HOMEWORK #7 IN ALGEBRAIC STRUCTURES 2

1. Prove Lemma 7.2.

2. Prove Lemma 7.4.

3. Show that the condition P2 implies P3 and the condition P3 implies P4 [see Lemma 7.5].

4. Let $\alpha \in \mathbb{C}$ be the positive 4-th root of 2, $L := \mathbb{Q}(\alpha)$ and $M \subset \mathbb{C}$ be the splitting field of $t^4 - 2$. Show that
a) $[L : \mathbb{Q}] = 4$ and the polynomial $t^4 - 2$ is irreducible in $\mathbb{Q}[t]$,
b) $M = L(i), i^2 = -1$,
c) the polynomial $t^4 - 2$ is irreducible in $\mathbb{Q}(i)[t]$,
d) there exists $\sigma \in Gal(M/\mathbb{Q})$ such that $\sigma(\alpha) = i\alpha$ and $\sigma(i) = i$,
e) $\sigma^2 \neq e, \sigma^4 = e$,
f) there exists $\tau \in Gal(M/L) \subset Gal(M/\mathbb{Q})$ such that $\tau(i) = -i$,
g) $\tau\sigma = \sigma^3\tau$
h) the group $Gal(M/\mathbb{Q})$ is generated by τ, σ and the table of the multiplication in $Gal(M/\mathbb{Q})$ can be deduced from the relations

$$\sigma^4 = e, \tau^2 = e, \tau\sigma = \sigma^3\tau$$

5. Let K be a field of characteristic $p > 0, \alpha \in \bar{K}$. Show that α is separable over K iff $[K(\alpha) : K] = [K(\alpha^p) : K]$