Multiplicative obstructions in global homotopy theory

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Introduction

The goal of this thesis is to study multiplicative aspects of global homotopy theory. Global homotopy theory means that we study phenomena in stable homotopy theory which are equivariant for all compact Lie groups G in a way compatible with variations in the group. An algebraic example of such a globally equivariant object would be the representation ring $R(G)$ of a compact Lie group. This is defined for any G, and it has functorial restrictions for any continuous homomorphism $K \to G$ as well as transfers for inclusions $H \leq G$ of closed subgroups. Topological examples include equivariant cohomology theories which exhibit the same structure of restrictions and transfers, for example equivariant K-theory or equivariant bordism. In the same way as non-equivariantly, such global cohomology theories are represented by spectra. For our model of global homotopy theory, we choose orthogonal spectra, which can be evaluated on all finite representations of compact Lie groups (see [Sch18]).
Many of these cohomology theories are endowed with additional multiplicative structures, and the question arises whether this can be lifted to the representing spectra. This question classically is approached by obstruction theory, finding conditions for an algebraic multiplication to be extended step by step to a topological multiplication. Our goal is to establish a similar obstruction theory for global homotopy theory.
In classical obstruction theory, the obstructions to extending structure often lie in some sort of cohomology of the corresponding objects. The first step will thus be to find an appropriate cohomology theory for the objects we consider, which are global analogues of commutative rings.

Spectra as a model for global homotopy theory

Definition 1. An orthogonal spectrum is a collection of pointed topological spaces X_n, endowed with an action of the orthogonal group $O(n)$, together with structure maps $\sigma_{n,m}\colon S^n \wedge X_m \to X_{n+m}$, which are $O(n) \times O(m)$-equivariant, unital and associative. We denote the category of orthogonal spectra by $\mathcal{S}p$.

If we have a orthogonal spectrum X, a compact Lie group G and an n-dimensional G-representation V, we define the value of X at V as $X(V) = \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$. Here $\mathbf{L}(\mathbb{R}^n, V)$ is the space of linear isometric embeddings, hence is a free and transitive $O(n)$-space. Moreover, on $X(V) \cong X_n$, we have a natural G-action by the G-action on V.
Using these G-equivariant evaluations, we define G-equivariant homotopy groups for any compact Lie group: We fix a complete G-universe $\mathcal{U}$. Then we define for $k \geq 0$ the G-equivariant homotopy groups as

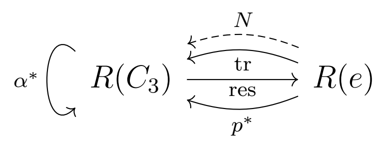
$$\pi_k^G(X) = \operatorname*{colim}_{V \subset \mathcal{U}} [S^{V \oplus \mathbb{R}^k}, X(V)]^G,$$

where $[\_,\_]^G$ denotes G-equivariant homotopy classes. If $k \leq 0$, we similarly define $\pi_k^G(X) = \operatorname{colim}_{V \subset \mathcal{U}} [S^V, X(V \oplus \mathbb{R}^{-k})]^G$.
Definition 2. A morphism $f\colon X \to Y$ of orthogonal spectra is a global equivalence if it induces an isomorphism $\pi_*^G(f)$ for all compact Lie groups G. Inverting these weak equivalences, we obtain the so called global homotopy category $\mathcal{GH}$.

The homotopy groups of an orthogonal spectrum come equipped with structure relating the values at different compact Lie groups. If we have a morphism $\alpha\colon K \to G$ of compact Lie groups or an inclusion $H \leq G$ of a closed subgroup, we have a restriction and a transfer map

$$\alpha^*\colon \pi_*^G(X) \to \pi_*^K(X),\ \mathrm{tr}_H^G\colon \pi_*^H(X) \to \pi_*^G(X),$$

respectively. These morphisms have to satisfy certain compatibility conditions, and we collect this data in the notion of a global functor. Thus, the collection of homotopy groups $\underline{\pi}_0(X) = (\pi_0^G(X))_G$ is the prototypical example of a global functor. Global functors are the analogues of abelian groups in non-equivariant homotopy theory and of Mackey functors for G-equivariant homotopy theory for a fixed compact Lie group G.



The structure of a global (power) functor [Sch18].

The category of orthogonal spectra has a symmetric monoidal structure, called the smash product $\wedge$. We define ultra-commutative ring spectra as commutative monoids in this category. This name is chosen since these ring spectra encode a rich type of structure. For example, on the homotopy groups $\underline{\pi}_0(R)$ of an ultra-commutative ring spectrum, there are norm maps

$$N_H^G\colon \pi_0^H(R) \to \pi_0^G(R) \text{ for } H \leq G$$

in addition to a multiplication. This additional structure makes $\underline{\pi}_0(R)$ into a so called global power functor, the global analogue of a Tambara functor. Thus, the notion of an ultra-commutative ring spectrum is stronger than that of an E_∞-ring spectrum, since these do not support those norm maps.

A cohomology theory for commutative rings

Let R be a commutative ring. We follow [Qui70] in defining André-Quillen cohomology of commutative R-algebras. For this, we consider abelian group objects in the category $\mathrm{Alg}_R^{\mathrm{aug}}$ of augmented R-algebras. This is an augmented R-algebra $S \xrightarrow{\varepsilon} R$ together with a unit map $\eta\colon R \to S$, an addition map $\alpha\colon S \times_R S \to S$ and an inversion map $\iota\colon S \to S$, which satisfy the usual conditions for an abelian group. The category $\mathrm{Ab}(\mathrm{Alg}_R^{\mathrm{aug}})$ is an abelian category equivalent to the category of R-modules. Indeed, the square-zero extension functor

$$R \ltimes (\_)\colon \mathrm{Mod}_R \to \mathrm{Alg}_R^{\mathrm{aug}},\ R \mapsto R \ltimes M = R \oplus M$$

preserves products, and hence abelian group objects. But any R-module M is an abelian group object in Mod_R by the addition map, thus we obtain a functor $R \ltimes (\_)\colon \mathrm{Mod}_R \to \mathrm{Ab}(\mathrm{Alg}_R^{\mathrm{aug}})$.
This is an equivalence, and the inverse is given by the kernel functor

$$K\colon \mathrm{Alg}_R^{\mathrm{aug}} \to \mathrm{Mod}_R,\ (S \xrightarrow{\varepsilon} R) \mapsto \ker(\varepsilon).$$

That these functors are indeed inverse follows from the fact that on the kernel of an abelian group object S in $\mathrm{Alg}_R^{\mathrm{aug}}$, all products vanish.

We can now consider the inclusion $\mathrm{Ab}(\mathrm{Alg}_R^{\mathrm{aug}}) \to \mathrm{Alg}_R^{\mathrm{aug}}$. We obtain a left adjoint to this functor by looking at derivations. Recall that if we have an augmented R-algebra $\varepsilon\colon S \to R$, we obtain an isomorphism $\varepsilon \ltimes (\_)\colon \mathrm{Der}_R(S, M) \cong \mathrm{Alg}_R^{\mathrm{aug}}(S, R \ltimes M)$. Moreover, the functor $\mathrm{Der}_R(S, \_)$ is represented by the module of Kähler differentials $\Omega^1_{S/R}$ together with the universal derivation $d\colon S \to \Omega^1_{S/R}$. Thus, we obtain the natural isomorphism

$$\mathrm{Alg}_R^{\mathrm{aug}}(S, R \ltimes M) \cong \mathrm{Der}_R(S, M) \cong \mathrm{Mod}_S(\Omega^1_{S/R}, M) \cong \mathrm{Mod}_R(R \otimes_S \Omega^1_{S/R}, M).$$

This together with the equivalence of R-modules and abelian group objects in $\mathrm{Alg}_R^{\mathrm{aug}}$ exhibits $R \otimes_{(\_)} \Omega^1_{(\_)/R}$ as left adjoint to the inclusion of abelian group objects. It thus provides a universal way of passing from $\mathrm{Alg}_R^{\mathrm{aug}}$ into an abelian category. Note that in all the above, we can also replace the category $\mathrm{Alg}_R^{\mathrm{aug}}$ by the category Alg_R/S of R-algebras over S, if we work relative to a fixed S.
Let R be a commutative ring and S be an R-algebra. We resolve S as an R-algebra by a simplicial projective object $P_\bullet \to S$. Then we apply the functor $S \otimes_{(\_)} \Omega^1_{(\_)/R}$ to the resolution $P_\bullet$ and obtain a complex $\mathbb{L}_{S/R}$ in Mod_S, called the cotangent complex. For an S-module M, we then define the André-Quillen (co-)homology groups of S over R with coefficients in M as

$$AQ^i(S/R, M) = H^i(\mathrm{Hom}(\mathbb{L}_{R/S}, M)) \cong H^i(\mathrm{Der}_R(P_\bullet, M)),\ AQ_i(S/R, M) = H_i(\mathbb{L}_{S/R} \otimes_S M).$$

These invariants detect properties of the commutative R-algebra S. It can be used for example to detect étale maps of rings.

Topological André-Quillen cohomology

Building on the definition of algebraic André-Quillen cohomology, Basterra defined a topological version for commutative ring spectra in [Bas99] as follows, following ideas of Kriz.
Let R be a commutative ring spectrum. Then we have model structures on the categories of R-modules, (augmented) R-algebras and non-unital R-algebras. We recall that the Kähler differentials are defined as $\Omega^1_{S/R} = I/I^2$, where $I = \ker(\mu\colon S \otimes_R S \to S)$ is the kernel of the multiplication map. Thus, we define for any augmented R-algebra S the augmentation ideal $I(S)$ as the fibre of the augmentation. This is a non-unital R-algebra. Then, for any non-unital algebra J, we define the module of indecomposables $Q(J)$ as the cofiber of the multiplication map $J \wedge_R J \to J$. These functors are right respectively left Quillen functors, and thus descend to the homotopy categories.
Definition 3. For an commutative ring spectrum R and a commutative R-algebra S, the S-module of derived Kähler differentials is

$$\Omega^1_{S/R} = \mathbf{LQR}I(S \wedge_R^{\mathbf{L}} S).$$

Moreover, for an S-module M, we define topological André-Quillen (co-)homology as

$$TAQ^i(S/R, M) = \pi_{-i}(F_S(\Omega^1_{S/R}, M)),\ TAQ_i(S/R, M) = \pi_i(\Omega^1_{S/R} \wedge_S M),$$

where F_S denotes the S-module function spectrum.

On Eilenberg-MacLane spectra, this recovers the algebraic definition in that

$$TAQ^*(HS/HR; HM) \cong AQ^*(R/S; M).$$

Moreover, we also have a description of this theory by stabilization, analogous to the relation of algebraic André-Quillen cohomology to abelianization of augmented algebras. We can form a stable model category $\mathcal{S}p(\mathcal{C})$ of spectra for any pointed model category $\mathcal{C}$. Then, the suspension spectrum functor $\mathcal{C} \to \mathcal{S}p(\mathcal{C})$ is in fact a universal left Quillen functor into a stable model category. As the category of R-modules is already stable, $\mathcal{S}p(\mathrm{Mod}_R)$ is Quillen equivalent to Mod_R itself. As in the algebraic case, the functor $\Omega^1_{(\_)/R}$ now induces a Quillen equivalence

$$\mathcal{S}p(\Omega^1_{(\_)/R})\colon \mathcal{S}p(\mathrm{Alg}_R^{\mathrm{aug}}) \to \mathcal{S}p(\mathrm{Mod}_R) \cong \mathrm{Mod}_R$$

(see [BM05]).

Finally, this cohomology theory for commutative ring spectra actually gives rise to an obstruction theory. One prominent example is the work by Goerss and Hopkins, who developed in [GH04] an obstruction theory for extending algebras over a homotopy ring spectrum to an actual algebra spectrum. They use this to show that the Lubin-Tate (or Morava E-theory) spectra E_n, related to deformations of finite-height formal group laws, have the structure of a commutative ring spectrum which depends functorially on the formal group law.

Towards a global theory

We now indicate how we want to modify this theory in order to give an obstruction theory for ultra-commutative ring spectra. First, we consider the algebraic side of the problem. The homotopy groups of an ultra-commutative ring spectrum R support multiplications and norm maps. Hence, a theory of derivations and a resulting André-Quillen cohomology theory has to incorporate these norm maps. However, the homotopy groups of an R-module do not interact with the norm maps. Using a corresponding definition of a module over a global power functor, we arrive at a working notion of derivations, which has the same relations to Kähler differentials and square-zero extensions as in the non-equivariant context. Hence, this gives rise to an André-Quillen cohomology theory. However, it is an observation by Strickland [Str] in the G-equivariant case that this notion of modules is not equivalent to abelian group objects in augmented algebras anymore. Hence, we need to either replace our notion of modules or the approach via abelian group objects.
On the topological side, the notion of an ultra-commutative ring spectrum is stronger than that of an E_∞-ring spectrum. It is however reasonable to expect that the equivalence $\mathrm{Mod}_R \cong \mathcal{S}p(\mathrm{Alg}_R^{\mathrm{aug}})$ translates to an equivalence for E_∞-algebra spectra, not for ultra-commutative ring spectra. The reason for this is that the stabilization does not incorporate any of the equivariant structure leading to the norm maps. Hence, we want to investigate how one can perform a genuine equivariant stabilization of the category of augmented algebras, which leads to an analogous identification with the category of R-modules. This is related to the first issue, since spectra are the non-equivariant analogue to abelian groups, and we want to translate this analogy into a global context.
After these foundational questions are addressed, we want to consider the construction of an obstruction theory, akin to that of Goerss-Hopkins, and to apply this to examples.

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