated in the following theorem, completely settles these questions. Recall that the auxiliary algebraic data needed to define the potential consists of a non-zero scalar (from a fixed field) for each puncture.

Theorem. *Let (S, M) be a surface with marked points and empty boundary.*

- (a) *If (S, M) is not a sphere with 4 punctures, then for any choice of scalars the quiver with potential associated to any ideal triangulation of (S, M) is not rigid and its (completed) Jacobian algebra is finite-dimensional and symmetric.*
- (b) *If (S, M) is a sphere with 4 punctures, then the same conclusion holds provided that the product of the scalars is not equal to 1.*

The theorem provides in particular an explicit construction of infinitely many families of symmetric, finite-dimensional Jacobian algebras.

As a consequence of the theorem we can associate a Hom-finite cluster category to any marked surface with empty boundary in a similar way as in the case of non-empty

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boundary [2, §3.4]. It is the generalized cluster category of Amiot [1] associated to the Jacobian algebra corresponding to (any) ideal triangulation. In the case of a sphere with 4 punctures, this category is a tubular cluster category studied by Barot and Geiss [4].

Corollary. *Let (S, M) be a surface with marked points and empty boundary. Then there is a Hom-finite triangulated 2-Calabi-Yau category $\mathcal{C}_{(S, M)}$ with a cluster-tilted object for each ideal triangulation.*

We outline our strategy for proving the theorem. Since the properties of non-rigidity and finite-dimensionality of Jacobian algebras are preserved under mutations of quivers with potentials [5] and any two ideal triangulations of a surface with marked points can be connected by a sequence of flips, it suffices to consider only one triangulation. Therefore we can avoid technical complications by dealing only with those triangulations which are suitably “nice”.

In Section 2 we consider triangulations with at least three arcs incident to every puncture and develop a combinatorial model for the associated quiver with potential. In Section 2.3 we introduce the additional conditions (\star) and (\diamond) on the quiver, and we express them in terms of combinatorial properties of the corresponding triangulation.

Then, in Section 3 we investigate the relations in the Jacobian algebra of a quiver with potential within the framework of our model. Some relations always hold, whereas additional relations are obtained by assuming additional hypotheses involving either condition (\star) or (\diamond) . Under these hypotheses we carry out the actual proof in Section 4, where we show that the potential is not rigid (Proposition 4.4) and the Jacobian algebra is finite-dimensional (Proposition 4.2) and symmetric (Proposition 4.7).

The existence of “nice” triangulations is shown in Section 5. It implies that for any surface with marked points and empty boundary there is a triangulation whose associated quiver with potential satisfies one of the conditions (\star) or (\diamond) , thus allowing us to conclude the proof.

In addition we also compute the Cartan matrices and the centers of the Jacobian algebras of the quivers with potentials considered in Section 4. For the precise statements see Proposition 4.8, Corollary 4.9 and Proposition 4.11. In particular, the rank of the Cartan matrix is bounded by the number of punctures, its determinant always vanishes and the center is the quotient of a polynomial ring (with as many variables as the arcs in the triangulation) by the ideal generated by all monomials of degree 2.

Since the property of a finite-dimensional algebra being symmetric, as well as its center and the rank of its Cartan matrix are all invariant under derived equivalence, the extension of the above results to all the quivers with potentials arising from triangulations of marked surfaces with empty boundary is now a consequence of the following result:

All the Jacobian algebras associated to the ideal triangulations of a given surface with marked points and empty boundary are derived equivalent.

We defer the proof of this result to a subsequent paper dealing with (weakly) symmetric Jacobian algebras in a broader framework.

2. COMBINATORIAL MODEL FOR THE QUIVER WITH POTENTIAL

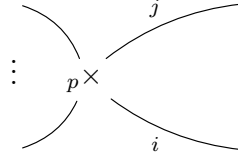
Let (S, M) be a closed surface with marked points. Recall that S is a compact, connected, oriented Riemann surface with empty boundary and M is a finite set of

points in S , called also punctures. In this section we consider a fixed ideal triangulation T of (S, M) with the property that

(T3) at each puncture $p \in M$ there are at least three arcs of T incident to p

(where an arc starting and ending at the same puncture is counted twice). In particular, such a triangulation T does not contain any self-folded triangles. As we shall see in Section 5, any marked closed surface has such a triangulation.

2.1. The quiver. Let Q be the adjacency quiver of T as defined by Fomin, Shapiro and Thurston [6]. Recall that Q is constructed in the following way: its vertices are the arcs of T , and we add an arrow from the arc i to the arc j if they are incident to a common puncture p and the arc j immediately follows i in the counterclockwise order around p .



Remark 2.1. *A-priori*, this process may create 2-cycles that then have to be removed when forming the adjacency quiver. However, due to our assumption (T3), this never happens.

The next proposition lists some basic properties of the quiver Q which will be crucial in our considerations. Denote by Q_0 the set of vertices of Q and by Q_1 the set of its arrows.

Proposition 2.2. *Let Q be the adjacency quiver of the triangulation T satisfying (T3). Then:*

- (a) Q is connected, and there are no loops or 2-cycles in Q .
- (b) For any $i \in Q_0$, there are exactly two arrows in Q_1 starting at i and two arrows ending at i .
- (c) There are invertible maps $f, g : Q_1 \rightarrow Q_1$ with the following properties:
 - For any $\alpha \in Q_1$, the set $\{f(\alpha), g(\alpha)\}$ consists of the two arrows that start at the vertex which α ends at;
 - f^3 is the identity on Q_1 .

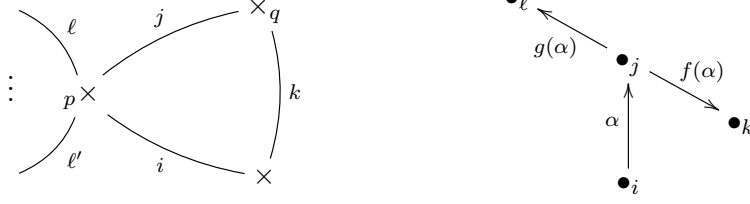
Proof. Part (a) is evident from the construction.

To show (b), observe that any arc i is a side of exactly two triangles of T , and each such triangle contributes one arrow starting at i and another ending at i .

For part (c), we define the maps f and g as follows. An arrow α corresponds to a pair i, j of consecutive arcs around a common puncture p , as in Figure 1, so that α starts at i and ends at j .

Let ℓ be the arc next to j in the counterclockwise order around p . We define $g(\alpha)$ to be the corresponding arrow $j \rightarrow \ell$.

Let q be the puncture at the other end of j and let k be the arc next to j in the counterclockwise order around q . We define $f(\alpha)$ to be the corresponding arrow $j \rightarrow k$. Observe that the arcs i, j, k enclose a triangle of T , hence $f^2(\alpha)$ is an arrow $k \rightarrow i$ and $f^3(\alpha)$ is the arrow α . In particular, the map f is invertible.

FIGURE 1. Definition of the maps f and g on the set of arrows.FIGURE 2. Cycles in Q arise in two ways: either from triangles of T (left) or from traversing the arcs around a puncture (right).

We note that the puncture q may coincide with the puncture p so that the arc ℓ may coincide with k , for example in a triangulation of a once punctured torus. In this case both arrows $f(\alpha)$ and $g(\alpha)$ start at j and end at $k = \ell$.

Finally, the map g is invertible; indeed, if ℓ' is the arc immediately preceding i in the counterclockwise order around p , then by applying g on the corresponding arrow $\ell' \rightarrow i$ we get α . \square

Since the map g is invertible, it induces a partition of the arrows in Q_1 into g -orbits, where the g -orbit of an arrow $\alpha \in Q_1$ is by definition the set of all arrows of the form $g^i(\alpha)$ for some $i \in \mathbb{Z}$. Let n_α be the size of the g -orbit of α , that is,

$$n_\alpha = \min \{r \in \mathbb{Z}_{>0} : g^r(\alpha) = \alpha\}$$

Obviously, the function $Q_1 \rightarrow \mathbb{Z}_{>0}$ sending α to n_α is constant on g -orbits.

Similarly, the invertible map f induces a partition of the arrows into f -orbits. Since the arrows $f(\alpha)$ and $g(\alpha)$ start where α ends, the arrows of any f -orbit or g -orbit can be arranged in a sequence whose concatenation is a cycle in Q .

The relations between these orbits and the triangulation are given in the next lemma.

Lemma 2.3. *Let f, g be the invertible maps corresponding to the triangulation T .*

- (a) *The f -orbits are of size 3; they are in one-to-one correspondence with the triangles of T .*
- (b) *The g -orbits are of size at least 3; they are in one-to-one correspondence with the punctures.*

Proof. Since $f^3 = id$, the f -orbits are of size 1 or 3. The first case is impossible since always $f(\alpha) \neq \alpha$ as these arrows start at different vertices.

Any triangle in T with sides i, j, k arranged in a clockwise order as in the left drawing of Figure 2 gives rise to a 3-cycle $i \rightarrow j \rightarrow k \rightarrow i$ in Q which can be written as

$$(2.1) \quad \alpha \cdot f(\alpha) \cdot f^2(\alpha).$$

The arrows $\{\alpha, f(\alpha), f^2(\alpha)\}$ form an f -orbit and any f -orbit is obtained in this way.

Fix a puncture, and let $i_0, i_1, \dots, i_{n-1}, i_n = i_0$ be the sequence of arcs incident to that puncture traversed in a counterclockwise order, as in the right drawing of Figure 2. We get a cycle $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{n-1} \rightarrow i_0$ in Q which by construction of the map g can be written as a path

$$(2.2) \quad \beta \cdot g(\beta) \cdot \dots \cdot g^{n-1}(\beta)$$

whose arrows form a g -orbit. Moreover, any g -orbit is obtained in this way. \square

2.2. The potential. In [10] Labardini associates to an ideal triangulation of a marked bordered surface a quiver with potential, using auxiliary data consisting of a non-zero scalar for every puncture (from a fixed field K). In the case of a triangulation of a marked closed surface satisfying (T3), by the correspondence of Lemma 2.3 between the punctures and the g -orbits on the set of arrows Q_1 in the adjacency quiver, we may view the auxiliary data as a function $c : Q_1 \rightarrow K^\times$ which is constant on g -orbits.

Recall that a potential on a quiver Q is a (possibly infinite) linear combination of cycles in the complete path algebra \widehat{KQ} of Q . An explicit form of the potential associated to the triangulation T in terms of the combinatorics of its adjacency quiver exploited in Proposition 2.2 is provided by the next proposition.

Proposition 2.4. *Let (Q, W) be the quiver with potential associated to T . Then the quiver Q is the adjacency quiver of T described above and the potential W is given by the formula*

$$(2.3) \quad W = \sum \alpha \cdot f(\alpha) \cdot f^2(\alpha) - \sum c_\beta \beta \cdot g(\beta) \cdot \dots \cdot g^{n_\beta-1}(\beta)$$

where the first sum is taken over representatives α of the f -orbits in Q_1 and the second sum is taken over representatives β of g -orbits in Q_1 .

Proof. The triangulation T satisfies (T3), hence it does not contain self-folded triangles and moreover in the formation of the adjacency quiver no 2-cycles had to be removed (see Remark 2.1). Therefore no reduction is needed which means that the associated quiver is identical to the adjacency quiver Q of T described above.

The associated potential W is by definition the sum of all 3-cycles in Q corresponding to the triangles of T together with scalar multiples of the cycles of Q “around” each puncture (see again Figure 2). By Lemma 2.3 and its proof, these cycles are precisely of the forms (2.1) and (2.2) corresponding to the f -orbits and g -orbits, respectively. \square

2.3. The conditions (\star) and (\diamond) . In order to prove the finite-dimensionality of the Jacobian algebra of (Q, W) in full generality we need to introduce a mild condition on the quiver Q concerning the size of its g -orbits. This condition is stated as follows:

$$(\star) \quad \text{For any } \alpha \in Q_1 \text{ we have } n_\alpha \geq 4 \text{ or } n_{f(\alpha)} \geq 4.$$

Let T be an ideal triangulation satisfying (T3). Then by Lemma 2.3 the size of any g -orbit is at least 3, that is, $n_\alpha \geq 3$ for any $\alpha \in Q_1$. The condition (\star) just says that

there are not too many g -orbits containing just 3 arrows; in other words, there are not too many punctures with just three arcs around them.

As the next lemma shows, the following condition on a triangulation T guarantees that its adjacency quiver satisfies (\star) :

(T3 $\frac{1}{2}$) T has (T3) and any arc has an endpoint with at least four arcs incident to it.

Lemma 2.5. *Let T be a triangulation with property (T3). Then condition (\star) is satisfied for its adjacency quiver if and only if T has property (T3 $\frac{1}{2}$).*

Proof. Assume that T has property (T3 $\frac{1}{2}$). Let $\alpha \in Q_1$ and set $\beta = f(\alpha)$. Then α ends at some vertex j where β starts at. The endpoints of the arc j in the triangulation correspond to the g -orbits of α and of β (which may coincide). Now the condition (T3 $\frac{1}{2}$) together with Lemma 2.3 imply that at least one of these orbits contains at least four arrows.

Conversely, assume that T does not have property (T3 $\frac{1}{2}$) and let j be an arc such that both of its endpoints have only three incident arcs. Take an arrow $\alpha \in Q_1$ ending at the vertex j . Then $n_\alpha = n_{f(\alpha)} = 3$. \square

It is much easier to verify the following property (T4) of a triangulation T , which obviously implies the property (T3 $\frac{1}{2}$):

(T4) at each puncture $p \in M$ there are at least four arcs of T incident to p .

Indeed, in Section 5 we will prove that with only two exceptions, namely the sphere with 4 or 5 punctures, any marked closed surface has a triangulation with the property (T4), and that the sphere with 5 punctures has a triangulation with property (T3 $\frac{1}{2}$).

However, the sphere with 4 punctures does not have a triangulation with property (T3 $\frac{1}{2}$), so that an argument involving the condition (\star) for the adjacency quiver of a triangulation satisfying (T3) would not be applicable. In order to deal with this particular case, we replace the condition (\star) by the condition (\diamond) on the g -orbits stated as follows:

(\diamond) For any $\alpha \in Q_1$ we have $n_\alpha = 3$

(or equivalently, $g^3 = id$ on Q_1) which holds for any triangulation of a sphere with 4 punctures having property (T3). Under the additional assumption that the product of the scalars associated to the punctures is not equal to 1, we are able to prove the finite-dimensionality in this case as well by using similar techniques.

3. RELATIONS IN THE JACOBIAN ALGEBRA

In this section we consider quivers with potential (Q, W) of the following form: Q is any quiver with the combinatorial properties described in Proposition 2.2, and W is the potential given by the formula (2.3) in the statement of Proposition 2.4. As shown in the previous section, this includes in particular the quivers with potential associated to triangulations of a marked closed surface which have property (T3).

3.1. **$\mathrm{PSL}_2(\mathbb{Z})$ -action on the quiver.** In view of Proposition 2.2(b) we can make the following definition.

Definition 3.1. For an arrow $\alpha \in Q_1$, denote by $\bar{\alpha}$ the other arrow starting at the same vertex as α .

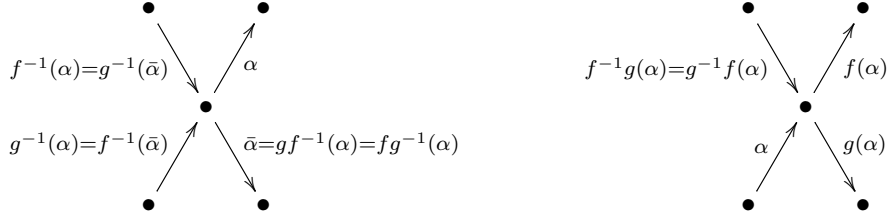
In the next lemma we record the basic relations between the functions f , g and $\bar{\cdot}$.

Lemma 3.2. *Let $\alpha \in Q_1$.*

- (a) *The set $\{f^{-1}(\alpha), g^{-1}(\alpha)\}$ consists of the two arrows that end at the vertex which α starts at.*
- (b) *$f(\bar{\alpha}) = g(\alpha)$ and $g(\bar{\alpha}) = f(\alpha)$.*
- (c) *$gf^{-1}(\alpha) = fg^{-1}(\alpha) = \bar{\alpha}$.*
- (d) *$f^{-1}(\bar{\alpha}) = g^{-1}(\alpha)$ and $g^{-1}(\bar{\alpha}) = f^{-1}(\alpha)$.*
- (e) *$f^{-1}g(\alpha) = g^{-1}f(\alpha)$ and is equal the other arrow ending at the same vertex as α .*

Proof. All these claims follow from the properties of the maps f and g described in Proposition 2.2(c). For example, both arrows $f^{-1}(\alpha)$ and $g^{-1}(\alpha)$ end at the vertex which α starts at. If they were identical, then applying f or g would give the same arrow, namely α , a contradiction.

The other statements follow similarly. They are best illustrated in the following pictures.



□

Proposition 3.3. *The group $\mathrm{PSL}_2(\mathbb{Z})$ acts transitively on the set of arrows Q_1 and its subgroup consisting of all the elements acting trivially is normal of finite index.*

Proof. The group $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation by two generators x, y and relations $x^2 = (xy)^3 = 1$. Its action on Q_1 is obtained by letting x, y act on an arrow $\alpha \in Q_1$ via

$$x(\alpha) = \bar{\alpha}, \quad y(\alpha) = g(\alpha)$$

and noting that $(xy)(\alpha) = f(\alpha)$ by the previous lemma.

Observe that any arrow starting or ending at a vertex which α starts or ends at belongs to the $\mathrm{PSL}_2(\mathbb{Z})$ -orbit of α . Since Q is connected, this implies that the action is transitive. □

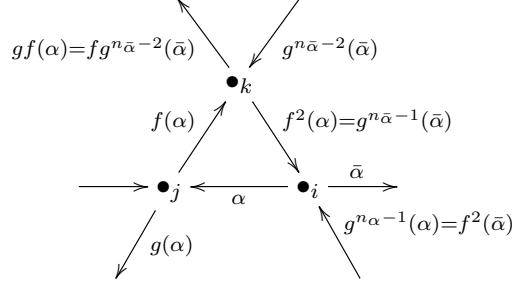
Lemma 3.4. *Let $\alpha \in Q_1$, and let i, j, k be the starting vertices of the arrows $\alpha, f(\alpha)$ and $f^2(\alpha)$, respectively.*

- (a) *The three vertices i, j, k are different and the six arrows $\alpha, \bar{\alpha}, f(\alpha), g(\alpha), f^2(\alpha), gf(\alpha)$ are all distinct.*
- (b) *$f^2(\alpha) = g^{n_{\bar{\alpha}}-1}(\bar{\alpha})$.*
- (c) *$gf(\alpha) = fg^{n_{\bar{\alpha}}-2}(\bar{\alpha})$.*

Proof. If any two of the vertices i, j, k were identical, then at least one of the arrows α , $f(\alpha)$ or $f^2(\alpha)$ would be a loop, a contradiction.

Now $\alpha, \bar{\alpha}$ are the two distinct arrows starting at i , and similarly $f(\alpha), g(\alpha)$ are those starting at j and $f^2(\alpha), gf(\alpha)$ those starting at k . As i, j, k are different, we get that these six arrows are all distinct.

We illustrate the situation in the following picture



Applying Lemma 3.2, we get $f^2(\alpha) = f^{-1}(\alpha) = g^{-1}(\bar{\alpha}) = g^{n\bar{\alpha}-1}(\bar{\alpha})$, hence also

$$gf(\alpha) = (gf^{-1})f^2(\alpha) = (fg^{-1})g^{n\bar{\alpha}-1}(\bar{\alpha}) = fg^{n\bar{\alpha}-2}(\bar{\alpha}).$$

□

3.2. Basic relations. The quiver with potential (Q, W) gives rise to the (completed) Jacobian algebra $\Lambda = \mathcal{P}(Q, W)$ which is our main object of study. It is defined as the quotient of the completed path algebra \widehat{KQ} by the closure of the two-sided ideal generated by the directional derivatives of W with respect to all arrows.

Lemma 3.5. *For any $\beta \in Q_1$ we have the following relation in Λ .*

$$\beta \cdot f(\beta) = c_{\bar{\beta}} \bar{\beta} \cdot g(\bar{\beta}) \cdot \dots \cdot g^{n_{\bar{\beta}}-2}(\bar{\beta}).$$

Proof. Since each arrow belongs to exactly one f -orbit and one g -orbit, we see that each arrow appears exactly once in each of two sums comprising W in (2.3).

By computing the directional derivative of W with respect to the arrow $\alpha = f^{-1}(\beta)$ we see that

$$(3.1) \quad f(\alpha) \cdot f^2(\alpha) = c_{\alpha} g(\alpha) \cdot g^2(\alpha) \cdot \dots \cdot g^{n_{\alpha}-1}(\alpha)$$

and the lemma follows by noting that $\bar{\beta} = g(\alpha)$ and hence $n_{\bar{\beta}} = n_{\alpha}$. □

Proposition 3.6. *For any $\alpha \in Q_1$ we have the following relations in Λ .*

$$(3.2) \quad \begin{aligned} \alpha \cdot f(\alpha) \cdot f^2(\alpha) &= c_{\alpha} \alpha \cdot g(\alpha) \cdot g^2(\alpha) \cdot \dots \cdot g^{n_{\alpha}-1}(\alpha) \\ &= c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot g^2(\bar{\alpha}) \cdot \dots \cdot g^{n_{\bar{\alpha}}-1}(\bar{\alpha}) = \bar{\alpha} \cdot f(\bar{\alpha}) \cdot f^2(\bar{\alpha}) \end{aligned}$$

$$(3.3) \quad \alpha \cdot g(\alpha) \cdot fg(\alpha) = c_{f(\alpha)} \alpha \cdot f(\alpha) \cdot gf(\alpha) \cdot g^2 f(\alpha) \cdot \dots \cdot g^{n_{f(\alpha)}-2} f(\alpha)$$

$$(3.4) \quad \alpha \cdot f(\alpha) \cdot gf(\alpha) = c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot \dots \cdot g^{n_{\bar{\alpha}}-3}(\bar{\alpha}) \cdot g^{n_{\bar{\alpha}}-2}(\bar{\alpha}) \cdot fg^{n_{\bar{\alpha}}-2}(\bar{\alpha})$$

Proof. The first equality in (3.2) follows from (3.1), whereas the second follows from Lemma 3.5 with $\beta = \alpha$ and Lemma 3.4(b). We get the last equality from the first one by interchanging α with $\bar{\alpha}$.

The relation (3.3) follows from Lemma 3.5 with $\beta = g(\alpha)$, noting that $\bar{\beta} = f(\alpha)$. Finally, (3.4) follows from Lemma 3.5 with $\beta = \alpha$ and Lemma 3.4(c). \square

Definition 3.7. Let $i \in Q_0$ and let $\alpha, \bar{\alpha}$ be the arrows starting at i . In view of (3.2), the two 3-cycles

$$\alpha \cdot f(\alpha) \cdot f^2(\alpha), \quad \bar{\alpha} \cdot f(\bar{\alpha}) \cdot f^2(\bar{\alpha})$$

as well as the scalar multiples of the n_α -cycle and $n_{\bar{\alpha}}$ -cycle

$$c_\alpha \alpha \cdot g(\alpha) \cdot g^2(\alpha) \cdot \dots \cdot g^{n_\alpha-1}(\alpha), \quad c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot g^2(\bar{\alpha}) \cdot \dots \cdot g^{n_{\bar{\alpha}}-1}(\bar{\alpha})$$

starting and ending at i are all equal in Λ . We denote their common value by z_i .

3.3. Additional relations. In this section we derive additional relations in the Jacobian algebra under further hypotheses on the quiver. They are summarized in the next proposition.

Proposition 3.8. *Assume one of the following hypotheses:*

- Q satisfies the condition (\star) ; or
- Q satisfies the condition (\diamond) and $\prod_{\alpha \in \Omega} c_\alpha \neq 1$, where Ω contains one representative from each g -orbit;

Then for any arrow $\alpha \in Q_1$, we have

$$\alpha \cdot g(\alpha) \cdot fg(\alpha) = 0 \quad \text{and} \quad \alpha \cdot f(\alpha) \cdot gf(\alpha) = 0$$

in the completed Jacobian algebra $\Lambda = \mathcal{P}(Q, W)$.

The proof of the proposition is given by the series of lemmas below. The case of condition (\star) is dealt with in Lemma 3.9 and Lemma 3.10, and the case of condition (\diamond) is considered in Lemma 3.11 and Lemma 3.12.

Lemma 3.9. *Assume that Q satisfies (\star) . Then for any arrow $\alpha \in Q_1$, there is an arrow $\alpha' \in Q_1$ and (scalar multiples of) paths q, q' not both of length zero such that*

$$(3.5) \quad \alpha \cdot g(\alpha) \cdot fg(\alpha) = \alpha \cdot f(\alpha) \cdot gf(\alpha) \cdot q$$

$$(3.6) \quad \alpha \cdot f(\alpha) \cdot gf(\alpha) = q' \cdot \alpha' \cdot g(\alpha') \cdot fg(\alpha').$$

Proof. The equation (3.5) follows from (3.3) whereas (3.6) follows from (3.4), taking $\alpha' = g^{n_{\bar{\alpha}}-3}(\bar{\alpha})$. The path q is of length $n_{f(\alpha)} - 3$ whereas q' is of length $n_{\bar{\alpha}} - 3$. Since $n_{\bar{\alpha}} = n_{f^2(\alpha)}$, the condition (\star) (for the arrow $f(\alpha)$) implies that $n_{f(\alpha)} \geq 4$ or $n_{\bar{\alpha}} \geq 4$. \square

Lemma 3.10. *Assume that Q satisfies (\star) . Then for any arrow $\alpha \in Q_1$, we have*

$$\alpha \cdot g(\alpha) \cdot fg(\alpha) = 0 \quad \text{and} \quad \alpha \cdot f(\alpha) \cdot gf(\alpha) = 0$$

in the completed Jacobian algebra Λ .

Proof. We show that the first expression vanishes in Λ . The proof for the second is similar. Indeed, invoking (3.5) and (3.6) we see that

$$\alpha \cdot g(\alpha) \cdot fg(\alpha) = \alpha \cdot f(\alpha) \cdot gf(\alpha) \cdot q = q' \cdot \alpha' \cdot g(\alpha') \cdot fg(\alpha') \cdot q$$

for some arrow α' and paths q, q' not both trivial. Thus, the path at the right hand side is strictly longer than the left hand side and contains a subpath of the same form.

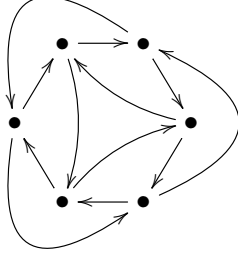


FIGURE 3. The quiver satisfying condition (\diamond) . It arises from an ideal triangulation of a sphere with 4 punctures having property (T3).

Set $\alpha_1 = \alpha'$. By repeating this process we get a sequence $\{\alpha_m\}_{m \geq 1}$ of arrows and (scalar multiples of) paths q_m, q'_m whose lengths sum to at least m such that

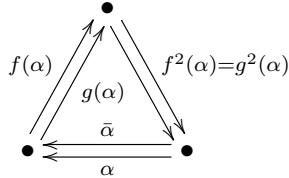
$$\alpha \cdot g(\alpha) \cdot fg(\alpha) = q'_m \cdot \alpha_m \cdot g(\alpha_m) \cdot fg(\alpha_m) \cdot q_m.$$

Since $\alpha \cdot g(\alpha) \cdot fg(\alpha)$ is equal in Λ to an arbitrarily long path, we deduce that it must vanish in Λ . \square

Lemma 3.11. *Assume that Q satisfies (\diamond) . Then Q is isomorphic to the quiver shown in Figure 3. In particular, it has 6 vertices, 12 arrows and 4 g -orbits. Moreover, for any arrow $\alpha \in Q_1$, the arrows $\alpha, \bar{\alpha}, f(\alpha)$ and $f(\bar{\alpha})$ belong to different g -orbits.*

Proof. Since g^3 acts on Q_1 as the identity, the $\mathrm{PSL}_2(\mathbb{Z})$ -action on Q_1 described in Proposition 3.3 induces a transitive action of the alternating group on four elements A_4 which has the presentation $\langle x, y : x^2 = (xy)^3 = y^3 = 1 \rangle$.

Therefore the number of arrows in Q divides 12. By Lemma 3.4 there are at least 6 arrows in Q . The case of 6 arrows is impossible since then Q would look like as



and $g^3(\alpha) = \bar{\alpha}$ would imply that Q does not satisfy (\diamond) (as a side remark we note that such Q arises from a triangulation of a once-punctured torus, and then the $\mathrm{PSL}_2(\mathbb{Z})$ -action induces an action of $\mathbb{Z}/6\mathbb{Z}$ on Q_1).

Therefore Q has 12 arrows and Q_1 is a A_4 -torsor. The assertions now follow. For example, the last one follows from the fact that in the presentation of A_4 given above, the elements $1, x, xy, xyx$ belong to different right cosets of the cyclic subgroup generated by y . \square

Lemma 3.12. *Assume that Q satisfies (\diamond) and that $\prod_{\alpha \in \Omega} c_\alpha \neq 1$, where Ω contains one representative from each g -orbit. Then for any arrow $\alpha \in Q_1$, we have*

$$\alpha \cdot g(\alpha) \cdot fg(\alpha) = 0 \quad \text{and} \quad \alpha \cdot f(\alpha) \cdot gf(\alpha) = 0$$

in the completed Jacobian algebra Λ .

Proof. By the previous lemma, there are four g -orbits and $\alpha, \bar{\alpha}, f(\alpha), f(\bar{\alpha})$ lie in different g -orbits, therefore by our assumption $c_\alpha c_{\bar{\alpha}} c_{f(\alpha)} c_{f(\bar{\alpha})} \neq 1$.

Now by repeatedly applying (3.3) and (3.4) we get

$$\begin{aligned} \alpha \cdot g(\alpha) \cdot fg(\alpha) &= c_{f(\alpha)} \alpha \cdot f(\alpha) \cdot gf(\alpha) = c_{f(\alpha)} c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot fg(\bar{\alpha}) \\ &= c_{f(\alpha)} c_{\bar{\alpha}} c_{f(\bar{\alpha})} \bar{\alpha} \cdot f(\bar{\alpha}) \cdot gf(\bar{\alpha}) = c_{f(\alpha)} c_{\bar{\alpha}} c_{f(\bar{\alpha})} c_\alpha \alpha \cdot g(\alpha) \cdot fg(\alpha) \end{aligned}$$

and the result follows. \square

4. JACOBIAN ALGEBRAS FROM “NICE” TRIANGULATIONS

As in the previous section, we consider quivers with potential (Q, W) of the following form: Q is any quiver with the combinatorial properties described in Proposition 2.2, and W is the potential given by the formula (2.3) in the statement of Proposition 2.4.

In this section, we investigate the (completed) Jacobian algebra $\Lambda = \mathcal{P}(Q, W)$ under one of the following additional hypotheses:

- Q satisfies the condition (\star) ; or
- Q satisfies the condition (\diamond) and $\prod_{\alpha \in \Omega} c_\alpha \neq 1$, where Ω contains one representative from each g -orbit;

(so that the conclusion of Proposition 3.8 holds) and show that Λ is finite-dimensional, symmetric, and the potential is not rigid. In addition we compute the Cartan matrix and the center of Λ .

4.1. Finite dimensionality.

Lemma 4.1. *For any arrow $\alpha \in Q_1$ we have in the completed Jacobian algebra Λ*

$$\alpha \cdot f(\alpha) \cdot f^2(\alpha) \cdot \alpha = 0 \quad \text{and} \quad \alpha \cdot g(\alpha) \cdot \dots \cdot g^{n_\alpha - 1}(\alpha) \cdot \alpha = 0.$$

Proof. In view of (3.2), it is enough to show that one of these expressions vanishes, since the other is a scalar multiple of it. Using (3.2) again, we get

$$\alpha \cdot f(\alpha) \cdot f^2(\alpha) \cdot \alpha = \bar{\alpha} \cdot f(\bar{\alpha}) \cdot f^2(\bar{\alpha}) \cdot \alpha$$

which vanishes by Proposition 3.8 applied to $f(\bar{\alpha})$, noting that $\alpha = gf^2(\bar{\alpha})$. \square

Proposition 4.2. *The algebra Λ is finite-dimensional. It has a basis consisting of the paths*

$$\{e_i\}_{i \in Q_0} \cup \{\alpha \cdot g(\alpha) \cdot \dots \cdot g^r(\alpha)\}_{\alpha \in Q_1, 0 \leq r < n_\alpha - 1} \cup \{z_i\}_{i \in Q_0}.$$

Proof. To show the finite-dimensionality of Λ , it is enough to show that the image of any sufficiently long path in \widehat{KQ} vanishes.

Indeed, by Proposition 2.2(c) such a path can be written as $\alpha_0 \cdot \alpha_1 \cdot \dots \cdot \alpha_N$ where for every $0 \leq j < N$ we have $\alpha_{j+1} = f(\alpha_j)$ or $\alpha_{j+1} = g(\alpha_j)$.

Now Proposition 3.8 and Lemma 4.1 tell us that the only paths whose image in Λ is possibly non-zero are of the form $\alpha \cdot g(\alpha) \cdot \dots \cdot g^r(\alpha)$ for some $0 \leq r \leq n_\alpha - 1$ or paths of the form $\alpha \cdot f(\alpha)$ or $\alpha \cdot f(\alpha) \cdot f^2(\alpha)$. As there are only finitely many such paths, this shows the finite-dimensionality of Λ .

By Lemma 3.5 and (3.2), any path $\beta \cdot f(\beta)$ or $\beta \cdot f(\beta) \cdot f^2(\beta)$ can be expressed as (scalar multiple) of a path of the form $\alpha \cdot g(\alpha) \cdot \dots \cdot g^r(\alpha)$ for suitable $\alpha \in Q_1$ and $r \geq 0$. Therefore the algebra Λ is spanned by the trivial paths e_i for each vertex $i \in Q_0$ together

with the paths $\alpha \cdot g(\alpha) \cdot \dots \cdot g^r(\alpha)$ for $\alpha \in Q_1$ and $0 \leq r \leq n_\alpha - 1$. The only relations among these paths are the commutativity relations in (3.2), hence when forming a basis for Λ we may take only those paths with $r < n_\alpha - 1$ and add the cycle z_i for each $i \in Q_0$. \square

Remark 4.3. The algebra Λ can therefore be written as a quiver with relations $\Lambda \simeq KQ/I$. The description of the ideal $I \subseteq KQ$ depends on our hypothesis on Q ; if Q satisfies the condition (\diamond) , then

$$I = \langle \alpha \cdot f(\alpha) - c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot \dots \cdot g^{n_{\bar{\alpha}}-2}(\bar{\alpha}) : \alpha \in Q_1 \rangle,$$

whereas if Q satisfies the condition (\star) , then

$$I = \langle \alpha \cdot f(\alpha) - c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot \dots \cdot g^{n_{\bar{\alpha}}-2}(\bar{\alpha}), \beta \cdot f(\beta) \cdot gf(\beta) : \alpha \in Q_1, \beta \in \Theta \rangle$$

where $\Theta \subseteq Q_1$ is a set of representatives of h -orbits for the (invertible) map $h: Q_1 \rightarrow Q_1$ defined by $h(\beta) = g^{-3}(\beta)$.

4.2. Non-rigidity.

Proposition 4.4. *The potential W is not rigid.*

Proof. By [5, §8], in order to show that W is not rigid, one has to find a potential on Q which is not cyclically equivalent to an element in the Jacobian ideal of W .

Indeed, consider a potential W' in \widehat{KQ} which is a 3-cycle $W' = \alpha \cdot f(\alpha) \cdot f^2(\alpha)$ for some arrow $\alpha \in Q_1$ starting at some vertex i . The image of W' in Λ is $z_i \neq 0$, hence it does not belong to the Jacobian ideal of W . Since this holds for any such 3-cycle, we deduce that W' is not cyclically equivalent to an element of the Jacobian ideal of W , hence W is not rigid. \square

4.3. Symmetry. For a finite-dimensional algebra Λ the space $D\Lambda = \text{Hom}_K(\Lambda, K)$ of K -linear functionals on Λ is a Λ - Λ -bimodule via

$$(\varphi\lambda)(x) = \varphi(\lambda x) \qquad (\lambda\varphi)(x) = \varphi(x\lambda)$$

for $\varphi \in D\Lambda$ and $\lambda, x \in \Lambda$. The algebra Λ is called *symmetric* if $D\Lambda$ and Λ are isomorphic as Λ - Λ -bimodules. For an element $\lambda \in \Lambda$, define a dual element $\lambda^\vee \in D\Lambda$ by

$$\lambda^\vee(x) = \begin{cases} a & \text{if } x = a\lambda \text{ for some } a \in K \\ 0 & \text{otherwise} \end{cases}$$

so that $(c\lambda)^\vee = c^{-1}\lambda^\vee$ for any $c \in K^\times$.

Let $\Lambda = \mathcal{P}(Q, W)$. There is a duality between paths which can be extended to a K -linear isomorphism $\Phi: D\Lambda \xrightarrow{\sim} \Lambda$ defined in the following way. Any non-zero path in Λ has the form $p = \alpha \cdot g(\alpha) \cdot \dots \cdot g^{r-1}(\alpha)$ for some $\alpha \in Q_1$ and $0 \leq r \leq n_\alpha$ (here, $r = 0$ means the path of length 0 corresponding to the starting vertex of α). The path p can be completed to a cycle $p \cdot q$ along a g -orbit and we define $\Phi(p^\vee)$ to be the multiple of q by the scalar corresponding to that g -orbit. More precisely,

$$(4.1) \qquad \Phi((\alpha \cdot g(\alpha) \cdot \dots \cdot g^{r-1}(\alpha))^\vee) = c_\alpha g^r(\alpha) \cdot \dots \cdot g^{n_\alpha-1}(\alpha)$$

which is well-defined by the identity (3.2). In particular, $\Phi(e_i^\vee) = z_i$ and $\Phi(z_i^\vee) = e_i$ for any $i \in Q_0$. Since $D\Lambda$ has a basis

$$\{e_i^\vee\}_{i \in Q_0} \cup \left\{ (\alpha \cdot g(\alpha) \cdot \dots \cdot g^{r-1}(\alpha))^\vee \right\}_{\alpha \in Q_1, 1 \leq r \leq n_\alpha - 1} \cup \{z_i^\vee\}_{i \in Q_0}$$

Φ can be extended by linearity to a K -linear isomorphism $\Phi : D\Lambda \rightarrow \Lambda$.

The next lemma shows that Φ has a similar completion property with respect to f -orbits as well.

Lemma 4.5. *Let $\alpha \in Q_1$. Then*

$$\Phi(\alpha^\vee) = f(\alpha) \cdot f^2(\alpha) \qquad \Phi((\alpha \cdot f(\alpha))^\vee) = f^2(\alpha).$$

Proof. This follows from (3.1), Lemma 3.5 and Lemma 3.4(b). \square

Lemma 4.6. *Let $i \in Q_0$, $\alpha \in Q_1$. Then $\alpha \cdot z_i = 0$ and $z_i \cdot \alpha = 0$.*

Proof. We show only that $z_i \cdot \alpha = 0$. The proof of the other claim is similar. Since z_i is a cycle starting and ending at i , it is enough to consider an arrow α starting at i . But then we can write $z_i = \alpha \cdot f(\alpha) \cdot f^2(\alpha)$ and the result follows from Lemma 4.1. \square

Proposition 4.7. *The Jacobian algebra Λ of (Q, W) is symmetric.*

Proof. We show that the isomorphism of K -vector spaces $\Phi : D\Lambda \rightarrow \Lambda$ is an isomorphism of Λ - Λ -bimodules. In other words, we need to verify that for any path p in Λ and any $i \in Q_0$, $\beta \in Q_1$ we have

$$(4.2) \qquad \Phi(p^\vee \cdot e_i) = \Phi(p) \cdot e_i \qquad \Phi(e_i \cdot p^\vee) = e_i \cdot \Phi(p)$$

$$(4.3) \qquad \Phi(p^\vee \cdot \beta) = \Phi(p^\vee) \cdot \beta \qquad \Phi(\beta \cdot p^\vee) = \beta \cdot \Phi(p^\vee).$$

If p starts at i and ends at j , then $\Phi(p)$ is a multiple of a path starting at j and ending at i . This shows (4.2). For (4.3), we start by noting that $p^\vee \cdot \beta = 0$ if p cannot be written as a linear combination of paths starting at β and $p^\vee \cdot \beta = q^\vee$ if p can be written uniquely as $p = \beta q$, and similarly for $\beta \cdot p^\vee$.

Let $p = e_i$ for $i \in Q_0$ and let $\beta \in Q_1$. Then $e_i^\vee \cdot \beta = 0$, $\beta \cdot e_i^\vee = 0$ and (4.3) follows from Lemma 4.6.

Let $p = z_i$ for some $i \in Q_0$ and let $\alpha, \bar{\alpha}$ be the arrows starting at i . Then

$$z_i^\vee \cdot \beta = \begin{cases} (c_\alpha g(\alpha) \cdot \dots \cdot g^{n_\alpha - 1}(\alpha))^\vee & \text{if } \beta = \alpha, \\ (c_{\bar{\alpha}} g(\bar{\alpha}) \cdot \dots \cdot g^{n_{\bar{\alpha}} - 1}(\bar{\alpha}))^\vee & \text{if } \beta = \bar{\alpha}, \\ 0 & \text{otherwise} \end{cases}$$

$$\beta \cdot z_i^\vee = \begin{cases} (c_\alpha \alpha \cdot g(\alpha) \cdot \dots \cdot g^{n_\alpha - 2}(\alpha))^\vee & \text{if } \beta = g^{n_\alpha - 1}(\alpha), \\ (c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot \dots \cdot g^{n_{\bar{\alpha}} - 2}(\bar{\alpha}))^\vee & \text{if } \beta = g^{n_{\bar{\alpha}} - 1}(\bar{\alpha}), \\ 0 & \text{otherwise} \end{cases}$$

and (4.3) follows from (4.1).

Let $p = \alpha \cdot g(\alpha) \cdot \dots \cdot g^{r-1}(\alpha)$ for some $\alpha \in Q_1$. If $1 \leq r < n_\alpha - 1$, then

$$p^\vee \cdot \beta = \begin{cases} (g(\alpha) \cdot \dots \cdot g^{r-1}(\alpha))^\vee & \text{if } \beta = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

$$\beta \cdot p^\vee = \begin{cases} (\alpha \cdot g(\alpha) \cdot \dots \cdot g^{r-2}(\alpha))^\vee & \text{if } \beta = g^{r-1}(\alpha), \\ 0 & \text{otherwise} \end{cases}$$

and (4.3) follows from (4.1). Finally, if $r = n_\alpha - 1$ then $p = c_\alpha^{-1} \bar{\alpha} \cdot f(\bar{\alpha})$ by Lemma 3.5 so that $p^\vee = c_\alpha f^2(\bar{\alpha}) = c_\alpha g^{n_\alpha-1}(\alpha)$ and

$$p^\vee \cdot \beta = \begin{cases} (g(\alpha) \cdot \dots \cdot g^{n_\alpha-2}(\alpha))^\vee & \text{if } \beta = \alpha, \\ c_\alpha f(\bar{\alpha})^\vee & \text{if } \beta = \bar{\alpha}, \\ 0 & \text{otherwise} \end{cases}$$

$$\beta \cdot p^\vee = \begin{cases} (\alpha \cdot g(\alpha) \cdot \dots \cdot g^{n_\alpha-3}(\alpha))^\vee & \text{if } \beta = g^{n_\alpha-2}(\alpha), \\ c_\alpha \bar{\alpha}^\vee & \text{if } \beta = f(\bar{\alpha}), \\ 0 & \text{otherwise} \end{cases}$$

thus (4.3) follows from (4.1), Lemma 3.4(b) and Lemma 4.5. \square

4.4. The Cartan matrix. Recall that the Cartan matrix of Λ is a $|Q_0| \times |Q_0|$ matrix whose (i, j) -entry is the dimension of the space of paths in Λ starting at the vertex i and ending at j .

Any puncture $p \in M$ defines a column vector $v_p \in \mathbb{Z}^{Q_0}$ in the following way. Let $i_0, i_1, \dots, i_{n-1}, i_n = i_0$ be the sequence of arcs incident to p traversed in a counterclockwise order, so that $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{n-1} \rightarrow i_0$ is a cycle whose arrows form a g -orbit. For any arc i set $v_p(i)$ to be the number of times i appears in the sequence $(i_0, i_1, \dots, i_{n-1})$. Set also $n_p = n$, or equivalently $n_p = \sum_{i \in Q_0} v_p(i)$.

Proposition 4.8. *The Cartan matrix of Λ is given by the formula*

$$C_\Lambda = \sum_{p \in M} v_p \cdot v_p^T$$

or equivalently, $(C_\Lambda)_{ij} = \sum_{p \in M} v_p(i)v_p(j)$. Moreover, $(C_\Lambda)_{ii} \in \{2, 4\}$ and $(C_\Lambda)_{ij} \in \{0, 1, 2, 4\}$ for any $i, j \in Q_0$. In particular, $\dim_K \Lambda = \sum_{p \in M} n_p^2$.

Proof. Consider two different vertices i and j . Every non-zero path from i to j is of the form $\alpha \cdot g(\alpha) \cdot \dots \cdot g^r(\alpha)$ for suitable $\alpha \in Q_1$ and $r \geq 0$, and all these paths are linearly independent. Hence the entry $(C_\Lambda)_{ij}$ is just the number of such paths.

Any such path corresponds to traversal of the arcs around a puncture starting at the arc i and ending at j going in a counterclockwise direction without completing a full round. For a given puncture p , the number of such traversals is therefore $v_p(i)v_p(j)$, hence the number of all such paths is $\sum_{p \in M} v_p(i)v_p(j)$.

If $i = j$ then in this way we have not counted the trivial path e_i , but on the other hand we counted the cycle z_i twice in view of the commutativity relations (3.2), so the formula $(C_\Lambda)_{ii} = \sum_{p \in M} v_p(i)v_p(i)$ still holds.

The remaining assertions on the entries $(C_\Lambda)_{ij}$ follow from the fact that for any $i \in Q_0$, $v_p(i) \geq 0$ for $p \in M$ and $\sum_{p \in M} v_p(i) = 2$. \square

Corollary 4.9. *We have $\text{rank } C_\Lambda \leq |M|$ and $\det C_\Lambda = 0$.*

Proof. The rank of each of the $|M|$ summands $v_p v_p^T$ of C_Λ is 1, hence the first claim. The second claim follows now from the fact that always $|M| < |Q_0|$. Indeed, the number of arcs in a triangulation of a closed surface with genus g and P punctures is $6g - 6 + 3P$ which always exceeds P . \square

Remark 4.10. The vanishing of $\det C_\Lambda$ comes in stark contrast to the situation for the Jacobian algebras arising from triangulations of bordered surfaces without punctures. Indeed, these algebras are gentle [3] and their Cartan determinants are always powers of 2 by [7].

4.5. The center.

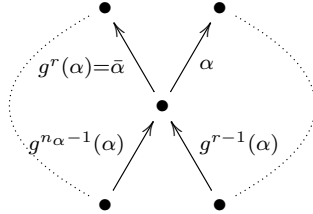
Proposition 4.11. *The center $Z(\Lambda)$ of Λ is isomorphic to the truncated polynomial algebra $K[\{x_i\}_{i \in Q_0}]/(\{x_i x_j\}_{i,j \in Q_0})$.*

Proof. We show that a basis of $Z(\Lambda)$ is given by 1 together with the cycles z_i for each $i \in Q_0$. The relation $z_i \cdot z_j = 0$ would follow from Lemma 4.6.

Let $z \in Z(\Lambda)$. Since z commutes with the idempotents e_i , it must be a sum of cycles. Let us describe the non-zero cycles starting at a given vertex $i \in Q_0$. Obviously, e_i and z_i are such cycles. Let α be an arrow starting at i . If $\bar{\alpha}$ and α are not in the same g -orbit, then these are all such cycles, otherwise write $\bar{\alpha} = g^r(\alpha)$ and then

$$w_i = \alpha \cdot g(\alpha) \cdot \dots \cdot g^{r-1}(\alpha), \quad w'_i = g^r(\alpha) \cdot g^{r+1}(\alpha) \cdot \dots \cdot g^{n_\alpha-1}(\alpha)$$

are also non-zero cycles starting at i as in the following picture,



and together with e_i and z_i they form all such cycles.

Assume that α and $\bar{\alpha}$ are in the same g -orbit. We want to show that in z the coefficients of the cycles w_i and w'_i must vanish. Indeed, write

$$z = \lambda_i e_i + \mu_i z_i + \rho_i w_i + \rho'_i w'_i + \dots$$

for some scalars $\lambda_i, \mu_i, \rho_i, \rho'_i$ where we ignore all cycles not starting at i .

Since there are no 2-cycles in Q , we have $3 \leq r \leq n_\alpha - 3$ and $w'_i \cdot \bar{\alpha} = 0$ by Proposition 3.8. Thus, if $\rho_i \neq 0$, then by Lemma 4.6

$$z \cdot \bar{\alpha} = \lambda_i \bar{\alpha} + \rho_i w_i \cdot \bar{\alpha} = \lambda_i \bar{\alpha} + \rho_i \alpha \cdot \dots \cdot g^r(\alpha)$$

whereas $\bar{\alpha} \cdot z$ is a sum of paths all starting at $\bar{\alpha}$. Since $\alpha \cdot \dots \cdot g^r(\alpha)$ cannot be written as a sum of paths starting at $\bar{\alpha}$, we get that $z \cdot \bar{\alpha} \neq \bar{\alpha} \cdot z$, a contradiction. We deduce that $\rho_i = 0$. A similar argument with multiplication by α shows that $\rho'_i = 0$ as well.

Finally note that all the coefficients λ_i must be equal, since Q is connected, whereas there is no restriction on the coefficients μ_i in view of Lemma 4.6. \square

5. EXISTENCE OF “NICE” TRIANGULATIONS

Proposition 5.1. *Let (S, M) be a marked closed surface. Then:*

- (a) *If (S, M) is not a sphere with 4 or 5 punctures, it has a triangulation satisfying (T4).*
- (b) *If (S, M) is a sphere with 5 punctures, it has a triangulation satisfying $(T3\frac{1}{2})$, but no triangulation satisfying (T4).*
- (c) *If (S, M) is a sphere with 4 punctures, it has a triangulation satisfying (T3), but no triangulation satisfying $(T3\frac{1}{2})$.*

The proof is by induction on the number of punctures, and follows by combining the statements of the next lemmas.

Lemma 5.2. *Let (S, M) be a marked closed surface and (S, M') the marked closed surface obtained from (S, M) by adding one more puncture.*

- (a) *If (S, M) has a triangulation satisfying (T3), then so does (S, M') .*
- (b) *If (S, M) has a triangulation satisfying $(T3\frac{1}{2})$, then so does (S, M') .*
- (c) *If (S, M) has a triangulation satisfying (T4), then so does (S, M') .*

Proof. Let T be a triangulation of (S, M) without self-folded triangles. We may place the additional puncture p of M' on an arc of T and obtain a triangulation T' of (S, M') by adding four arcs incident to p as in the right picture below:



In T' there are 4 arcs incident to p and the number of arcs incident to each other puncture has not decreased. The lemma thus follows. \square

Lemma 5.3. *Any triangulation of a once-punctured closed surface of genus $g \geq 1$ has property (T4).*

Proof. When counting the arcs incident to the puncture, each arc of the triangulation is counted twice. Since there are $6g - 3$ arcs in the triangulation, the puncture has $12g - 6$ arcs incident to it. \square

- Lemma 5.4.**
- (a) *A sphere with 6 punctures has a triangulation satisfying (T4).*
 - (b) *A sphere with 5 punctures has a triangulation satisfying $(T3\frac{1}{2})$, but no triangulation satisfying (T4).*
 - (c) *A sphere with 4 punctures has a triangulation satisfying (T3), but no triangulation satisfying $(T3\frac{1}{2})$.*

Proof. Figure 4 presents triangulations of spheres with 4, 5 and 6 punctures with the required properties. Note that they can be viewed as the faces of a tetrahedron, triangular bipyramid and an octahedron, respectively.

No triangulation of a sphere with 4 or 5 punctures can satisfy (T4), since the number of arcs (6 and 9, respectively) is less than twice the number of punctures. Moreover, a

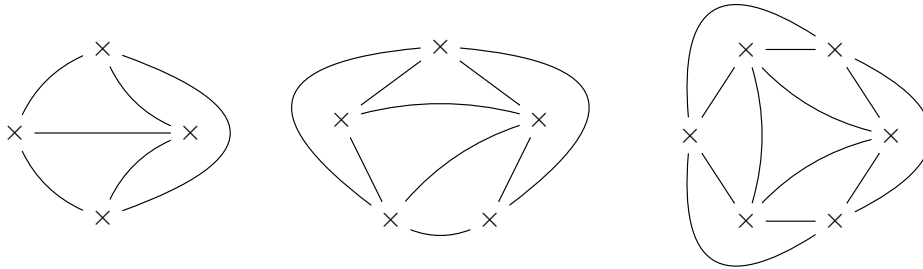


FIGURE 4. Triangulations of spheres with 4, 5 and 6 punctures.

triangulation of a sphere with 4 punctures which satisfies (T3) cannot satisfy $(T3)^{\frac{1}{2}}$, since at all the punctures there are exactly three incident arcs. \square

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