UNIQUE CONTINUATION AND INVERSE PROBLEMS

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Contents

1 Recovery: Reconstructions from boundary measurements - Talk 2: Sections 4,5,6

A summary written by Alex Amenta

Abstract

We continue the reconstruction of the conductivity γ in the electrical prospection problem, and more generally of the compressibility κ and density ρ in the reduced acoustic equation for time-harmonic waves. We also show how to reconstruct κ and ρ from surface point source data.

1.1 Introduction

Recall that in Calderón's problem we consider the equation

$$L_\gamma(u) := \nabla \cdot (\gamma \nabla u) = 0 \tag{1}$$

on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. The problem is to reconstruct the conductivity $\gamma \in C^{1,1}(\overline{\Omega})$ from knowledge of the Dirichlet-to-Neumann map Λ_γ on the boundary $\partial\Omega$. More generally, we can consider the reduced acoustic equation for time-harmonic waves with frequency ω,

$$\nabla \cdot \left(\frac{1}{\rho(x)} \nabla p(x) \right) + \omega^2 \kappa(x) p(x) = 0, \tag{2}$$

and in this case we want to reconstruct the compressibility $\kappa \in L^\infty(\Omega)$ and the density $\rho \in C^{1,1}(\overline{\Omega})$ from the knowledge of the associated Dirichlet-to-Neumann map $\Lambda_{\omega,\kappa,\rho}$ at two suitable frequencies ω. In my talk I will only consider the more general problem (2) to save doubling up on notation.

1.2 Overview of the reconstruction procedure

Substituting $p = \rho^{1/2} w$ in (2) leads to the Schrödinger equation

$$-\Delta w + qw = 0 \qquad \text{in } \Omega \tag{3}$$

with potential

$$q = \rho^{1/2} \nabla \rho^{-1/2} - \omega^2 \kappa \rho.$$

Letting Λ_q denote the Dirichlet-to-Neumann map associated with (3), we have

$$\Lambda_q = \rho^{1/2}\Lambda_{\omega,\kappa,\rho}\rho^{1/2} - \frac{1}{2}\rho^{-1}\frac{\partial\rho}{\partial\nu}. \tag{4}$$

Note that in this expression, other than $\Lambda_{\omega,\kappa,\rho}$, we only use the values of ρ and its normal derivative on the boundary of Ω.

Nachman's reconstruction method consists of four steps, summarised in [1, Theorem 5.2]. We let ω_1, ω_2 be two frequencies such that ω_1^2 and ω_2^2 are not Dirichlet eigenvalues of $-\kappa^{-1}\nabla\cdot(\rho^{-1}\nabla)$ in Ω, and we suppose that we are given the Dirichlet-to-Neumann maps $\Lambda_{\omega_j,\kappa,\rho}$, $k = 1, 2$.

Step 1: *Calculate the boundary values* $\rho|_{\partial\Omega}$ *and* $(\partial\rho/\partial\nu)|_{\partial\Omega}$ *from* $\Lambda_{\omega_1,\kappa,\rho}$. Once we have $\rho|_{\partial\Omega}$ and $(\partial\rho/\partial\nu)|_{\partial\Omega}$, equation (4) shows that we have the maps Λ_{q_j}. A sketch of this step of the reconstruction is given in the next section.

Step 2: *Compute the scattering transforms* $t_j = t_j(q_j)$ *from* Λ_{q_j}. These are given by

$$t_j(\xi,\zeta) := \int_{\mathbb{R}^n} e^{-ix\cdot(\xi+\zeta)}q_j(x)\psi_j(x,\zeta)\,dx$$

for $\xi \in \mathbb{R}^n$ and $\zeta \subset \mathbb{C}^n$ with $\zeta\cdot\zeta = 0$, where ψ are certain 'generalised eigenfunctions' of (3) (with potential $q = q_j$). The fact that these can be calculated in terms of Λ_{q_j} is contained in [1, Theorem 1.4]. Details are given in the previous talk.

Step 3: *Calculate* q_j *from* t_j. This is handled by [1, Theorem 3.4], which gives an explicit expression for the Fourier transform $\hat{q}_j$. Again, details for this step are contained in the previous talk.

Step 4: *Calculate* ρ *and* κ *from* q_1 *and* q_2. This is the easiest step of the procedure. Once we know q_1, q_2, and $\rho|_\Omega$, we can solve the Dirichlet problem

$$\Delta v - \tilde{q}v = 0 \quad \text{in } \Omega, \qquad v|_{\partial\Omega} = \rho^{-1/2}|_{\partial\Omega}, \tag{5}$$

with potential $\tilde{q} := (\omega_2^2 q_1 - \omega_1^2 q_2)/(\omega_2^2 - \omega_1^2)$. The assumptions guarantee that (5) has a unique solution, and this solution is $\rho^{-1/2}$. We can then find $\kappa = (q_1 - q_2)/(\rho(\omega_2^2 - \omega_1^2))$.

1.3 Reconstructing ρ on the boundary

Here we provide a brief sketch of Step 1 in the reconstruction procedure. Nachman considers the operator

$$L_{\gamma,\beta}u := \nabla\cdot(\gamma\nabla u) + \beta u$$

with corresponding Dirichlet-to-Neumann map $\Lambda_{\gamma,\beta}$. The problem is then to calculate $\gamma|_{\partial\Omega}$ and $(\partial\gamma/\partial\nu)|_{\partial\Omega}$ in terms of $\Lambda_{\gamma,\beta}$. The solution is the following theorem ([1, Theorem 4.3])

Theorem 1. *Suppose that Ω has a $C^{1,1}$ boundary and $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Assume $\gamma \in C^{1,1}(\overline{\Omega};\mathbb{R})$ has a positive lower bound, $\beta \in L^\infty(\Omega;\mathbb{R})$, and 0 is not a Dirichlet eigenvalue of $L_{\gamma,\beta}$ in Ω. Then*

$$\lim_{|\eta|\to\infty} \left\| 2e^{-i\langle\cdot,\eta\rangle} S_0 \Lambda_{\gamma,\beta} e^{i\langle\cdot,\eta\rangle} - \gamma \right\|_{L^2(\partial\Omega)} = 0 \tag{6}$$

and

$$\lim_{|\eta|\to\infty} \left\| 2e^{-i\langle\cdot,\eta\rangle} (\gamma\Lambda_{1,0} + \Lambda_{1,0}\gamma - 2\Lambda_{\gamma,\beta}) e^{i\langle\cdot,\eta\rangle} - \frac{\partial\gamma}{\partial\nu} \right\|_{L^2(\partial\Omega)} = 0. \tag{7}$$

This theorem makes use of the single layer potential

$$S_0 f(x) := \int_{\partial\Omega} G_0(x,y) f(y)\, d\sigma(y), \tag{8}$$

where G_0 is the classical Green's function

$$G_0(x,y) = \frac{1}{(n-2)\omega_n}|x|^{2-n}, \qquad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

The trace double layer potential,

$$B_0 f(x) := \text{p.v.} \int_{\partial\Omega} \frac{\partial G_0}{\partial\nu(y)}(x,y) f(y)\, d\sigma(y),$$

also plays a role.

The proof of (6) relies on the identity

$$S_0\Lambda_{\gamma,\beta} - \frac{1}{2}\gamma = B_0\gamma - R(\mathbf{G}_0\beta + \nabla\mathbf{G}_0 \cdot \nabla\gamma)P_{\gamma,\beta}, \tag{9}$$

where R is the trace map, $\mathbf{G}_0$ and $\nabla\mathbf{G}_0$ denote the integral operators with kernels G_0 and ∇G_0 respectively, and $P_{\gamma,\beta}$ is the solution operator for the Dirichlet problem for $L_{\gamma,\beta}$. This identity is proven by applying the divergence theorem to a regularised version of the integral expression for the left hand side. One applies (9) to the functions $e^{i\langle\cdot,\eta\rangle}$ for $\eta \in \mathbb{R}^n$, and shows that the right hand side disappears as $|\eta| \to \infty$. This uses compactness of B_0 on $L^2(\partial\Omega)$ (which is classical), boundedness of $P_{\gamma,\beta}$ from $H^{-1/2}(\partial\Omega)$ to $L^2(\partial\Omega)$ ([1, Lemma 4.2]), the trace theorem, and boundedness of $\nabla\mathbf{G}_0$ and $\mathbf{G}_0$ from $L^2(\Omega)$ to $H^1(\Omega)$ (which is related to [1, Lemma 2.11]) The limit (7) is proven similarly, but requires a bit more work.

1.4 Reconstruction from surface point source data

The reconstruction procedure for the acoustic equation can be used to solve a further reconstruction problem: that of computing ρ and κ from *surface point source data*. In this problem one is given the scattering solutions $P(x, y, \omega)$ with frequency ω for all $x, y \in \partial\Omega$, i.e. solutions to the equation

$$\nabla_x \cdot \left(\frac{1}{\rho(x)} \nabla_x P(x, y, \omega) \right) + \omega^2 \kappa(x) P(x, y, \omega) = -\delta(x - y) \qquad \text{in } \mathbb{R}^n,$$

with ρ and κ equal to known constants ρ_0, κ_0 outside of Ω. One can then compute ρ and κ from the surface data $P(\cdot, \cdot, \omega)$ for two 'admissible' frequencies [1, Theorem 1.1].

This is done by computing the Dirichlet-to-Neumann maps $\Lambda_{q_j - k_j^2}$ from the point source data, where

$$q_j(x) = \rho^{1/2}(x) \nabla \rho^{-1/2}(x) + \omega_j^2(\kappa_0 \rho_0 - \kappa(x)\rho(x)) \quad \text{and} \quad k_j^2 = \omega_j^2 \kappa_0 \rho_0$$

for $j = 1, 2$. In [1, Theorem 1.6] Nachman proves the formula

$$\Lambda_{q-k^2} = \Lambda_{-k^2} + \mathcal{S}_k^{-1} - S_k^{+\,-1}. \tag{10}$$

The single layer potentials S_k^+ and $\mathcal{S}_k$ are defined as in (8), but with kernels

$$G_k^+(x, y) := \frac{i}{4} \left(\frac{|k|}{2\pi|x - y|} \right)^{(n-2)/2} H^{(1)}_{(n-2)/2}(|k||x - y|)$$

(the outgoing Green's function for the Helmholtz equation) and $\mathcal{G}_k$, the integral kernel for the operator

$$(H - k^2 - i0)^{-1} := \lim_{\varepsilon \downarrow 0} (H - k^2 - i\varepsilon)^{-1},$$

with $H := -\Delta - q$. This limit exists as an operator between weighted L^2-spaces $L^2_\delta(\mathbb{R}^n) \to L^2_{-\delta}(\mathbb{R}^n)$, $\delta > 1/2$ [1, Proposition 6.1], and the kernel satisfies the integral equation

$$\mathcal{G}_k(x, y) = G_k(x, y)^+ - \int G_k^+(x, z) q(z) \mathcal{G}_k(z, y)\, dz.$$

Once the identity (10) is established, one can use it to recover the maps $\Lambda_{q_j - k_j^2}$, $j = 1, 2$, corresponding to two frequencies; all components of the right hand side of (10) are known, since the integral kernel of $\mathcal{S}_k$ can be written in terms of the surface point source data,

$$\mathcal{G}_k(x, y) = \rho_0^{-1} P(x, y, \omega).$$

Once one has the maps $\Lambda_{q_j - k^2}$ at hand, one can compute ρ and κ from the reconstruction procedure outlined in the last section.

References

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2 A property of measures in $\mathbb{R}^N$ and an application to unique continuation (after T.H. Wolff [1])

A summary written by Adolfo Arroyo-Rabasa

Abstract

The author discusses unique continuation results for functions which satisfy an elliptic differential inequality and where the governing side can be written in terms certain p-norms of the lower order terms. The proof is constructive, and based on a fundamental property of measures with rapid decay.

2.1 Introduction

We focus in unique continuation results when *the Laplacian* is bounded in terms of the function itself and its gradient, that is,

$$|\Delta u| \leq A|u| + B|\nabla u|,$$

where A, B are positive weights, each of which satisfies adequate local L^p-bounds. The precise statement is the following.

Theorem 1. *Suppose $d \geq 3$ and $\frac{1}{p} = \frac{1}{p'} + \frac{2}{d}$.*[1] *Let Ω be domain in $\mathbb{R}^d$ and assume $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ satisfies*

$$|\Delta u| \leq A|u| + B|\nabla u| \qquad \text{for some } A \in L^{d/2}_{\mathrm{loc}},\ B \in L^d_{\mathrm{loc}}\,. \tag{1}$$

Then if u vanishes on an open set it must vanish identically.

2.2 The difficulty of the problem

When the right-hand side of (1) only depends on u ($B = 0$), the unique continuation property follows from a classical Carleman estimate (see [2]). The fundamental principle of the proof goes as follows. Assume that u vanishes

[1] i.e., $p = 2d/(d+2)$ and $p' = 2d/(d-2)$

in the half-space $\{x_d < 0\}$ and let S_ρ be the strip $\{0 \le x_d < \rho\}$. Carleman's classical estimate and assumption (1) give

$$\begin{aligned}\|e^{-\lambda e_d\cdot x}u\|_{L^{p'}(S_\rho)} &\le \|e^{-\lambda e_d\cdot x}\Delta u\|_{L^p}\\ &\le \|A\|_{L^{d/2}(S_\rho)}\cdot\|e^{-\lambda e_d\cdot x}u\|_{L^{p'}(S_\rho)} + \|e^{-\lambda e_d\cdot x}\Delta u\|_{L^p(\mathbb{R}^d\setminus S_\rho)},\end{aligned}$$

which (for $\rho \ll 1$) means we obtain the bound

$$\|e^{-\lambda e_d\cdot(x-\rho)}u\|_{L^{p'}(S_\rho)} \lesssim \|\Delta u\|_{L^p(\mathbb{R}^d\setminus S_\rho)} \qquad \forall\, \lambda > 0. \tag{2}$$

This allows us to deduce $u \equiv 0$ in S_ρ, which is in turn equivalent to the unique continuation property.

Notice we only used the smallness of A on very thin ρ-stripes. To carry the same reasoning into the general case of (1) we would require a **uniform Carleman estimate for the gradient**. Unfortunately such bounds do not exist and hence we can only expect a bound of the form

$$\|e^{-\lambda e_d\cdot(x-\rho)}u\|_{L^{p'}(S_\rho)} \lesssim \|e^{-\lambda e_d\cdot(x-\rho)}\nabla u\|_{L^p(S_\rho)} + \|\Delta u\|_{L^p(\mathbb{R}^d\setminus S_\rho)} \tag{3}$$

for all $\lambda > 0$; which is clearly insufficient to deduce the strong continuation of u.

2.3 A weighted Carleman estimate for gradients

Wolff proposes a somewhat alternative approach which rests in the following **weak Carleman estimate** for differential forms in terms of their exterior and interior derivatives.

Lemma 2 (weak Carleman estimate)**.** *Let $|E| \ge |k|^{-1}$. Let ω be a differential form in $\mathbb{R}^d$ with compact support and $W^{1,p}$ coefficients, then*

$$\|e^{k\cdot x}\omega\|_{L^2(E)} \lesssim |k||E|^{\frac{1}{d}}\cdot\left(\|e^{k\cdot x}d\omega\|_{L^p} + \|e^{k\cdot x}d^*\omega\|_{L^p}\right).$$

In the case of 1-forms this yields a **weighted Carleman-estimate for gradients**. Indeed, since $d^2 = 0$ and $d^*d = \Delta$, then

$$\|e^{k\cdot x}\nabla u\|_{L^2(E)} \lesssim |k||E|^{\frac{1}{d}}\cdot\|e^{k\cdot x}\Delta u\|_{L^p}. \tag{4}$$

Notice it is impossible to simply plug-in this weaker inequality directly into the original argument. Indeed, the strength of (2) comes from the estimate being uniform with respect to λ whilst (4) worsens as $|k|$ tends to infinity. In the context of (2) this is equivalent to taking $\rho \searrow 0$, whence the proof fails.

2.4 The alternative argument of Wolff's proof

We have established that a *direct deductive proof* of the unique continuation property using (4) is incompatible with the spirit of the proof itself. This is because one needs to work with a **strictly larger** neighborhood of an open set where u vanishes. The author is well aware of this and instead he proposes a *proof by contradiction.* In turn, he avoids passing to the dangerous limit $k \to \infty$. The idea is to contradict a version of (4) for a particular vector k and a particlular set E. The proof is based in the following Ansatz.

Proposition 3 (Ansatz)**.** *Let u be as in the assumptions of Theorem 1. Assume u does not vanish identically in Ω and suppose that we can find $E \subset \Omega$ and $k \in \mathbb{R}^d$ with the following properties.*

1. *On the one hand, E is sufficiently* ***large*** *so that*
$$\|e^{k\cdot x}\big(A|u| + B|\nabla u|\big)\|_{L^p(E)} \;\gtrsim\; \|e^{k\cdot x}\big(A|u| + B|\nabla u|\big)\|_{L^p}\,, \tag{5}$$
2. *on the other hand E remains* ***small*** *in the sense that* [2]
$$|E| \;\lesssim\; \frac{1}{|k|^d}\,, \tag{6}$$
$$\|A\|_{L^{d/2}(E)} + \|B\|_{L^d(E)} \;\ll\; 1\,. \tag{7}$$

Then u vanishes in Ω.

Proof. By Hölder's inequality and the use of both the classical and the weighted Carleman estimates we deduce

$$\begin{aligned}
\|e^{k\cdot x}\big(A|u| + B|\nabla u|\big)\|_{L^p} &\overset{(4),(5)}{\lesssim} \Big[\|A\|_{L^{d/2}(E)} + (|k|\cdot|E_j|^{\frac{1}{d}})\|B\|_{L^d(E)}\Big] \\
&\qquad\cdot \|e^{k\cdot x}\Delta u\|_{L^p} \\
&\overset{(1)}{\leq} \Big[\|A\|_{L^{d/2}(E)} + (|k|\cdot|E_j|^{\frac{1}{d}})\|B\|_{L^d(E)}\Big] \\
&\qquad\cdot \|e^{k\cdot x}\big(A|u| + B|\nabla u|\big)\|_{L^p}
\end{aligned}$$

Hereby we conclude (recall that $u \neq 0$)

$$1 \overset{(6)}{\lesssim} \|A\|_{L^{d/2}(E)} \;+\; \|B\|_{L^d(E)} \overset{(7)}{\ll} 1 \quad\Rightarrow\quad \perp\,(\text{contradiction})\,.$$

□

[2]Notice that if the upper bound above holds, then (4) does behave as a **uniform** Carleman inequality.

2.5 The building blocks of the proof

Let us now briefly discuss the construction linking Theorem 1 to the Ansatz (Proposition 3). We emphasize the following angular stones:

1. a **reduction argument** of Theorem 1 to functions supported in a unitary ball, and
2. a **geometric property** of measures which have compact support.

The following result tells addresses the first point.

Lemma 4 (reduction argument)**.** *The conclusion of Theorem 1 is equivalent to the following statement. If* $u : \Omega \to \mathbb{R}$ *satisfies* (1) *and*

$$\Omega^{\complement} \subset B_{1/2}(-e_d) \tag{8}$$

$$\operatorname{supp} u \subset \overline{B_1(-e_d)}, \qquad 0 \in \operatorname{supp} u \tag{9}$$

$$u \in W^{2,p}_{\mathrm{loc}}(\Omega), \tag{10}$$

then $u \equiv 0$.

Proof. This lemma follows by applying a simple transformation which takes a ball where u vanishes into its polar (with respect to larger ball where u is no longer identically zero). □

The following result addresses the aforementioned geometrical property of measures; this is extracted from a stronger result (see Lemma 1 in [1]). Its proof relies on convex analysis methods.

Lemma 5 (geometric property)**.** *Suppose* μ *is a positive measure in* $\mathbb{R}^d$ *which has compact support. For* $k \in \mathbb{R}^d$ *define a measure* $d\mu_k(x) = e^{k\cdot x} d\mu(x)$*. Then for any* $k \in \mathbb{R}^d$ *there is a compact convex set* E_k *such that*

$$\mu_k(E_k) \geq \frac{1}{2}\|\mu_k\|\,.$$

Furthermore if $\mathcal{C} \subset \mathbb{R}^d$ *is compact convex then there is a pairwise disjoint sub-collection* $\{E_j\}, \{k_j\} \subset \mathcal{C}$ *such that*

$$\sum_j |E_j|^{-1} \geq C(d)|\mathcal{C}|\,.$$

2.5.1 Sketch of the construction

In light of Lemma 5 it suffices to show Proposition assuming (8)-(10). Let $u \in W^{2,p}(\mathbb{R}^d)$ satisfy (8)-(10). Our goal is to construct v satisfying (5)-(7).

Localization. Set K to be convex hull of supp $u \cap \{x : x_d \geq -\frac{1}{4}\}$. Choose a cut-off function ϕ of a small neighborhood of K supported on the larger strip $S(-\frac{1}{3}) := \{x : x_d \geq -\frac{1}{3}\}$ so that the corresponding norms of A and B are small:

$$\|A\|_{L^{d/2}(\text{supp}\phi)} + \|B\|_{L^d(\text{supp}\phi)} < \alpha \ll 1. \tag{11}$$

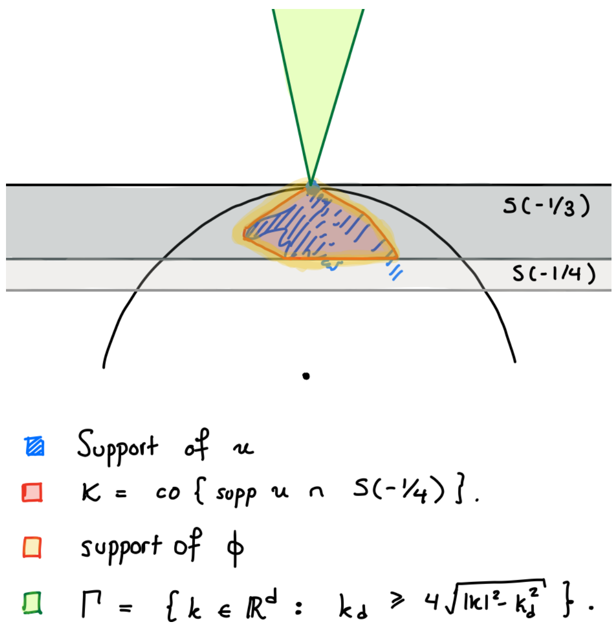

Figure 1: Elements of the construction

Localizing u as $v = \phi u$ we get the point-wise bound

$$|\Delta v| \leq A|v| + B|\nabla v| + B|u\nabla\phi| + \underbrace{|2\nabla\phi \cdot \nabla u + u\Delta\phi|}_{=:\chi}.$$

Hence the error term $\chi \in L^p$ is supported in the intersection union of the moon-piece $A_1 := D_1(-e_d) \cap S(-\frac{1}{3})$ and $A_2 := S(-\frac{1}{3}) \setminus S(-\frac{1}{4})$.

Bounding the error term. In analogy to Lemma 5, we define the positive measure

$$\mu := A|v| + B|\nabla v|,$$

which by construction is supported in $B_1(-e_d)$. Similarly, we set

$$\mu_k := e^{k\cdot x}\big(A|v| + B|\nabla v|\big)\,.$$

For k in the cone $\Gamma := \{k \in \mathbb{R}^d : k_d \geq 4\sqrt{|k|^2 - k_d^2}\}$ **of sufficiently large modulus**, we can bound the error term $e^{k\cdot x}\chi$ in terms of μ_k. More precisely,

$$\|e^{k\cdot x}\chi\|_{L^p} \leq \|\mu_k\|_{L^p} \qquad \text{whenever } k \in \Gamma, k \gg 1\,. \tag{12}$$

This estimate relies on a purely geometric argument and the fact that $0 \in \operatorname{supp} u$ (see Figure 1).

Conclusion. Let $M \gg 1$ so that (12) holds if $|k| \geq M$. We are now in possition to apply Lemma 1 to μ and $\mathcal{C} = B_{\varepsilon M}(-Me_d)$ (where ε is a small parameter) to find vectors $\{k_j\}$ and disjoint convex sets $\{E_j\}$ such that (after Hölder's inequality)

$$\|e^{k_j\cdot x}\big(A|v| + B|\nabla v|\big)\|_{L^p(E_j)} \ \geq\ 2^{-\frac{1}{p}}\,\|e^{k_j\cdot x}\big(A|v| + B|\nabla v|\big)\|_{L^p}\,,$$

and

$$\sum |E_j|^{-1} \ \gtrsim\ M^d \ \approx\ |k_j|^d.$$

These two inequalities and (12) suffice to guarantee (5)-(6); (7) holds provided that α in (11) is chosen to be sufficiently small. $\square$

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3 2D Unique Continuiation and Quasiconformal Maps - Talk 3: Strong unique continuation for general elliptic equations in 2D

A summary written by Constantin Bilz

Abstract

We prove the strong unique continuation property for two-dimensional elliptic operators in divergence form with lower order terms and bounded coefficients in the principal part. A key ingredient is a representation theorem due to Bers and Nirenberg [6] for solutions to Beltrami equations.

3.1 Result and strategy

Let $\Omega \subset \mathbb{R}^2$ be a connected open set. We consider the weak form of the elliptic equation

$$Lu = 0 \quad \text{in } \Omega, \tag{1}$$

where the operator L takes the form

$$Lu = -\operatorname{div}(A\nabla u + uB) + C \cdot \nabla u + du.$$

We assume the coefficients $A = (a_{ij})$, $B = (b_i)$, $C = (c_i)$ and d to be defined on all of $\mathbb{R}^2$.

The main result of the article under review is the following

Theorem 1. *Let $K \geq 1$, $q > 2$ and $\kappa > 0$. Assume that A is positive definite but possibly non-symmetric with $L^\infty(\mathbb{R}^2)$ entries and assume uniform ellipticity in the sense that*

$$A(x)\xi \cdot \xi \geq K^{-1}|\xi|^2$$
$$A^{-1}(x)\xi \cdot \xi \geq K^{-1}|\xi|^2$$

for every $\xi \in \mathbb{R}^2$ and almost every $x \in \mathbb{R}^2$. If furthermore $B \in L^q(\mathbb{R}^2; \mathbb{R}^2)$ and $d \in L^{q/2}(\mathbb{R}^2)$ with

$$\|B\|_{L^q(\mathbb{R}^2)} + \|C\|_{L^q(\mathbb{R}^2)} + \|d\|_{L^{q/2}(\mathbb{R}^2)} \leq \kappa,$$

then the operator L has the strong unique continuation property.

The strategy consists of three main steps. First, the reduction to the case of a pure divergence equation using two special multipliers. Then, the passage to a first order elliptic system of Beltrami type. Finally, the analysis of solutions to the Beltrami equations by using a representation theorem due to Bers and Nirenberg [6].

Similar proof strategies were used before in the predecessors [1] (see also [5]), [2] and [9], in each case using only one instead of two multipliers in the first step and hence obtaining results for more restricted classes of elliptic operators.

Regarding the last step, first applications of the mentioned representation theorem for Beltrami equations, which is Theorem 4 below, to unique continuation appeared in [6] and [7] where elliptic equations in non-divergence form were considered.

We will demonstrate the reduction to a pure divergence equation in Subsection 3.2. In Section 3.3 we will complete the proof of Theorem 1. In the talk we will also outline the proof of the representation theorem.

3.2 Reduction to a pure divergence equation

The following lemma is a slight variation of a gradient estimate first proved by N. G. Meyers.

Lemma 2 ([8, Theorem 1])**.** *There exists an exponent $p \in (2, q)$ only depending on K and q, and a radius $R_0 > 0$ only depending on K, q and κ such that given $R \leq R_0$ and functions $F \in L^p(B_R; \mathbb{R}^2)$ and $f \in L^s(B_R)$ with $\frac{1}{s} \leq \frac{1}{2} + \frac{1}{q}$, the equation*

$$Lu = -\operatorname{div} F + f$$

has a unique weak solution $u \in W_0^{1,p}(B_R)$ and we have for a constant C depending only on K, q and s:

$$\|\nabla u\|_{L^p(B_R)} \leq C\big(R^{2(\frac{1}{p}-\frac{1}{q})}\|F\|_{L^q(B_R)} + R^{2(\frac{1}{p}-\frac{1}{s})+1}\|f\|_{L^s(B_R)}\big)$$

and also

$$\|u\|_{L^\infty(B_R)} \leq CR^{1-\frac{2}{p}}\big(R^{2(\frac{1}{p}-\frac{1}{q})}\|F\|_{L^q(B_R)} + R^{2(\frac{1}{p}-\frac{1}{s})+1}\|f\|_{L^s(B_R)}\big).$$

We now let the first multiplier m be a weak solution to (1) in a ball B_{R_1} of radius $R_1 < R_0$ when the coefficient B is replaced by 0, i.e.

$$-\operatorname{div}(A\nabla m) + C \cdot \nabla m + dm = 0 \quad \text{in } B_{R_1}. \tag{2}$$

From Lemma 2 it can be seen that we can choose R_1 and m in such a way that

$$\frac{1}{2} \leq m \leq 2, \tag{3}$$

$$\|\nabla u\|_{L^p(B_{R_1})} \leq 1. \tag{4}$$

Now we can define an auxiliary elliptic operator $\tilde{L}$ by

$$\tilde{L}u = -\operatorname{div}(\tilde{A}\nabla u + u\tilde{B}) + \tilde{C} \cdot \nabla u$$

with coefficients $\tilde{A} = mA^T$, $\tilde{B} = mC - A\nabla m$ and $\tilde{C} = mB$. Similarly as L, the new operator $\tilde{L}$ again satisfies a uniform ellipticity condition and an L^p integrability condition for the lower order terms B and C.

Repeating the same procedure, we can choose a second multiplier w, a radius $R_2 < R_1$ and an exponent $t \in (2, p)$ only depending on K, κ and q such that w is a weak solution to

$$\tilde{L}w = 0 \quad \text{in } B_{R_2}, \tag{5}$$

and satisfies

$$\frac{1}{2} \leq w \leq 2, \tag{6}$$

$$\|\nabla w\|_{L^t(B_{R_2})} \leq 1. \tag{7}$$

This second multiplier w yields a second auxiliary operator

$$\hat{L} = -\operatorname{div}(\hat{A}\nabla u + u\hat{B})$$

with coefficients $\hat{A} = mwA$ and $\hat{B} = wA\nabla m + mwB - mA^T\nabla w - mwC$. Again we have uniform ellipticity for $\hat{L}$ and L^t integrability for B. Note that $\hat{L}$ is a pure divergence operator.

Now it can be verified by elementary calculations that the following reduction formula holds.

Proposition 3. *For any $v \in W^{1,2}(B_R)$ we have*

$$\hat{L}v = wL(mv)$$

in $W^{-1,2}(B_R)$.

3.3 A representation theorem for Beltrami equations

Let u be a weak solution to $Lu = 0$, and let $B_R \subset \Omega$ be a disk of radius $R < R_2$. We write

$$v = \frac{u}{m}$$

where m is the first multiplier defined in the previous subsection. By Proposition 3 we have

$$\hat{L}v = 0. \tag{8}$$

As $\hat{L}$ is a pure divergence operator, we can easily pass to a system of Beltrami type using a well known method. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By (8) there is a $\tilde{v} \in W^{1,t}(B_R)$ unique up to an additive constant such that $\nabla \tilde{v} = J(\hat{A}\nabla v + v\hat{B})$. We set $f = v + i\tilde{v}$ and can rewrite this equation in terms of the complex coordinate $z = x_1 + ix_2$ to obtain the Beltrami system

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} + \alpha f + \beta \overline{f} \quad \text{in } B_R \tag{9}$$

with coefficients μ, ν only depending on $\hat{A}$ and lower order coefficients α, β only depending on $\hat{A}$ and $\hat{B}$. It is easy to verify $|\mu| + |\nu| \leq k < 1$ in B_R.

This is where the previously mentioned representation theorem comes into play.

Theorem 4 ([6, p. 116]). *The function f has a representation of the form*

$$f = e^s F(\chi)$$

where $\chi : \mathbb{C} \to \mathbb{C}$ is a k-quasiconformal homeomorphism and η-H??lder continuous in B_R, $F : \chi(B_R) \to \mathbb{C}$ is holomorphic, and $s : B_R \to \mathbb{C}$ is η-Hölder continuous in B_R. Furthermore, χ^{-1} is η-Hölder continuous in $\chi(B_R)$ and the exponent η and the implied constants only depend on K, κ, and q.

By Hölder continuity of χ^{-1} we can now see that a nontrivial f may vanish only up to finite order. From this it is possible show that the same conclusion holds for u. Hence we finished a proof sketch of Theorem 1. We will outline a proof of Theorem 4 in the talk, but omit it here. Proofs can be found e.g. in [6] and [4].

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4 A nonlinear Plancherel theorem with applications to global well-posedness for the defocusing Davey-Stewartson equation and to the inverse boundary value problem of Calderón - Talk 1: Estimates for a $\bar{\partial}$-problem

Sections 2, 3 up to Lemma 3.7

A summary written by Gianmarco Brocchi

Abstract

We study solvability of the inhomogeneous problem $L_q u = f$, where $L_q u := \bar{\partial} u + q\bar{u}$, and $q \in L^2$. The authors prove new pointwise bounds for fractional integrals and pseudo-differential operators with non-smooth symbols, as well as new estimates for pointwise multiplier in negative Besov spaces.

4.1 Introduction

Consider the problem

$$\bar{\partial} u + q\bar{u} = f. \tag{1}$$

Indicate with L_q the operator $\bar{\partial} + q\,\bar{\cdot}$. We want to study the inverse operator L_q^{-1}, particularly the dependence on q of the operator norm $\|L_q^{-1}\|$.

Let $s \in [0,1)$. The operator $\bar{\partial} \colon \dot{H}^s(\mathbb{C}) \to \dot{H}^{s-1}(\mathbb{C})$, as well as the multiplication by q for $q \in L^{1/s}(\mathbb{C})$. When $s = \frac{1}{2}$, $q \in L^2$ and we have

$$L_q \colon \dot{H}^{\frac{1}{2}}(\mathbb{C}) \to \dot{H}^{-\frac{1}{2}}(\mathbb{C}).$$

Our aim is to prove the following theorem.

Theorem 1. *Given $q \in L^2$, for every $f \in \dot{H}^{-\frac{1}{2}}$ there exists an unique solution $u = L_q^{-1} f$ to the problem (1), obeying the following bound:*

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|L_q^{-1}\| \, \|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

Moreover, the operator norm only depends on the L^2-norm of q:

$$\|L_q^{-1}\| \lesssim C(\|q\|_2).$$

Solutions of equation (1) are related to the Scattering Transform used in [1] to study the defocusing Davey-Stewartson II equation. New bounds on $\bar{\partial}^{-1}$ and on pointwise multiplication are needed to apply Inverse-Scattering methods in this settings.

4.2 New bounds on fractional integrals

In the following, Mf is the Hardy-Littlewood maximal function.

Theorem 2. *Let $\alpha \in (0, d)$, and $p \in (1, 2]$. For any $f \in L^p(\mathbb{R}^d)$ we have:*

a) $|(-\Delta)^{-\frac{\alpha}{2}} f(x)| \lesssim_{d,\alpha} \lambda^{d-\alpha} M\hat{f}(0) + \lambda^{-\alpha} Mf(x)$ *for any* $\lambda > 0$*;*

b) $|(-\Delta)^{-\frac{\alpha}{2}} f(x)| \lesssim_{d,\alpha} \Big(M\hat{f}(0)\Big)^{\frac{\alpha}{d}} \Big(Mf(x)\Big)^{1-\frac{\alpha}{d}}.$

In order to apply the result to the Scattering transform, we rewrite point $b)$ using $e^{iy\xi} f(y)$ as function of y in place of f. Then

$$\text{b)}\ \ |(-\Delta)^{-\frac{\alpha}{2}}\big(e^{iy\xi} f(y)\big)(x)| \lesssim_{d,\alpha} \Big(M\hat{f}(\xi)\Big)^{\frac{\alpha}{d}} \Big(Mf(x)\Big)^{1-\frac{\alpha}{d}}.$$

We are mainly interested in the case $d = 2$, $\alpha = 1$.

Corollary 3. *For $q \in L^2(\mathbb{C})$ we have*

b') $|\bar{\partial}^{-1}(e^{-i(zk+\overline{zk})} q(z))(x)| \lesssim \Big(M\hat{q}(k)\Big)^{\frac{1}{2}} \Big(Mq(x)\Big)^{\frac{1}{2}}$

c) $\|\bar{\partial}^{-1}(e^{-i(zk+\overline{zk})} q(z))\|_{L^4} \lesssim \|q\|_{L^2}^{\frac{1}{2}} \Big(M\hat{q}(k)\Big)^{\frac{1}{2}}.$

We use Theorem 2 to show L^2-boundness for a class of pseudo-differential operators (PDOs) with non-smooth symbols.

Theorem 4. *Let $\alpha \in [0, d)$, and $a(x, \xi)$ be a symbol on $\mathbb{R}^d \times \mathbb{R}^d$ such that*

i) $a \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d \times \mathbb{R}^d)$*, and*

ii) $\|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x, \xi)\|_{L_\xi^{\frac{2d}{d+\alpha}}} \in L_x^{\frac{2d}{d-\alpha}}$

then the pseudo-differential operator

$$a(x,D)f(x) := \int_{\mathbb{R}^d} e^{ix\xi} a(x,\xi)\hat{f}(\xi)\frac{d\xi}{(2\pi)^d}$$

is bounded on L^2. *We have the following bounds:*

$$\|a(x,D)f\|_{L^2} \lesssim_{\alpha,d} \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi)\|_{L_x^{\frac{2d}{d-\alpha}} L_\xi^{\frac{2d}{d+\alpha}}} \|f\|_{L^2}$$

$$|a(x,D)f(x)| \lesssim_{\alpha,d} (Mf(x))^{\frac{\alpha}{d}} \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\cdot)\|_{L_\xi^{\frac{2d}{d+\alpha}}} \|f\|_{L^2}^{1-\frac{\alpha}{d}} \quad \textit{for a.e. } x \in \mathbb{R}^d.$$

4.3 Estimates on pointwise multiplier

By the Sobolev embedding $\dot{H}^r(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$, with $p^* = \frac{2d}{d-2r}$. We embed the dual space $L^{(p^*)'}(\mathbb{R}^d) \hookrightarrow \dot{H}^{-r}(\mathbb{R}^d)$. To show the continuity of the map

$$\begin{gathered} \dot{H}^r(\mathbb{R}^d) \to \dot{H}^{-r}(\mathbb{R}^d) \\ u \mapsto qu \end{gathered}$$

it is enough to prove that it maps continuously L^{p^*} into its dual. This follows from the embeddings above and Hölder's inequality:

$$\|qu\|_{\dot{H}^{-r}} \lesssim \|qu\|_{(p^*)'} \le \|q\|_p \|u\|_{p^*} \lesssim \|q\|_p \|u\|_{\dot{H}^r}. \tag{2}$$

It gives $q \in L^p$, with $p = d/2r$. In our case ($\mathbb{C} \cong \mathbb{R}^2$) from (2) we have

$$\|qu\|_{\dot{H}^{-r}(\mathbb{R}^2)} \lesssim \|q\|_{L^{\frac{1}{r}}} \|u\|_{\dot{H}^r(\mathbb{R}^2)}.$$

We can improve the above estimate, trading regularity with integrability, by putting q in a negative homogeneous Besov space. The norm of $\dot{B}_q^{s,p}$ is

$$\|f\|_{\dot{B}_q^{s,p}} = \left\| 2^{sk} \|P_k f\|_{L^p} \right\|_{\ell^q}$$

where P_k is the Littlewood-Paley projector. We have the following theorem.

Theorem 5. *Let* $r \in [0,1)$ *and* $\max\left\{2, \frac{d}{r}\right\} \le p < \frac{d}{2r}$. *Then*

$$\|qu\|_{\dot{H}^{-r}(\mathbb{R}^d)} \lesssim \|q\|_{\dot{B}_{p,\infty}^{\frac{d}{p}-2r}} \|u\|_{\dot{H}^r(\mathbb{R}^d)}.$$

Note: The space $\dot{B}^{\frac{d}{p}-2r}_{p,q}$ has the same scaling of $L^{\frac{d}{2r}}$, but negative regularity.

Sketch of the proof. Write

$$qu = \sum_{(k_1,k_2,k_3)\in\mathcal{A}} P_{k_1}\left((P_{k_2}q)(P_{k_3}u)\right),$$

where P_k is the frequency projector to $A_k = \{2^{j-1} < |\xi| < 2^{j+1}\}$, and

$$\mathcal{A} = \{(k_1,k_2,k_3) \in \mathbb{Z}^3 \,:\, A_{k_1} \cap (A_{k_2} + A_{k_3}) \neq 0\}.$$

Estimate $\|qu\|_{\dot{H}^{-r}}$ using Bernstein inequalities and Littlewood-Paley trichotomy. □

4.4 A $\bar{\partial}$-problem

In order to prove Theorem 1, we have to show that, for $q \in L^2(\mathbb{C})$, L_q is invertible from $\dot{H}^{\frac{1}{2}}(\mathbb{C})$ to $\dot{H}^{-\frac{1}{2}}(\mathbb{C})$. We recall two known bounds on $\bar{\partial}^{-1}$:

Lemma 6. i) *Let $p \in (1,2)$, and $1/p^* = 1/p - 1/2$. For $h \in L^p$ we have*

$$\|\bar{\partial}^{-1}h\|_{L^{p*}} \lesssim_p \|h\|_{L^p} \tag{3}$$

ii) *Let $1 < p_1 < 2 < p_2$ and $f \in L^{p_1} \cap L^{p_2}$, then*

$$\|\bar{\partial}^{-1}f\|_\infty \lesssim_{p_1,p_2} \|f\|_{L^{p_1}} + \|f\|_{L^{p_2}}.$$

The inverse operator L_q^{-1} is well defined from $L^{\frac{4}{3}}$ to L^4.

Lemma 7. *Let $q \in L^2(\mathbb{C})$. Then $L_q u = f$ has an unique solution for $f \in L^{\frac{4}{3}}$. In particular, the operator $L_q \colon L^4 \to L^{\frac{4}{3}}$ is invertible.*

Idea. By the previous result, $\bar{\partial}^{-1} \colon L^{\frac{4}{3}} \to L^4$ continuously. We write

$$L_q = \bar{\partial}(I + \bar{\partial}^{-1}(q\bar{\cdot})) =: \bar{\partial} \circ \mathcal{B}.$$

Then it is enough to show the existence of an unique solution to $\mathcal{B}u = \bar{\partial}^{-1}f$ for $f \in L^{\frac{4}{3}}$. In other words, if $\mathcal{B} \colon L^4 \to L^4$ is invertible, the unique solution to $L_q u = f$ is given by $u = \mathcal{B}^{-1}\bar{\partial}^{-1}f$.

Proof. Since the operator $\bar{\partial}^{-1}(q\bar{\cdot})$ is compact from L^4 to itself, the operator $\mathcal{B} := (I+\bar{\partial}^{-1}(q\bar{\cdot}))$ is Fredholm, in particular $\mathcal{B}$ is injective iff is surjective.We prove that $\mathcal{B}$ is injective. Let $u \in L^4$ such that $\mathcal{B}u = 0$, i.e. $\bar{\partial}u = -q\bar{u}$. Write $q = q_n + q_s$, where q_s has small L^2-norm to be determined. We can choose $\nu \in L^\infty$ such that (!) holds[3] in the following

$$\bar{\partial}(u\nu) = (\bar{\partial}u)\nu + u\bar{\partial}\nu \overset{(!)}{=} (\bar{\partial}u + q_n\bar{u})\nu \overset{(*)}{=} (-q_s\bar{u})\nu$$

where $(*)$ holds since $\bar{\partial}u = -q\bar{u}$. Then, using bound (3) on $\bar{\partial}^{-1}$, we have

$$\|u\nu\|_{L^4} \leq c\|\bar{\partial}(u\nu)\|_{L^{\frac{4}{3}}} = c\|q_s\bar{u}\nu\|_{L^{\frac{4}{3}}} \leq c\|q_s\|_{L^2}\|u\nu\|_{L^4} \leq \frac{1}{2}\|u\nu\|_{L^4}$$

where in the last inequality we chose q_s with $\|q_s\|_{L^2} \leq (2c)^{-1}$. This shows that, if $\mathcal{B}(u\nu) = 0$, $u\nu = 0$, so $\ker(\mathcal{B}) = \{0\}$. $\square$

The same result holds when we consider L_q^{-1} on Sobolev spaces.

Lemma 8. *For $q \in L^2(\mathbb{C})$ the operator $L_q\colon \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$ is invertible and*

$$\|L_q^{-1}f\|_{\dot{H}^{\frac{1}{2}}} \leq C(q)\|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

We now study the dependence of L_q^{-1} and $C(q)$ on q.

Lemma 9. *The constant $C(q)$ has a local Lipschitz dependence on q. Given $q_0 \in L^2$, there exists $\epsilon > 0$ such that for every $q_1, q_2 \in B(q_0, \epsilon)$.*

$$\begin{aligned} \|L_{q_1}^{-1} - L_{q_2}^{-1}\| &\lesssim C(q_0)^2\|q_1 - q_2\|_{L^2} \\ |C(q_1) - C(q_2)| &\lesssim C(q_0)^2\|q_1 - q_2\|_{L^2}. \end{aligned}$$

It remains to prove that the bound on $C(q)$ is uniform for q in a bounded set in L^2. Denote with

$$C(R) := \sup\{C(q) \ : \ \|q\|_2 \leq R\}, \qquad C\colon \mathbb{R}_+ \to [0, \infty].$$

[3] Choose ν as solution of the equation

$$\bar{\partial}\nu = q_n\frac{\bar{u}}{u}\nu.$$

The previous lemma, taking $q_0 = 0$, showed that $C(R)$ is finite for small R. We shall prove that it is finite for all $R > 0$. Argue by contradiction: let

$$R_0 := \inf\{R \in \mathbb{R}_+ \,:\, C(R) = +\infty\}.$$

Then $\lim_{R\to R_0} C(R) = +\infty$, and there exists a sequence $\{q_n\}_{n\in\mathbb{N}} \subset B_{R_0}$ such that $\|q_n\|_2 \to R_0$, with $\|L_{q_n}^{-1}\| \overset{n\to\infty}{\longrightarrow} +\infty$. If we were able to extract a convergent subsequence $\{q_{n_k}\}$ we would have

$$q_{n_k} \overset{L^2}{\to} q, \text{ and } \|L_{q_k}^{-1}\| \overset{k\to\infty}{\longrightarrow} \|L_q^{-1}\| < +\infty$$

leading to a contradiction, since R_0 was minimal. Unfortunately, we cannot hope to extract a subsequence converging in L^2.

Symmetries: obstruction to compactness Translations and dilations are symmetries of the problem that preserve the L^2-norm. Indicate with $S(\lambda, y)q(x) = \lambda q(\lambda(x-y))$. One has

$$C(q) = C(S(\lambda, y)q).$$

To prove finiteness of $C(R)$, it would suffices to show *compactness up to symmetries* of $\{q_n\}$ in a *weaker* topology.

Definition 10. *The sequence $\{q_n\}$ is compact up to symmetries if there exists a sequence $\{(\lambda_n, y_n)\}_{n\in\mathbb{N}}$ such that $\{S(\lambda_n, y_n)q_n\}$ is compact.*

Using Theorem 5, we can extend the result of Lemma 9 to a larger space: the Besov space $\dot{B}_{\infty}^{-\frac{1}{3},3}$.

Lemma 11. *Given $q_0 \in L^2$, there exists $\epsilon = \epsilon(C(q_0)) > 0$ such that for $q_1, q_2 \in \{q \,:\, \|q - q_0\|_{B_\infty^{-\frac{1}{3},3}} < \epsilon\}$ we have*

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\| \lesssim C(q_0)^2 \|q_1 - q_2\|_{B_\infty^{-\frac{1}{3},3}}$$
$$|C(q_1) - C(q_2)| \lesssim C(q_0)^2 \|q_1 - q_2\|_{B_\infty^{-\frac{1}{3},3}}.$$

Using profile decomposition we can split $\{q_n\}$ in different pieces driven by different symmetries and conclude the proof.

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5 L^2 Carleman estimates - Talk 1: Carleman Inequalities

A summary written by Daniel Campos

Abstract

We present Carleman's method of weighted L^2-inequalities and applications to Cauchy uniqueness and unique continuation problems. These kind of estimates are based essentially in a lower bound for the commutator of the symmetric and skew-symmetric parts of the conjugated operator.

5.1 Introduction

The Cauchy initial value problem asks for a solution to a differential equation with prescribed data over a hypersurface. For linear equations, the theorems of Cauchy-Kovalevskaya (1875) and Holmgren (1901) concern the existence and uniqueness of solutions under analyticity conditions on the coefficients of the equation and the Cauchy data. The method of weighted inequalities, introduced by Carleman (1939), allowed for the first time to remove the analyticity condition on the coefficients.

Let $\Omega \subset \mathbb{R}^n$ be an open connected set, let $D_j = i^{-1}\partial/\partial x_j$, and consider the linear differential operator of order m, $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$. We define its principal part by $P_m(x, D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$, and its principal symbol by $p_m(x, \xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. We will be interested in properties of functions satisfying differential inequalities of the form

$$|P(x, D)u| \leq \sum_{0\leq j<m} V_j(x)|\nabla^j u(x)|, \quad V_j \in L^\infty_{loc}(\Omega). \tag{1}$$

Definition 1. *The operator P has the unique continuation property if every $u \in H^m_{loc}$ that satisfies the inequality (1) pointwise a.e. and vanishes in an open set of Ω must vanish identically.*

Let Σ be a hypersurface of Ω given by the equation $\rho(x) = 0$, for a $C^1(\Omega)$ real-valued function ρ with $\nabla\rho \neq 0$ on Σ.

Definition 2. *The operator P has Cauchy uniqueness across Σ if every $u \in H^m_{loc}$ that satisfies the inequality (1) pointwise a.e. and vanishes in the set $\{\rho < 0\}$ must vanish in a neighborhood of Σ.*

Definition 3. *The characteristics of P are simple with respect to Σ if whenever $p_m(x, \xi - i\lambda\nabla\rho) = 0$ we have that $\nabla_\xi p_m(x, \xi - i\lambda\nabla\rho) \cdot \nabla_x \rho(x) \neq 0$, for $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$.*

Under this condition, Euler's identity for homogeneous polynomials implies that $p_m(x, \nabla\rho) \neq 0$, i.e. the hypersurface is non-singular with respect to P. The following theorems will be the guiding motivation of our presentation.

Theorem 4. *(Calderón, 1958) Let $\Omega \subset \mathbb{R}^n$ be an open set and let P be a differential operator of order m with real smooth coefficients in the principal part and L^∞_{loc} complex-valued for the lower order terms. If Σ is a C^1 hypersurface as before and the characteristics of P are simple with respect to Σ, then P has Cauchy uniqueness across Σ.*

Theorem 5. *Let P be a second-order elliptic operator with real smooth coefficients in the principal part and L^∞_{loc} complex valued for the lower order terms. Then the characteristics of P with respect any hypersurface are simple and thus has the unique continuation property.*

5.2 Example of Carleman's method

Consider the following example of an elliptic equation in $\mathbb{R}^2$: let $u \in C^1$ solve

$$(\partial_x + i\partial_y)u = a(x,y)u, \quad u|_{y<0} = 0,$$

with $a \in L^\infty$. We will show that u has to vanish in a neighborhood of the x-axis. This is ensured by Holmgren's theorem if a is analytic, but that may not be the case in our problem. Carleman's idea was to construct a special real-valued weight ϕ such that for a large parameter $\lambda > 0$ we have an inequality of the form

$$\|e^{-\lambda\phi}(\partial_x + i\partial_y)v\| \geq C\|e^{-\lambda\phi}v\|, \tag{2}$$

valid for any $v \in C^1_c$. The purpose of this is to compete with the differential inequality $|(\partial_x + i\partial_y)u| \leq C|u|$ obtained from the equation. To determine a more precise form for it we rewrite it as $\|e^{-\lambda\phi}(\partial_x + i\partial_y)e^{\lambda\phi}w\| \geq C\|w\|$ and observe that the conjugated operator equals

$$e^{-\lambda\phi}(\partial_x + i\partial_y)e^{\lambda\phi}w = (\lambda\partial_x\phi + i\partial_y)w + i(\lambda\partial_y\phi - i\partial_x)w =: Aw + iBw,$$

where A and B are formally self-adjoint operators. To exploit this decomposition it may be convenient to consider L^2 estimates, so that we obtain

$$\begin{aligned}\|(A+iB)w\|_{L^2}^2 &= \|Aw\|_{L^2}^2 + \|Bw\|_{L^2}^2 + \langle Aw, iBw\rangle + \langle iBw, Aw\rangle \\ &= \|Aw\|_{L^2}^2 + \|Bw\|_{L^2}^2 + i\langle [A,B]w, w\rangle \geq i\langle [A,B]w, w\rangle.\end{aligned}$$

A simple computation yields $i\langle [A,B]w, w\rangle = \lambda\langle(-\Delta\phi)w, w\rangle$. With a choice of ϕ that ensures $-\Delta\phi \geq c > 0$, we see that (2) could take the form

$$\|e^{-\lambda\phi}(\partial_x + i\partial_y)v\|_{L^2} \geq C\lambda^{1/2}\|e^{-\lambda\phi}v\|_{L^2}. \tag{3}$$

The fact that the constant in the smaller side increases with λ is fundamental and will allow to absorb smaller terms from the other part.

Let us show that such an estimate and a convexity property of the weight imply that the solution to the equation must vanish in a neighborhood of the origin. The translation invariance of the problem then gives the same result for any point on the x-axis, thus yielding the Cauchy uniqueness. Let χ be a smooth cutoff function with $\chi \equiv 1$ if $|(x,y)| \leq 1$ and $\chi \equiv 0$ if $|(x,y)| \geq R$, for some $R > 1$ to be determined. From (3) and the equation it follows that

$$\begin{aligned}C\lambda^{1/2}\|e^{-\lambda\phi}\chi u\|_{L^2} &\leq \|e^{-\lambda\phi}(\partial_x + i\partial_y)(\chi u)\|_{L^2} \\ &\leq \|e^{-\lambda\phi}(\partial_x\chi + i\partial_y\chi)u\|_{L^2} + \|e^{-\lambda\phi}\chi a u\|_{L^2}.\end{aligned}$$

The boundedness of a allows to absorb the second term into the left-hand side for large λ, so that we obtain

$$C\lambda^{1/2}\|e^{-\lambda\phi}\chi u\|_{L^2} \leq \|e^{-\lambda\phi}(\partial_x\chi + i\partial_y\chi)u\|_{L^2},$$

for a new constant C and λ large (say $C\lambda^{1/2} > 2\|a\|_{L^\infty}$). We are interested in the weight ϕ satisfying the convexity properties $-\Delta\phi \geq c > 0$ and $\phi \geq c_1 > 0$ on $U := \operatorname{supp}[(\partial_x\chi + i\partial_y\chi)u] \subset \{1 \leq |(x,y)| \leq R, y \geq 0\}$. If we bound $\phi \leq c_2 \leq c_1$ in a small ball $B = \{|(x,y)| \leq r\}$ near the origin, then this would give $C\lambda^{1/2}e^{-\lambda c_2}\|u\|_{L^2(B)} \leq e^{-\lambda c_1}\|\chi\|_{C^1}\|u\|_{L^2}$, which yields that u must vanish in B. For the choice of weight

$$\phi(x,y) = y - \frac{y^2}{2} + \alpha x^2,$$

it is not difficult to choose the appropriate parameters $R > 1 > r$ and $\alpha > 0$.

5.3 Derivation of a Carleman estimate

5.3.1 Conjugation identities

Above we saw that the main step in proving Cauchy uniqueness was a Carleman estimate. Since we will consider operators with L^∞_{loc} coefficients for the lower part, the analogs of (3) we are interested in have the form

$$\|e^{-\lambda\phi}P_m v\|_{L^2} \geq C_K \sum_{|\alpha|<m} \lambda^{m-|\alpha|-1/2}\|e^{-\lambda\phi}D^\alpha v\|_{L^2}, \tag{4}$$

for $\lambda \geq \lambda_0 \geq 1$ and $v \in C_c^\infty(K)$, with compact $K \subset \Omega$. As in the previous section we conjugate this expression. We have that $e^{-\lambda\phi}D_j e^{\lambda\phi} = D_j - i\lambda\partial_j\phi$. It can be useful to think of the parameter λ as having the same "weight" as a derivative. In the expansion of $e^{-\lambda\phi}D^\alpha e^{\lambda\phi}w = (D - i\lambda\nabla\phi)^\alpha w$ we observe that the function w is hit every time by at most one derivative or one factor of λ. Using the commutator relation $[D, i\lambda\Phi] = \lambda\nabla\Phi$ for any function Φ, and the conjugation identity $e^{-\lambda\phi}P_m(x,D)e^{\lambda\phi} = P_m(x, D - i\lambda\nabla\phi)$, we can show that (4) is equivalent to

$$\|P_m(x, D - i\lambda\nabla\phi)w\|_{L^2} \geq C_K\lambda^{1/2}\|w\|_{H^{m-1}_\lambda}, \tag{5}$$

where the Sobolev space H^k_λ is defined as the space of functions that satisfy

$$\|f\|_{H^k_\lambda} := \left(\int_{\mathbb{R}^n} (\lambda^2 + |\xi|^2)^k |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} < \infty.$$

Definition 6. *The class of symbols of order k, denoted by S^k, is the set of smooth functions $a(x,\xi,\lambda) : \Omega \times \mathbb{R}^n \times [1,+\infty)$ which are polynomials, in ξ and λ, of degree less than or equal to k.*

The symbol of the operator $P_m(x, D-i\lambda\nabla\phi)$ is $p_m(x, \xi-i\lambda\nabla\phi)+r_{m-1}(x,\xi,\lambda)$, with p_m the principal symbol, as in the introduction, and $r_{m-1} \in S^{m-1}$. The following proposition allows us to disregard the lower order terms.

Proposition 7. *If a is in S^k, then $\|op(a)w\|_{L^2} = O(\|w\|_{H^k_\lambda})$.*

This gives that $\|P_m(x, D - i\lambda\nabla\phi)w\|^2_{L^2} \geq C\|Q_m w\|^2_{L^2} + O(\|w\|_{H^{m-1}_\lambda})$, where Q_m is the operator with homogeneous symbol $q_m(x,\xi,\lambda) := p_m(x, \xi - i\lambda\nabla\phi)$.

5.3.2 Computations

As in the example we write $Q_m = A_m + iB_m$, where $A = (Q_m + Q_m^*)/2$ and $B = (Q_m - Q_m^*)/2i$ are formally self-adjoint operators, to obtain

$$\|Q_m w\|_{L^2}^2 = \|A_m w\|_{L^2}^2 + \|B_m w\|_{L^2}^2 + i\langle [A_m, B_m]w, w\rangle.$$

The symbol of the adjoint operator Q_m^* is $\overline{q_m} + S^{m-1}$, and so the principal symbols of A_m and B_m are $a_m := \mathrm{Re}\ q_m$ and $b_m := \mathrm{Im}\ q_m$, respectively. We will later use that these are real-valued. From Proposition 7 we obtain that

$$\|Q_m w\|_{L^2}^2 \geq C(\|\mathrm{op}(a_m)w\|_{L^2}^2 + \|\mathrm{op}(b_m)w\|_{L^2}^2) + i\langle [A_m, B_m]w, w\rangle + O(\|w\|_{H_\lambda^{m-1}}^2).$$

The following two propositions allow us to deal with the previous expression.

Proposition 8. *If $a \in S^j$ and $b \in S^k$, then there exists a symbol $c \in S^{j+k}$ such that $op(a)op(b) = op(c)$, and we have the expansion*

$$c = ab + i^{-1}\nabla_\xi a \cdot \nabla_x b + S^{j+k-2}.$$

In particular, the symbol of the commutator $[op(a), op(b)]$ equals

$$i^{-1}\{a, b\} + S^{j+k-2} := i^{-1}(\nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b) + S^{j+k-2}.$$

Proposition 9. *If q is in S^{2k}, then $|\langle op(q)w, w\rangle| = O(\|w\|_{H_\lambda^k}^2)$.*

These propositions and the self-adjointness of $i[A, B]$ yield

$$\begin{aligned} i\langle [A_m, B_m]w, w\rangle &= \mathrm{Re}\ i\langle [A_m, B_m]w, w\rangle \\ &= \mathrm{Re}\ i\langle [\mathrm{op}(a_m), \mathrm{op}(b_m)]w, w\rangle + O(\|w\|_{H_\lambda^{m-1}}^2) \\ &= \mathrm{Re}\ \langle c_{2m-1,\phi} w, w\rangle + O(\|w\|_{H_\lambda^{m-1}}^2), \end{aligned}$$

where $c_{2m-1,\phi}$ is a homogeneous polynomial symbol in S^{2m-1} that equals

$$\begin{aligned} \{a_m, b_m\} &= \frac{1}{2i}\{\overline{q_m}, q_m\} = \mathrm{Im}\ (\overline{\nabla_\xi q_m} \cdot \nabla_x q_m) \\ &= \mathrm{Im}\ (\overline{\nabla_\xi p_m(x,\zeta)} \cdot \nabla_x p_m(x,\zeta)) - \lambda(\nabla^2\phi \overline{\nabla_\xi p_m(x,\zeta)}) \cdot \nabla_\xi p_m(x,\zeta), \end{aligned}$$

where $\zeta := \xi - i\lambda\nabla\phi$ and $\nabla^2\phi$ is the Hessian matrix of ϕ. The symmetry of the Hessian implies that c_{2m-1} is real-valued if ϕ is.

5.3.3 Carleman estimate and Cauchy uniqueness

In the past two subsections we proved that

$$O(\|w\|^2_{H^{m-1}_\lambda}) + \|P_m(x,\zeta)\|^2_{L^2} \geq C_1(\|\mathrm{op}(a_m)w\|^2_{L^2} + \|\mathrm{op}(b_m)w\|^2_{L^2}) + C_2 \mathrm{Re}\langle \mathrm{op}(c_{2m-1,\phi})w, w\rangle,$$

for some constants $C_1 < C_2 < 1$. Analogous results to Propositions 7, 8, 9 for pseudodifferential operators give that if $\lambda \geq \mu > 0$ then

$$\begin{aligned}\|\mathrm{op}(a_m)w\|^2_{L^2} &\geq \mu\|\mathrm{op}(\lambda^2+|\xi|^2)^{-1/4}\mathrm{op}(a_m)w\|^2_{L^2} \\ &\geq \mathrm{Re}\langle \mathrm{op}[C\mu(\lambda^2+|\xi|^2)^{-1/2}a_m^2]w, w\rangle + \mu O(\|w\|^2_{H^{m-3/2}_\lambda}).\end{aligned}$$

We are left to bound the homogeneous symbol $C\mu(\lambda^2+|\xi|^2)^{-1/2}|q_m|^2+c_{2m-1,\phi}$. We carry out the rest of the proof in a simple setting. Denote the points in $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$ by $x=(y,t)$, and assume that the height function is given by $\rho(y,t) = t$ with hypersurface $\Sigma = \{\rho = 0\}$. Denote the dual variables $\xi = (\eta,\tau) \in \mathbb{R}^{n-1}\times\mathbb{R} = \mathbb{R}^n$. As before, we will look for a weight of the form

$$\phi(y,t) = t - \frac{\mu t^2}{2} + \frac{|y|^2}{2\mu},$$

with $\mu > 0$ a large parameter to be fixed. With this choice we obtain that $\zeta = \xi - i\lambda\nabla\phi = (\eta - i\lambda y/\mu, \tau - i\lambda(1-2t))$ and

$$c_{2m-1,\phi} = \mathrm{Im}\,(\overline{\nabla_\xi p_m(x,\zeta)}\cdot\nabla_x p_m(x,\zeta)) - \frac{\lambda}{\mu}|\nabla_\eta p_m(x,\zeta)|^2 + \lambda\mu|\partial_\tau p_m(x,\zeta)|^2.$$

The last term has the largest coefficient in the previous expression. The condition on the characteristics being simple prevents $p_m(x,\zeta)$ and $\partial_\tau p_m(x,\zeta)$ from vanishing simultaneously. This allows to bound the symbol we want.

Lemma 10. *Let p_m, $c_{2m-1,\phi}$, and Σ be as before. Suppose that the characteristics of p_m are simple with respect to Σ and that the coefficients in p_m are real-valued. There exists a (large) constant $\mu > 0$ such that if $|x| \leq \mu^{-2}$, then for $(\xi,\lambda) \in \mathbb{R}^n \times (0,+\infty)$ we have*

$$C\mu(\lambda^2+|\xi|^2)^{-1/2}|p_m(x,\xi-i\lambda\nabla\phi)|^2 + c_{2m-1,\phi}(x,\xi,\lambda) \geq \mu^{-1}\lambda(\lambda^2+|\xi|^2)^{m-1}.$$

The bound $O(\|w\|^2_{H^{m-1}_\lambda}) + \|P_m(x,\zeta)\|^2_{L^2} \geq \mu^{-1}\lambda\|w\|^2_{H^{m-1}_\lambda}$ follows by Gårding's inequality on quadratic forms associated to positive symbols. This gives (5) for large λ and w supported in $\{|x| \leq \mu^{-2}\}$. The weight ϕ has the convexity properties as in the example, so we conclude the Cauchy uniqueness as before.

5.4 Application: a unique continuation problem

Let us prove Theorem 5. Let Σ be given by $\rho = 0$, where $\rho \in C^1(\Omega;\mathbb{R})$ with $\nabla\rho \neq 0$ on Σ. Suppose that for some $x_0 \in \Omega$ and $(\xi_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$ we have that $p_2(x_0, \xi_0 - i\lambda_0\nabla\rho(x_0)) = \nabla_\xi p_2(x_0, \xi_0 - i\lambda_0\nabla\rho(x_0)) \cdot \nabla_x\rho(x_0) = 0$. Assume without loss of generality that $\nabla_x\rho(x_0) = e_n$, so that we have

$$p_2(x_0, (\eta_0, \tau_0 - i\lambda_0)) = \partial_\tau p_2(x_0, (\eta_0, \tau_0 - i\lambda_0)) = 0.$$

The ellipticity condition gives that $p_2(x,\xi) \neq 0$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$. In particular it implies that $\lambda_0 \neq 0$. It also gives that $p_2(x_0, (0,\tau)) \neq 0$, and so $\tau \mapsto p_2(x_0, (\eta_0, \tau))$ is a (non-trivial) polynomial of degree 2 with real coefficients. This contradicts that $\tau_0 - i\lambda_0 \notin \mathbb{R}$ is a double root of it.

To show the unique continuation property we use the Cauchy uniqueness with respect to spheres to extend the vanishing domain. Suppose $u \in H^2_{loc}$ satisfies (1) and vanishes in a non-empty open set of the open connected set Ω. Let $K = \operatorname{supp} u \subset \Omega$. If ∂K is empty, then K is closed and open, and so K is empty (since K^c is non-empty). Thus we can assume that there is $x_0 \in \partial K$. Take a radius $R > 0$ such that $B(x_0, 3R) \subset \Omega$ and let $x_1 \in B(x_0, R) \cap K^c$. Note that $x_0 \in B(x_1, R) \subset B(x_0, 2R) \subset \Omega$. Let $r \leq R$ be the supremum of the radii such that $B(x_1, s) \subset K^c$. This implies that u vanishes in the closed ball $\overline{B}(x_1, r)$, and theorem 4 gives that u must vanish in a bigger ball, contradicting the maximality of r.

References

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6 A counterexample to unique continuation for elliptic equations in divergence form with $C^{0,1-}$ coefficients (after N. Mandache [4])

A summary written by Mihajlo Cekić

Abstract

We review a sharp counterexample (on the Hölder scale) of N. Mandache to unique continuation for second order elliptic equations in divergence form in dimension 3 with $C^{0,1-}$ coefficients.

6.1 Introduction

We consider the uniqueness problem for the uniformly elliptic equation in divergence form in $\mathbb{R}^n$:

$$Pu := \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) = 0 \tag{1}$$

We will also write $A = (a_{ij})$ for the matrix of coefficients; it is assumed that A is positive definite, symmetric and has eigenvalues in $[C, C^{-1}]$ with $C > 0$.

The unique continuation property (UCP in short) for the operator P holds if any solution u to equation (1) vanishing on an open non-empty subset, must vanish on $\mathbb{R}^n$. We introduce the limiting Hölder space via

$$C^{0,1-}(\mathbb{R}^n) := \cup_{\alpha<1} C^{\alpha}(\mathbb{R}^n) \tag{2}$$

For $n = 2$, it is known by a Theorem of Bers and Nirenberg [1] that UCP holds for P with only measurable coefficients; for $n \geq 3$, it is known that the UCP holds for P with Lipschitz coefficients (see e.g. Hörmander [2]).

For negative results, first there was a counterexample by Pliś [6] for elliptic operators in general (non-divergence) form with $C^{0,1-}$ coefficients, then by Miller [5] for P when $n = 3$ and a^{ij} of regularity $C^{\frac{1}{6}}$. Mandache [4] proves the optimality on the Hölder scale, by proving the non-UCP for P, when $n = 3$ and a^{ij} in $C^{0,1-}$ – his construction builds on the one by Miller.

For the sharp counterexamples in the L^p scale of coefficients, see Koch and Tataru [3] and references therein.

Here is the main theorem we discuss, Theorem 1. in [4]:

Theorem 1. *There exist a smooth function u, smooth functions b_{11}, b_{12}, b_{22}, and continuous functions d_1, d_2 defined on $\mathbb{R}^3 \ni (t, x, y)$, such that:*

1. *u is the solution of the equation:*

$$\partial_t^2 u + \partial_x((b_{11} + d_1)\partial_x u) + \partial_y(b_{12}\partial_x u) + \partial_x(b_{12}\partial_y u) + \partial_y((b_{22} + d_2)\partial_y u) = 0. \quad (3)$$

2. *There is a $T > 0$ such that supp $u = (\infty, T] \times \mathbb{R}^2$.*

3. *u, b_{ij} and d_i are periodic in x and in y with period 2π.*

4. *For any $t \in \mathbb{R}$, $u(t, \cdot, \cdot)$ satisfies the Neumann boundary condition on $(0, 2\pi)^2$ with respect to equation (3) (as an equation in x and y).*

5. *d_1 and d_2 do not depend on x and y and are of Hölder class $C^{0,1-}$.*

6.

$$\frac{1}{2} < \begin{pmatrix} b_{11} + d_1 & b_{12} \\ b_{12} & b_{22} + d_2 \end{pmatrix} < 1 \quad (4)$$

Furthermore, there are also functions as above, satisfying conditions 1.-6. except that equation (3) is replaced with the parabolic equation:

$$\partial_t u = \partial_x((b_{11} + d_1)\partial_x u) + \partial_y(b_{12}\partial_x u) + \partial_x(b_{12}\partial_y u) + \partial_y((b_{22} + d_2)\partial_y u) \quad (5)$$

6.2 Idea of the proof

We denote $\Delta_{xy} = \partial_x^2 + \partial_y^2$ the Laplacian in 2D. Observe that $e^{-\lambda t}\cos\lambda x$ and $e^{-\lambda t}\cos\lambda y$ are harmonic functions, in the kernel of $\partial_t^2 + \Delta_{xy}$, for any $\lambda \in \mathbb{R}$. We will *alternately glue* infinitely many of the $e^{-\lambda_k t}\cos\lambda_k x$ and $e^{-\lambda_{k+1} t}\cos\lambda_{k+1}$ in the limit $\lambda_k \to \infty$, in increasingly shorter intervals. In the *gaps* between these intervals, we smoothly modify solution and the coefficients accordingly. With a careful choice of lengths of intervals, gaps and $\lambda_k \to \infty$ in the end, we obtain a smooth solution. We divide the construction in the following two steps.

Step 1. We construct the "typical" smooth modification in the gaps that alternates between the two harmonic functions above. More precisely, we construct $v, B_{ij}, D_i : [0, 5a] \times \mathbb{R}^2 \to \mathbb{R}$, where the parameter $a > 0$ plays the role of time and $[0, 5a]$ of the gap. We also need: $\lambda > \frac{1}{a}$ – the old frequency, $\lambda' > \lambda$ – the new frequency and $\rho \in (0, \frac{\lambda}{\lambda'})$ – a technical parameter.

We ask that for an $\varepsilon > 0$, we have $B_{ij} = \delta_{ij}$ and $D_i = 0$ for $i, j = 1, 2$ and $t \in [0, \varepsilon) \cup (5a - \varepsilon, 5a]$; also $v(t, x, y) = e^{-t\lambda}\cos\lambda x$ for $t \in [0, \varepsilon)$ and $v(t, x, y)$ proportional to $e^{-t\lambda'}\cos\lambda' y$ for $t \in (5a - \varepsilon, 5a]$.

Step 2. Put infinitely many occurrences of Step 1 together.

6.3 Sketch of the construction

We divide the construction in two steps, as described in the previous section.

6.3.1 Step 1: filling the gaps

We construct v, B_{ij}, D_i from Step 1 above for $i, j = 1, 2$. Let $\chi(t)$ be a smooth cut off function with $\chi = 0$ near $(-\infty, 0]$ and $\chi = 1$ near $[1, \infty]$. We divide the construction in this step in five cases, one for each interval of the form $[(i-1)a, ia]$ for $i = 1, 2, 3, 4, 5$ and describe the v, B_{kl} and D_l separately.

Case i = 1. Aim: smooth decay of $B_{22} + D_2$ from 1 to ρ^2. We let

$$v = e^{-\lambda t} \cos \lambda x, \;\; B_{11} = B_{22} = 1, \;\; B_{12} = D_1 = 0, \;\; D_2 = \chi\Big(\frac{t}{a}\Big)(\rho^2 - 1) \quad (6)$$

Case i = 2. Aim: introduce a component of solution oscillating in y, i.e.

$$v = e^{-\lambda t} \cos \lambda x + \tilde{c}\chi\Big(\frac{t-a}{a}\Big)e^{-\rho\lambda' t} \cos \lambda' y \quad (7)$$

Here $\tilde{c} = e^{\frac{5a}{2}(\rho\lambda' - \lambda)}$ is a constant. Furthermore, we let $B_{22} = 1$, $D_1 = 0$ and $\rho^2 - 1$. In what follows, we construct $B_{11} = 1 + \tilde{B}$ and B_{12} by hand. By imposing the equation (3) on v, we obtain after simplifying:

$$\begin{aligned}\tilde{\chi}(t) \cos \lambda' y = \lambda \partial_x(\tilde{B} \sin \lambda x) \\ + \lambda' \tilde{c}\chi\Big(\frac{t-a}{a}\Big)e^{(\lambda - \rho\lambda')t} \sin \lambda' y \partial_x B_{12} + \lambda \sin \lambda x \partial_y B_{12}\end{aligned} \quad (8)$$

Here we introduced the shorthand notation $\tilde{\chi}(t)$ for:

$$\tilde{\chi}(t) = \tilde{c}e^{(\lambda-\rho\lambda')t}\Big(\frac{1}{a^2}\chi''\Big(\frac{t-a}{a}\Big) - \frac{2\rho\lambda'}{a}\chi\Big(\frac{t-a}{a}\Big)\Big) \quad (9)$$

We consider an ansatz $B_{12} = \tilde{\chi}(t)\frac{2 \sin \lambda x \sin \lambda' y}{\lambda\lambda'}$. It is easy to see from eq. (8):

$$\tilde{B}(t, x, y) = \tilde{\chi}(t)\Big(\frac{\cos \lambda' y \cos \lambda x}{\lambda^2} - \tilde{c}e^{(\lambda-\rho\lambda')t}\chi\Big(\frac{t-a}{a}\Big)\frac{2 \sin^2 \lambda' y}{\lambda^2}\Big) \quad (10)$$

Case $i = 3$**.** Aim: propagate two components with different speeds. Let

$$v = e^{-\lambda t} \cos \lambda x + \tilde{c}e^{-\rho\lambda' t} \cos \lambda' y \quad (11)$$

For the coefficients, we just pick $B_{11} = B_{22} = 1$, $B_{12} = D_1 = 0$ and $D_2 = \rho^2 - 1$. Note that the second component is decaying faster as $\rho\lambda' < \lambda$.

Case $i = 4$. Aim: remove the x-component, symmetric to $i = 2$. Let

$$v = \chi\Big(\frac{4a - t}{t}\Big)e^{-\lambda t}\cos\lambda x + \tilde{c}e^{-\rho\lambda' t}\cos\lambda' y \tag{12}$$

We let $B_{11} = 1$, $D_1 = 0$, $D_2 = \rho^2 - 1$. Similarly to the case $i = 2$, we define B_{12} and $B_{22} = 1 + B'$ by imposing equation (3) on v. We skip the details.

Case $i = 5$. Aim: increase $B_{22} + D_2$ from ρ^2 to 1. Let $\chi_1(t) = \int_0^t \chi(s)ds$, so that $\chi_1 = 0$ near $(-\infty, 0]$ and $\chi_1 = t + \chi_1(1) - 1$ near $[1, \infty)$. Take:

$$v = \tilde{c}\cos\lambda' y \exp\Big(-\lambda'\rho t - \lambda'(1-\rho)a\chi_1\Big(\frac{t - 4a}{a}\Big)\Big) \tag{13}$$

Then we reverse-engineer the coefficients by imposing that v satisfies equation (3). We get $B_{11} = B_{22} = 1$, $B_{12} = D_1 = 0$ and

$$D_2 = -\frac{\partial_t^2 v}{\partial_y^2 v} - 1 = \Big(\rho + (1-\rho)\chi\Big(\frac{t-4a}{a}\Big)\Big)^2 - \frac{1-\rho}{a\lambda'}\chi'\Big(\frac{t-4a}{a}\Big) - 1 \tag{14}$$

It is straightforward to check that for $t \in (5a - \varepsilon, 5a]$ for some $\varepsilon > 0$:

$$v(t, x, y) = \alpha(a, \lambda, \lambda')e^{-\lambda'(t-5a)}\cos\lambda' y \tag{15}$$

Here $\alpha(a, \lambda, \lambda')$ is a constant bounded above by $e^{\frac{5a\lambda}{2}}$.

Derivative estimates for v, b_{ij} and d_i. We sketch the estimate for b_{12} and the rest follow similarly. We first estimate $\tilde{\chi}(t)$ (eq. (9)) on $[a, 2a]$:

$$\begin{aligned}|\partial_t^k\tilde{\chi}(t)| \leq \tilde{c}e^{(\lambda-\lambda'\rho)t}\sum_{j=0}^{k}\binom{k}{j}(\lambda-\lambda'\rho)^j\Big(\frac{1}{a^{k-j+2}}\Big|\chi^{(k-j+2)}\Big|\Big(\frac{t-a}{a}\Big)\\ + \frac{2\rho\lambda'}{a^{k-j+1}}\Big|\chi^{(k-j+1)}\Big|\Big(\frac{t-a}{a}\Big)\Big) \leq e^{-a(\lambda-\rho\lambda')/2}\cdot 3\cdot 2^k\cdot C_{\chi,k+2}\lambda^{k+2}\end{aligned} \tag{16}$$

Here we used $\lambda > \rho\lambda'$, $\tilde{c}e^{t(\lambda-\rho\lambda')} \leq e^{-a(\lambda-\rho\lambda')/2}$ for $t \in [a, 2a]$, binomial formula and $\lambda' > \lambda > 1/a$. From this, definition of B_{12}, similar reasoning for $i = 4$ and more generally for B_{ij}, v and D_i, we have for all $k + l + m > 0$

$$|\partial_t^k\partial_x^l\partial_y^m B_{ij}| \leq e^{-a(\lambda-\rho\lambda')/2}C'_{\chi,k,m}\lambda^{k+l}\lambda'^m \tag{17}$$

$$|\partial_t^k\partial_x^l\partial_y^m v| \leq C''_{\chi,k}\lambda'^{k+m}\lambda^l \quad\text{and}\quad |\partial_t D_i| \leq 5C_\chi\frac{1-\frac{\lambda^2}{\lambda'^2}}{a} \tag{18}$$

Boundary conditions. Aim: check that v satisfies conditions 3. and 4. in the theorem. It is straightforward to check it suffices to have $\lambda, \lambda' \in \mathbb{N}$.

6.3.2 Step 2: the glueing construction

Let $\{a_k\}_{k\geq 1}$ and $\{\lambda_k\}_{k\geq 1}$ be increasing sequence of positive integers s.t. $\sum_{i=1}^{\infty} a_i < \infty$ and $1/a_k < \lambda_k < \lambda_{k+1}$. Denote the partial sums $T_k = 5\sum_{i=1}^{k} a_i$ and let $\rho_k := \lambda_k^2/\lambda_{k+1}^2$. Let $k_0 \in 2\mathbb{N}$. We specify these parameters later. Define the main function as:

$$u(t,x,y) := \begin{cases} e^{-(t-T_k)\lambda_k}\cos\lambda_k x, & t\in(-\infty, T_{k_0}] \\ c_k v_{a_k,\lambda_k,\lambda_{k+1}}(t,x,y), & t\in[T_k,T_{k+1}] \text{ and } k \text{ even} \\ c_k v_{a_k,\lambda_k,\lambda_{k+1}}(t,y,x), & t\in[T_k,T_{k+1}] \text{ and } k \text{ odd} \\ 0, & t\in[T,\infty] \end{cases} \tag{19}$$

The coefficients b_{ij}, d_i for $i=1,2$ are defined similarly using the construction in Step 1, so that u satisfies equation (3). To make u smooth, we need to take $c_{k_0}=1$ and $c_{k+1}=c_k\cdot\alpha(a_k,\lambda_k,\lambda_{k+1})$ for $k>k_0$ (see eq. (15)). Again by (15), we get $c_k \leq e^{-\frac{5}{2}\sum_{j=k_0}^{k-1} a_j\lambda_j}$. Note u, b_{ij} and d_i are smooth for $t\neq T$.

Hölder continuity of d_i. We claim there is an estimate of the form:

$$|d_i(t_1)-d_i(t_2)| \leq 10C_\chi \sup_{k\geq k_0}\Big((1-\frac{\lambda_k^2}{\lambda_{k+1}^2})\cdot\min\big(5,\frac{|t_1-t_2|}{a_k}\big)\Big) \tag{20}$$

The estimate is proved by considering cases on t_1 and t_2: both are in $[T_k,T_{k+1}]$ for some k, one of them is in $\mathbb{R}\setminus[T_{k_0},T]$ or they are in $[T_k,T_{k+1}]$ for distinct k. For example, the first case follows from the derivative estimate (18) and as $|t_1-t_2|\leq 5a_k$. To check Hölder continuity, ask for $C_\alpha>0$ for $\alpha<1$ s.t. $(1-\frac{\lambda_k^2}{\lambda_{k+1}^2})\cdot\min(5,|t|/a_k)\leq C_\alpha t^\alpha$ for $k\geq k_0$ and $t\geq 0$. In turn, it suffices

$$1-\frac{\lambda_k^2}{\lambda_{k+1}^2}\leq C_\alpha a_k^\alpha \tag{21}$$

Smoothness of u and b_{ij} at $t=T$. It suffices to prove all the derivatives converge to zero at $t=T$. For b_{ij}, it suffices to have for all $m\in\mathbb{N}$, by (17):

$$\lim_{k\to\infty} e^{-a_k\left(\lambda_k-\frac{\lambda_k^2}{\lambda_{k+1}}\right)/2}\lambda_{k+1}^m = 0 \tag{22}$$

The continuity of d_i implies $\lim_{k\to\infty}\rho_k=1$, so $\lim_{k\to\infty}\frac{\lambda_k}{\lambda_{k+1}}=1$. This, combined with derivative estimates (18) and the fact $-\frac{5}{2}\sum_{j=k_0}^{k}a_j\lambda_j\leq -a_k\lambda_k/2\leq -a_k(\lambda_k-\frac{\lambda_k^2}{\lambda_{k+1}})/2$, we see it suffices to prove the estimate (22).

Choice of a_k, λ_k and k_0. Combining the facts above, we need to have: $\sum_{i=1}^{\infty} a_i < \infty$ (finiteness), $1/a_k < \lambda_k < \lambda_{k+1}$ (technical condition), $\lambda_k \in \mathbb{N}$ (periodicity and boundary condition), limit condition (22) (smoothness of b_{ij} and u), estimate (21) (d_i are in $C^{0,1-}$). One such choice is given by $\lambda_k = (k+1)^3$ and $a_k = \big(k \log^2(k+1)\big)^{-1}$. It is an exercise to show these satisfy the conditions above.

Note that b_{ij} and d_i are uniformly continuous, so for large k_0 we get the uniform ellipticity condition 6. in the theorem. This ends the construction.

Remark 2. *We see that as $\prod_{k\geq k_0} \rho_k^2 = 0$, we have $\sum_{k\geq k_0}(1-\rho_k^2) = \infty$. So one of d_1 and d_2 has unbounded variation and is not an element of $C^{0,1}$.*

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7 Stability - Talk 1: Stable determination of conductivity by boundary measurements

A summary written by Dimitrije Cicmilović

Abstract

We consider the problem of determining the scalar coefficient γ in the elliptic equation $\operatorname{div}(\gamma \nabla u) = 0$ in Ω when, for every Dirichlet $u = \phi$ on $\partial\Omega$, the Neumann datum $\gamma(\frac{\partial}{\partial n})u = \Lambda_\gamma \phi$ is known. We discuss Alessandrini's proof of a continuous dependence result.

7.1 Introduction

We consider the Dirichlet problem of the conductivity equation

$$\operatorname{div}(\gamma \nabla u) = 0 \tag{1}$$
$$u|_{\partial\Omega} = \phi \tag{2}$$

in $\Omega \subset \mathbb{R}^n$ and for a fixed real valued function (conductivity) $\gamma : \Omega \to \mathbb{R}$. We always assume that γ is strictly positive and that the boundary of Ω is smooth. We want to recover the conductivity from the Dirichlet-to-Neumann map

$$\Lambda_\gamma : \phi \mapsto \gamma \frac{\partial u}{\partial n},$$

where n denotes the outer normal at boundary $\partial\Omega$ and u is a solution to the Dirichlet problem (1), (2), and to study its stability.

Prior to this result, Kohn and Vogelius proved the unique determinedness from Dirichlet-to-Neumann map when the conductivity is analytic ([2]), and extended it to a piecewise analytic case ([3]). Sylvester and Uhlmann then proved the determinedness in the case of smooth conductivity for $n \geq 3$ ([4]), and a continuous dependence from $\gamma|_\Omega$ to Λ_γ ([5]).

We now present Alessandrini's continuous dependence result for the case of space dimension $n \geq 3$. The space restriction is due to the fact that Alessandrini used the approach in [4], where techniques used imply $n \geq 3$, when studying the stability of (1).

Firstly, we assume that Ω is bounded and that for some constants $E > 0$ and $s > \frac{n}{2}$, the conductivities satisfy a priori estimates

$$E^{-1} \leq \|\gamma\|_{L^\infty(\Omega)}, \tag{3}$$
$$\|\gamma\|_{H^{s+2}(\Omega)} \leq E. \tag{4}$$

Given initial data $\phi \in H^{\frac{1}{2}}(\Omega)$, we denote by u_k ($k = 1, 2$) the $H^1(\Omega)$ solutions of (1), with respective conductivities γ_k, and define the respective Dirichlet-to-Neumann maps

$$\Lambda_k : \phi \mapsto \gamma_k \frac{\partial u_k}{\partial n}.$$

Then one has the following result:

Theorem 1. *Let γ_1, γ_2 satisfy (3), (4). The following estimate holds*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C_E \omega(\|\Lambda_1 - \Lambda_2\|)$$

where the function ω is such that

$$\omega(t) \leq |\log t|^{-\delta}, \tag{5}$$

for every t, $0 < t < \frac{1}{e}$, and δ, $0 < \delta < 1$, depends only on n and s.

The constant C_E depends only on Ω, s, n and E. Note that $s > \frac{n}{2}$ implies that $\gamma \in C^2(\overline{\Omega})$. We remark that one cannot drop assumption (4), which would result in an ill-posedness of the Dirichlet problem (1), (2).

The proof of Theorem 1 is based on analysis of stability for inverse problem for the Schrödinger equation obtained by modifying (1), and for which the bounds for continuous dependence can be derived.

7.2 Reduction to Schrödinger equation

Defining

$$v = \sqrt{\gamma} u \tag{6}$$
$$q = \sqrt{\gamma}^{-1} (\Delta \sqrt{\gamma}) \tag{7}$$

one derives from (1) the following equation

$$\Delta v - q v = 0 \text{ in } \Omega. \tag{8}$$

Let v be solution to (8) for Dirichlet data $\psi \in H^{\frac{1}{2}}(\partial\Omega)$. We define Dirichlet-to-Neumann map $\tilde{\Lambda} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ by

$$\tilde{\Lambda} : \psi \mapsto \frac{\partial}{\partial n} v. \tag{9}$$

We remark that (6) implies the existence and uniqueness result in $H^1(\Omega)$ for Dirichlet problem (8), hence $\tilde{\Lambda}$ is well-defined.

In view of previous discussion, different conductivities (γ_k) lead to different potentials (q_k) in Schrödinger equation. First step towards proving Theorem 1 is the following proposition

Proposition 2. *Let γ_k satisfy (3), (4), and let q_k be given by (7). The following estimate holds*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C_E \omega(\|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|), \tag{10}$$

where the function ω satisfies (5) with δ, $0 < \delta < 1$, depending only on n.

Aside from the ansatz in the following section, the proof will rely on the identity

$$\langle (\tilde{\Lambda}_1 - \tilde{\Lambda}_2) v_2, v_1 \rangle = \int_\Omega (q_1 - q_2) v_1 v_2 dx, \tag{11}$$

which follows from Green's formula and exploiting the structure of Dirichlet problem (8).

7.2.1 Ansatz - complex geometrical optics (CGO)

As mentioned, the stability issue of (1) is derived to stability issue of (8), which is investigated by looking at specific type of its solutions, introduced in [4]. Alessandrini relies on the following lemma from [4]:

Lemma 3. *Let γ_k satisfy (3), (4), and let q_k be given by (7). There exists $C_E > 0$ such that, for every $\xi_k \in \mathbb{C}^n$, satisfying*

$$\xi_k \cdot \xi_k = 0, \ |\xi_k| \geq C_E \tag{12}$$

there exists solution v_k to (8) of the form

$$v_k(x) = e^{\xi_k \cdot x}(1 + \psi_k(x)), x \in \Omega, \tag{13}$$

such that

$$\|\psi_k\|_{H^s(\Omega)} \leq C_E |\xi_k|^{-1}. \tag{14}$$

Combination of previous lemma with (11), and some complex analysis estimates, gives Proposition 2.

7.3 Final steps

The following proposition gives boundary measurements stability

Proposition 4. *Let γ_1 and γ_2 satisfy (3), (4). The following estimates hold*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C_E\|\Lambda_1 - \Lambda_2\|, \tag{15}$$

$$\|\frac{\partial}{\partial n}(\gamma_1 - \gamma_2)\|_{L^\infty(\partial\Omega)} \leq C_E(\|\Lambda_1 - \Lambda_2\|^\tau + \|\Lambda_1 - \Lambda_2\|) \tag{16}$$

where τ, $0 < \tau < 1$, depends only on n.

The proof of the first inequality was first presented in [5]. The role of the second one is to enable bridging the Dirichlet-to-Neumann maps for (1) and (8), and subsequently relate the stability decay from one to another. This bridging follows from the identity

$$\tilde{\Lambda}_k\psi = \sqrt{\gamma_k}^{-1}\Lambda_k(\sqrt{\gamma_k}^{-1}\psi) + \frac{1}{2}\gamma_k^{-1}\frac{\partial\gamma_k}{\partial n}\psi\Big|_{\partial\Omega}. \tag{17}$$

Short computation, (16) and application of a priori bounds (3),(4) gives the inequality

$$\|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\| \leq C(\|\Lambda_1 - \Lambda_2\| + \|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)}). \tag{18}$$

Now the application of Sobolev embedding, logarithmic convexity of Sobolev norms and trace theorem gives the control of second summand on RHS of last inequality by a power of L^2 norm, and hence L^∞ one. Then (15) gives the control of Dirichlet-to-Neumann maps.

We finalize the proof of Theorem 1. Direct computation gives us the identity

$$\sqrt{\gamma}^{-1}(\Delta\sqrt{\gamma}) = \Delta(\log\sqrt{\gamma}) + |\nabla(\log\sqrt{\gamma})|^2.$$

From the definition (7) one readily gets

$$\nabla\cdot((\gamma_1\gamma_2)\nabla f) = (\gamma_1\gamma_2)^{\frac{1}{2}}(q_1 - q_2) \text{ in } \Omega, \tag{19}$$

where $f = \log\frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}$ is a $C^2(\Omega)$ function. Now a priori estimates (3), (4) and maximum principle imply

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C_E\|f\|_{L^\infty(\Omega)} \leq C_E(\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} + \|q_1 - q_2\|_{L^\infty(\Omega)}). \tag{20}$$

Note that the first summand on the RHS is controlled by the norm of Dirichlet-to-Neumann map (15), whereas for the second one one wants to use the stability result (Prop. 2) for Schrödinger equation, meaning one has to relate L^2 and L^∞ norms of the potentials, which is done by the following interpolation inequality

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C\|q_1 - q_2\|^{\delta}_{L^2(\Omega)}\|q_1 - q_2\|^{1-\delta}_{H^s(\Omega)},$$

where $\delta = 1 - \frac{n}{2s}$. Using a priori bounds (3), (4) once again, (20) reads

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C_E(\|\Lambda_1 - \Lambda_2\| + \|q_1 - q_2\|^{\delta}_{L^2(\Omega)}).$$

Now the Proposition 2, with remark following (18), completes the proof.

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8 2D Unique Continuitation and Quasiconformal Maps - Talk 2: Uniqueness properties of solutions to nonlinear Beltrami equations and the Stoilow factorization

A summary written by Gael Yomgne Diebou

Abstract

We classify all solutions to the linear Beltrami equation within a suitable regularity class by means of the Stoilow factorization. In a slightly similar fashion, we equally classify solutions to the nonlinear Beltrami equation. Ultimately, we discuss the uniqueness properties of principal solutions of some general nonlinear first order elliptic system in the plane.

8.1 Introduction

The Beltrami equation occurs in the study of conformal mappings between two domains endowed with measurable Riemannian structures. This equation has a vast history (see [1], [3]) and plays a central role in the diverse interplays between the geometric theory of quasiconformal mappings, the nonlinear elliptic planar PDEs and complex analysis.

Let Ω be an open connected subset of the complex plane $\mathbb{C}$ and $\mu : \Omega \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The complex linear Beltrami equation reads

$$\frac{\partial f}{\partial \overline{z}} = \mu(z)\frac{\partial f}{\partial z} \tag{1}$$

where $\dfrac{\partial f}{\partial \overline{z}} = \overline{\partial} f = (\partial_x f + i\partial_y f)/2$, $\dfrac{\partial f}{\partial z} = \partial f = (\partial_x f - i\partial_y f)/2$, $z = x + iy$ and $\partial_x f$ and $\partial_y f$ are partial derivatives of f with respect to x and y, respectively. The function $\mu : \Omega \to \mathbb{C}$ in the equation (1) is called the Beltrami coefficient of f or more generally the complex dilatation of f.

The existence theory for equation (1) is well understood. Indeed, for arbitrary measurable μ which enjoys the property $\|\mu\|_\infty < 1$, there are homeomorphic solutions to equation (1). This statement is justified by the measurable Riemann mapping theorem and was first proven by Morrey [2]. Concerning the uniqueness theory, one can only speak of a unique solution to (1) if one additionally imposes a certain normalization condition ($f(z) = z + \mathcal{O}(1/z)$ near ∞) on f, see [1, p. 165]. This normalized solutions of equation (1) are called principal solutions. Then, it is reasonable to ask whether one can provide a complete classification of non principal solutions to equation (1). To this interrogation, there is a positive answer owing to the Stoilow factorization which has its origin in two dimensional topology and which is used to obtain all solutions of (1) within natural regularity classes provided one has the existence of one solution of (1) beforehand. We will present the Stoilow factorization in more details in section 2.

The Beltrami equation also exists in the nonlinear setup. It can be written in the general form given by the first order system

$$\overline{\partial} f = \mathcal{H}(z, f, \partial f) \tag{2}$$

The existence of solutions for the equation (2) is established under general conditions on the function $\mathcal{H}$.

A function f is a principal solution of (2) if $f \in W^{1,2}_{loc}(\mathbb{C})$ is a solution to equation (2) normalized outside a compact set by the condition $f(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots$ while a homeomorphic solution $f \in W^{1,1}_{loc}(\mathbb{C})$ to (2) is said to be normalized if the solution is normalized by its value at three given points, that is, $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

In [1, p. 238], the uniform ellipticity and the Lusin measurability assumptions on $\mathcal{H}$ afford the authors to prove that equation (2) admits both principal and normalized solutions. Analogously to the linear theory, the uniqueness issue for the equation (2) only deals with principal and normalized solutions. If a certain Lipschitz regularity assumption is granted on $\mathcal{H}$, one will see later that the equation (2) has a unique principal solution. In case the function $\mathcal{H}$ in (2) is real homogeneous with respect to the last variable and does not depend on f, we will discuss a particular connection between solutions of the corresponding equation (2) and the reduced distortion inequality.

8.1.1 Remark

The notion of principal solution of (2) introduced above is pertinent only when the function $H(z, w, \Lambda)$ vanishes for sufficiently large values of $|z|$.

Due to Theorem 3.6.3 in [1], the normalization $f(\infty) = \infty$ holds automatically. Note that all normalized solutions as defined above are actually

$W^{1,2}_{loc}(\mathbb{C})$-functions. Principal and normalized solutions of (2) have an interesting connection with the linear theory which in fact allows them to inherit all properties of solutions to (1). In effect, any principal or normalized solution f of (2), if it exists, solves its own linear equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}, \qquad \mu(z) = \frac{H(z, f(z), \partial f)}{\partial f}.$$

8.2 Stoilow factorization

Let Ω be defined as above. The classical Stoilow factorization in two dimensions states that any discrete open mapping $f : \Omega \to \mathbb{R}^2$ can be factorized in the following manner; there is an analytic function φ and a homeomorphism h such that $f = \varphi \circ h$ (a mapping f is called discrete if the preimage of a point in $\mathbb{C}$ is discrete in the domain Ω). We need to understand this result from the point of view of PDEs and analysis thus, requires a reformulation. Quasiregular mappings (see below for its definition) in the complex plane are discrete and open and the previous factorization indicates that they can be decomposed as $\varphi \circ h$ with φ analytic and h quasiconformal, see [4, p. 247]. This motivates the consideration of quasiregular mappings at the expense of open and discrete maps in establishing the factorization theorem. For the Beltrami equations, it stipulates that if the latter has a homeomorphic solution, then all other solutions within the same regularity class are obtained by composition with a holomorphic map. Multiple versions of the factorization theorem also exist for other first order PDEs, see [1]. The chosen regularity class here is the Sobolev space $W^{1,2}_{loc}$.

Definition 1. *A function $f \in W^{1,2}_{loc}(\Omega)$ is said to be K-quasiregular if it is orientation-preserving (its Jacobian $J(z, f) \geq 0$ almost everywhere) and the following estimate on its derivative known as the distortion inequality $|Df(z)|^2 \leq KJ(z, f)$ for almost every $z \in \Omega$ holds true.*

The factorization theorem will provide a clear understanding of planar quasiregular mappings.

Theorem 2 (Stoilow Factorization)**.** *Let $f : \Omega \to \Omega'$ be a homeomorphic solution to the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}, \qquad f \in W^{1,1}_{loc}(\Omega), \tag{3}$$

with $|\mu(z)| \leq k < 1$ *almost everywhere in* Ω.
Suppose $g \in W^{1,2}_{loc}(\Omega)$ *is any other solution to (3) on* Ω. *Then there exists a holomorphic function* $\psi : \Omega' \to \mathbb{C}$ *such that*

$$g(z) = \psi(f(z)), \qquad z \in \Omega.$$

Conversely, if ψ *is holomorphic on* Ω', *then the composition* $\psi \circ f \in W^{1,2}_{loc}(\Omega)$ *and solves (3).*

It may look surprising that one is considering $W^{1,1}_{loc}$-solutions in (3) instead of the $W^{1,2}_{loc}$ regularity as announced earlier. Note that any homeomorphism in the regularity class $W^{1,1}_{loc}(\Omega)$ has a locally integrable Jacobian as asserts Corollary 3.3.6 in [1] which readily implies the $W^{1,2}_{loc}$ regularity from the distortion inequality.
Now let us consider the nonlinear Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H}(z, \partial f/\partial z) \quad \text{in } \Omega \tag{4}$$

where $\mathcal{H}(z, \Lambda)$ satisfies the uniform ellipticity condition that for almost every $z \in \mathbb{C}$ and all $\Lambda_1, \Lambda_2 \in \mathbb{C}$,

$$|\mathcal{H}(z, \Lambda_1) - \mathcal{H}(z, \Lambda_2)| \leq k|\Lambda_1 - \Lambda_2|, \qquad 0 \leq k < 1,$$

and the real homogeneity condition

$$\mathcal{H}(z, \lambda\Lambda) = \lambda\mathcal{H}(z, \Lambda) \text{ for } \lambda \in \mathbb{R}.$$

It is natural to ask if one can establish a Stoilow-type factorization for solutions to (4). This case is rather subtle but still leads to an interesting result which involves a modified version of the distortion inequality in Definition 1.

Theorem 3. *Assume* f *is a homeomorphic solution to the equation*

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H}(z, \partial f/\partial z), \quad f \in W^{1,2}_{loc}(\Omega).$$

Then any other $W^{1,2}_{loc}$*-solution* g *to the equation in* Ω *takes the form*

$$g = G \circ f$$

where $G : f(\Omega) \to \mathbb{C}$ *satisfies the reduced distortion inequality*

$$\left|\frac{\partial G}{\partial \bar{w}}\right| \leq c \left|\mathcal{I}m \, \frac{\partial G}{\partial w}\right|, \qquad w \in f(\Omega)$$

where c *only depends on the ellipticity constant* k.

8.3 Uniqueness

Suppose that the function $\mathcal{H}$ in (4) also depends on f and consider the nonlinear first order system

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H}(z, f, \partial f/\partial z). \tag{5}$$

Observe that the difference of two principal solutions of (5) need not to be a principal solution of the same equation. To establish uniqueness of these, one needs some regularity on $\mathcal{H}$. Indeed, assuming that $\mathcal{H}$ has the following Lipschitz regularity

$$|\mathcal{H}(z, w_1, \Lambda) - \mathcal{H}(z, w_2, \Lambda)| \leq c|\Lambda||w_1 - w_2| \tag{6}$$

for some constant c independent of z and Λ, we obtain the following.

Theorem 4. *Suppose the function $\mathcal{H} : \mathbb{C}^3 \to \mathbb{C}$ is such that (6) holds and is compactly supported in the z-variable. Assume further the homogeneity condition $\mathcal{H}(z, w, 0) = 0$ for almost every $(z, w) \in \mathbb{C}^2$, the Lusin measurability on $\mathcal{H}$, the uniform elliptic condition, that for almost every $z, w \in \mathbb{C}$ and all $\Lambda, \Lambda' \in \mathbb{C}$,*

$$|\mathcal{H}(z, w, \Lambda) - \mathcal{H}(z, w, \Lambda')| \leq k|\Lambda - \Lambda'|, \quad 0 \leq k < 1.$$

Then the equation (5) has only one principal solution.

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9 The Calderón problem with partial data - Talk 1

A summary written by Marco Fraccaroli

Abstract

We introduce the tools, namely the Carleman estimates and the construction of CGO solutions, needed in order to prove that in dimension $n \geq 3$, the knowledge of the Cauchy data for the Schrödinger equation measured on possibly very small subsets of the boundary determines uniquely the potential.

9.1 Introduction

From now on we will always assume $n \geq 3$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with C^∞ boundary. For $q \in L^\infty(\Omega)$, we consider the operator $-\Delta + q$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ as a bounded perturbation of minus the usual Dirichlet Laplacian.

The operator has a discrete spectrum, and we assume

$$0 \text{ is not an eigenvalue of } -\Delta + q \colon H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega). \tag{1}$$

Under this condition, the Dirichlet problem for the operator is well-posed and the Dirichlet to Neumann map

$$\mathcal{N}_q \colon H^{\frac{1}{2}}(\partial\Omega) \ni v \mapsto \partial_\nu u_{|\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega), \tag{2}$$

is well defined, where ν denotes the exterior unit normal and u is the unique solution in

$$H_\Delta(\Omega)\colon = \{u \in H^1(\Omega), \Delta u \in L^2(\Omega)\} \tag{3}$$

of the problem

$$(-\Delta + q)u = 0 \text{ in } \Omega,\ u_{|\partial\Omega} = v. \tag{4}$$

Question: does the map $\mathcal{N}_q$ determine uniquely the potential q?

In [3], Sylvester and Uhlmann proved the uniqueness result under the condition that the Dirichlet to Neumann map is measured on the whole boundary.

Theorem 1. *Let Ω be defined as above. Let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials satisfying* (1) *and assume*

$$\mathcal{N}_{q_1} = \mathcal{N}_{q_2}. \tag{5}$$

Then $q_1 = q_2$.

It arises naturally the question about the uniqueness in the case in which only partial data of $\mathcal{N}_q$ are known. A first result was obtained by Bukhgeim and Uhlmann in [1], where, roughly speaking, it is proved a uniqueness result when the Dirichlet to Neumann map is measured on slightly more than half of the boundary.

In [2], Kenig, Sjöstrand and Uhlmann showed a uniqueness result when the Dirichlet to Neumann map is measured on an arbitrary open subset of the boundary.

Let $x_0 \in \mathbb{R}^n \setminus \overline{\mathrm{ch}(\Omega)}$, where $\mathrm{ch}(\Omega)$ denotes the convex hull of Ω. We define the front and the back faces of $\partial\Omega$ with respect to x_0 by

$$F(x_0) = \{x \in \partial\Omega\colon (x - x_0) \cdot \nu(x) \le 0\}, \tag{6}$$

$$B(x_0) = \{x \in \partial\Omega\colon (x - x_0) \cdot \nu(x) \ge 0\}. \tag{7}$$

Their main result is the following

Theorem 2. *Let $\Omega, x_0, F(x_0), B(x_0)$ be defined as above. Let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials satisfying* (1) *and assume that there exist open neighbourhoods $\widetilde{F}, \widetilde{B} \subset \partial\Omega$ of $F(x_0)$ and $B(x_0)$ respectively, such that*

$$\mathcal{N}_{q_1} u = \mathcal{N}_{q_2} u \text{ in } \widetilde{F}, \text{ for all } u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathcal{E}'(\widetilde{B}). \tag{8}$$

Then $q_1 = q_2$.

Here $\mathcal{E}'(\widetilde{B})$ denotes the space of compactly supported distributions in $\widetilde{B}$.

If $\widetilde{B} = \partial\Omega$, then we have the following

Theorem 3. *Let $\Omega, x_0, F(x_0), B(x_0)$ be defined as above. Let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials satisfying* (1) *and assume that there exist open neighbourhoods $\widetilde{F} \subset \partial\Omega$ of $F(x_0)$, such that*

$$\mathcal{N}_{q_1} u = \mathcal{N}_{q_2} u \text{ in } \widetilde{F}, \text{ for all } u \in H^{\frac{1}{2}}(\partial\Omega). \tag{9}$$

Then $q_1 = q_2$.

In particular, if Ω is "well shaped" with respect to one point $x_1 \in \partial\Omega$, $\widetilde{F}$ can be made arbitrary small by choosing x_0 arbitrarily close to x_1.

Along the line of the argument of Sylvester and Uhlmann in [3], the main tool will be the construction of special solutions, called complex geometrical optics (CGO) solutions to the Schrödinger equation associated to the operator $-\Delta + q$. However, in order to control the solutions on parts of the boundary, as in [1], we need some Carleman estimates. They allow us to construct a larger class of CGO solutions that better fits our purpose.

9.2 CGO solutions

The CGO solutions that we want to construct have the form

$$u = e^{\frac{1}{h}(-\varphi + i\psi)}(m + r), \tag{10}$$

where h is small, m is smooth and non-vanishing, r is a correction term with controlled norms, namely

$$\|r\|_{L^2(\Omega)} \leq O(h), \|r\|_{H^1(\Omega)} \leq O(1). \tag{11}$$

Moreover, the functions φ, ψ satisfy the conditions

$$\nabla\varphi \cdot \nabla\psi = 0, |\nabla\varphi|^2 = |\nabla\psi|^2, \tag{12}$$

φ has non-vanishing gradient and is a *limiting Carleman weight* on Ω.

To define the last condition we consider the conjugated operator $e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}}$. It has the Weyl symbol for the semiclassical quantization $a(x,\xi) + ib(x,\xi)$, where

$$a(x,\xi) = \xi^2 - (\nabla\varphi(x)), b(x,\xi) = 2\nabla\varphi(x) \cdot \xi. \tag{13}$$

Definition 4. *We say that φ is a* **limiting Carleman weight** *on Ω if, for every $x \in \Omega$, the Poisson bracket*

$$\{a,b\}(x,\xi) = a'_\xi \cdot b'_x - a'_x \cdot b'_\xi = 0, \text{ when } a(x,\xi) = b(x,\xi) = 0. \tag{14}$$

In particular, if φ is a limiting Carleman weight, then also $-\varphi$ is a limiting Carleman weight.

9.2.1 Carleman estimates

Proposition 5. *Let Ω be as above, φ be a limiting Carleman weight on a neighbourhood of Ω, $q \in L^\infty(\Omega)$. Then if $u \in C_c^\infty(\Omega)$, we have*

$$h(\|e^{\frac{\varphi}{h}}u\|_{L^2(\Omega)} + \|h\nabla(e^{\frac{\varphi}{h}}u)\|_{L^2(\Omega)}) \leq C\|e^{\frac{\varphi}{h}}(-h^2\Delta + h^2 q)u\|_{L^2(\Omega)}, \tag{15}$$

where C depends on Ω, and $h > 0$ is small enough so that $Ch\|q\|_{L^\infty(\Omega)} \leq \frac{1}{2}$.

Proposition 6. *Let Ω, φ, q as above. Let ν denote the exterior unit normal to $\partial\Omega$ and define*

$$\partial\Omega_\pm = \{x \in \partial\Omega \colon \pm\nabla\varphi \cdot \nu \geq 0\}. \tag{16}$$

Then there exists a constant $C_0 > 0$, such that for every $u \in C^\infty(\overline{\Omega})$ with $u_{|\partial\Omega} = 0$, we have for $0 < h \ll 1$,

$$\begin{aligned}-\frac{h^3}{C_0}((\nabla\varphi\cdot\nu)e^{\frac{\varphi}{h}}\partial_\nu u|e^{\frac{\varphi}{h}}\partial_\nu u)_{\partial\Omega_-} + \frac{h^2}{C_0}(\|e^{\frac{\varphi}{h}}u\|^2_{L^2(\Omega)} + \|e^{\frac{\varphi}{h}}h\nabla u\|^2_{L^2(\Omega)}) \\ \leq \|e^{\frac{\varphi}{h}}(-h^2\Delta + h^2 q)u\|^2_{L^2(\Omega)} + C_0 h^3((\nabla\varphi\cdot\nu)e^{\frac{\varphi}{h}}\partial_\nu u|e^{\frac{\varphi}{h}}\partial_\nu u)_{\partial\Omega_+}.\end{aligned} \tag{17}$$

9.2.2 Construction of CGO solutions

We outline the construction of CGO solutions needed to prove Theorem 3; the construction of the CGO solutions needed to prove Theorem 2 is left to second part of the exposition of this article.

We first prove an existence result for solutions of the inhomogeneous equation.

Theorem 7. *Let $0 \leq s \leq 1$. Then for $h \geq 0$ small enough, for every $v \in H^{s-1}(\Omega)$, there exists $u \in H^s(\Omega)$ such that*

$$e^{\frac{\varphi}{h}}(-h^2\Delta + h^2 q)e^{-\frac{\varphi}{h}}r = v, h\|r\|_{H^s} \leq C\|v\|_{H^{s-1}}. \tag{18}$$

The proof is given by a Hahn-Banach theorem argument, relying on the Carleman estimate of Proposition 5 (the correspondent result for Theorem 2 relies on the Carleman estimate of Proposition 6).

The construction of the CGO solution is now analogous to the WKB approximation for the solutions of the wave equation.

Once the limiting Carleman weight φ is fixed, the eikonal equations

$$\nabla\psi(x)^2 = \nabla\varphi(x)^2, \nabla\psi(x)\cdot\nabla\varphi(x) = 0, \tag{19}$$

determine ψ in the following way.

Theorem 8. *Let $\varphi \in C^\infty(\widetilde{\Omega})$ be a limiting Carleman weight on a neighbourhood of $\overline{\Omega}$ and define the hypersurface $G = \varphi^{-1}(C_0)$ for some fixed C_0. Assume that each integral curve of $\nabla\varphi \cdot \nabla$ through a point in Ω also intersects G and that the corresponding projection map $\Omega \to G$ is proper. Then we get a solution for the eikonal equations in $C^\infty(\Omega)$ by solving first $\nabla g(x)^2 = \nabla\varphi(x)^2$ on G and then defining ψ by $\psi_{|G} = g, \nabla\varphi \cdot \nabla\psi = 0$.*

The smooth non-vanishing function $m \in C^\infty$ is determined by the transport equation

$$(\nabla\psi \cdot \nabla + \nabla \cdot \nabla\psi + i(\nabla\varphi \cdot \nabla + \nabla \cdot \nabla\varphi))m = 0. \tag{20}$$

Due to the construction, we obtain that

$$e^{\frac{\varphi}{h}}(-h^2\Delta + h^2 q)e^{-\frac{\varphi}{h}} e^{\frac{i\psi}{h}} m = h^2 d, \tag{21}$$

with $d = O(1)$ in $L^2(\Omega)$.

Finally, the correction term r is constructed via Theorem 7 with φ replaced by $-\varphi$. We are able to find $r \in H^1(\Omega)$ with $h\|r\|_{H^1(\Omega)} \leq Ch^2$ such that

$$e^{\frac{\varphi}{h}}(-h^2\Delta + h^2 q)e^{-\frac{\varphi}{h}} e^{\frac{i\psi}{h}} r = -h^2 d, \tag{22}$$

and as a consequence

$$(-h^2\Delta + h^2 q)(e^{\frac{1}{h}(-\varphi + i\psi)}(m + r)) = 0. \tag{23}$$

The conditions on the different components of the CGO solutions we enlisted at the beginning of this section are then satisfied. In particular

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10 2D Unique Continuation and Quasifonformal - Talk 1: On Landis' Conjecture in the Plane

A summary written by María Ángeles García-Ferrero

Abstract

Landis' conjecture claims that a solution to $\Delta u - Vu = 0$ in $\mathbb{R}^n$ decaying superexponentially must be zero. We see that this holds in $\mathbb{R}^2$ and in an exterior domain for real-valued u and V, provided that $V \geq 0$ a.e. The proof relies on reducing the equation to a Beltrami system and estimates for that.

10.1 Introduction

Let u be a solution of the equation $\Delta u - u = 0$ in $\mathbb{R}^n$ (or outside a compact set). It is well known that if $u(x)$ is bounded and decays as $|x| \to \infty$ at the rate $e^{-(1+\epsilon)|x|}$, $\epsilon > 0$, then $u \equiv 0$. Landis posed the following question: Does this also happen for solutions to $\Delta u - Vu = 0$, with $\|V\|_{L^\infty(\mathbb{R}^n)} \leq 1$?

In $\mathbb{R}^2$, a counterexample is constructed in [3] with complex-valued V and u satisfying $|u(x)| \leq Ce^{-C|x|^{4/3}}$. In the case of real-valued u and V, Landis' conjecture is proved in [2], provided that $V \geq 0$ a.e. The same result holds if we add to the equation a term as $-\nabla(Wu)$ or $W \cdot \nabla v$, with $W = (W_1, W_2)$ a measurable real-valued vector. Actually, the result is given in a quantitative form, meaning that if u decays fast than some rate ($e^{-|z|\log|z|}$), then it must vanish.

Although Carleman estimates have been powerfully used in questions related to Landis' conjecture, here another approach is necessary since they would not distinguish between real- or complex-valued functions. The proof hinges on the relation between second order elliptic equations in the plane and the Beltrami system $\bar{\partial} u = \mu u$ ($\bar{\partial} = \frac{1}{2}(\partial_x + i\partial y)$).

In addition, [2] also provides a proof of Landis's conjecture in an exterior domain, what requires a Carleman estimate for $\bar{\partial}$.

Notation

We will denote by $z = (x, y)$ any point in $\mathbb{R}^2$, and sometimes it can be identified with $z = x + iy$ in $\mathbb{C}$. Clearly, $B_r(z_0)$ will denote the open disc of radius r and centre z_0, and $B_r = B_r(0)$.

The elliptic operators we will deal with in §10.2 are $Lu = \Delta u - \nabla(Wu) - Vu$ and its adjoint $L^*u = \Delta u + W \cdot \nabla u - Vu$. Since we will always be considering L^∞ norms, we will shorten the notation as $\|\cdot\|_\Omega = \|\cdot\|_{L^\infty(\Omega)}$.

10.2 Quantitative Landis' Conjecture in the Plane

The following result proves Landis' conjecture in the plane in quantitative form:

Theorem 1. *Assume that W_1, W_2 and V are measurable, real-valued and $V \geq 0$ a.e. in $\mathbb{R}^2$, furthermore*

$$\|W\|_{\mathbb{R}^2} \leq 1, \qquad \|V\|_{\mathbb{R}^2} \leq 1\,.$$

Let u be a real solution to $Lu = 0$ or $L^*u = 0$ in $\mathbb{R}^2$ such that $|u(z)| \leq e^{C_0|z|}$ and $u(0) = 1$. Then

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq e^{-CR\log R} \tag{1}$$

for $R \gg 1$ and C depending on C_0.

This result follows by a scaling argument from the following estimate for the maximal vanishing order of solutions in a bounded domain:

Theorem 2. *Assume that W_1, W_2 and V are measurable, real-valued and $V \geq 0$ a.e. in B_2, moreover, there exists $K \geq 1$, $M \geq 1$ such that*

$$\|W\|_{B_2} \leq K\,, \qquad \|V\|_{B_2} \leq M.$$

Let u be a real solution to $Lu = 0$ or $L^*u = 0$ in B_2 such that $\|u\|_{B_2} \leq e^{C_0\sqrt{M}+K}$ and $\|u\|_{B_1} \geq 1$. Then

$$\|u\|_{B_r} \geq r^{C\sqrt{M}+K}$$

for all sufficiently small r and C depending on C_0.

Proof of Theorem 1

Theorem 1 is proved from Theorem 2 after a suitable rescaling. Take $R \gg 1$ and z_0 such that $|z_0| = R$. Let us consider the function $u_R(z) = u(z_0 + Rz)$, which satisfies the equation $Lu_R = 0$ or $L^*u_R = 0$ with

$$W_R(x) = RW(z_0 + Rz)\,, \qquad V_R(x) = R^2V(z_0 + Rz)\,,$$

which then verify the estimates $\|W_R\|_{B_2} \leq R$, $\|V_R\|_{B_2} \leq R^2$. In addition, it is easy to see that $\|u_R\|_{B_2} \leq e^{C_0' R}$ and $\|u_R\|_{B_1} \geq 1$.

Consequently, Theorem 2 holds and $\|u_R\|_{B_r} \geq r^{CR}$ for r small enough and $C = C(C_0')$. Taking $r = R^{-1}$ we obtain the desired estimate (1).

Theorem 2 for the Beltrami system

To exemplify the main ideas of the proof of Theorem 2, we start with the simple case of the Beltrami system $\bar{\partial} u - Vu = 0$ in B_2, with $\|V\|_{B_2} \leq M$. Assuming that $\|u\|_{B_2} \leq e^{C_0 M}$ and $\|u\|_{B_1} \geq 1$, our goal is to prove that $\|u\|_{B_r} \geq r^{CM}$ for sufficiently small r and $C = C(C_0)$.

To start, we write the solution as $u = e^w h$, where h is holomorphic in B_2 and $\bar{\partial} w = V$. Hence we can write w as

$$w(z) = -\frac{1}{\pi} \int_{B_2} \frac{V(\xi)}{\xi - z} d\xi \,,$$

and deduce from here that $|w(z)| \leq C\|V\|_{B_2} \leq CM$ for $z \in B_2$.

The rest of the proof lies on the Hadamard's three-circle theorem, which states that if h is an holomorphic function in Ω, then

$$\|h\|_{B_{r_1}} \leq \|h\|_{B_r}^{\theta} \|h\|_{B_{r_2}}^{1-\theta} \tag{2}$$

where $r < r_1 < r_2$, $B_{r_2} \subset \Omega$, and $\theta = \frac{\log \frac{r_2}{r_1}}{\log \frac{r_2}{r}}$.

Taking into account that for any $\rho < 2$

$$e^{-CM} \|u\|_{B_\rho} \leq \|h\|_{B_\rho} \leq e^{CM} \|u\|_{B_\rho} \,,$$

from (2) with $r_1 = 1$ and $r_2 = \frac{3}{2}$ and assumptions on u, we obtain $e^{-CM} \leq \|u\|_{B_r}^{\theta}$, $C = C(C_0)$, and hence $\|u\|_{B_r} \geq r^{CM}$ as we want.

Proof of Theorem 2 with $W = 0$

Following the previous ideas, we prove Theorem 2 without the presence of W, i.e. for the equation $\Delta u - Vu = 0$ in B_2. We start writing the solution as $u = \phi v$, where

$$\Delta \phi - V\phi = 0 \,, \quad e^{-2\sqrt{M}} \leq \phi \leq e^{2\sqrt{M}} \quad \text{ in } B_2 \,,$$

and

$$\nabla \cdot (\phi^2 \nabla v) = 0 \quad \text{ in } B_2 \,. \tag{3}$$

The existence and bounds of ϕ are ensured by the existence of subsolution $e^{\sqrt{M}z}$ and supersolution $e^{2\sqrt{M}}$ and the assumption that $V \geq 0$ a.e.

In order to deal with v, we rewrite (3) as a Beltrami system as follows: Let $\tilde{v}$ be the stream function related to v, i.e. $\nabla^{\perp}\tilde{v} = \phi^2 \nabla v$ (or equivalently $\bar{\partial}\tilde{v} = -i\phi^2\bar{\partial}v$). Let $g = \phi^2 v + i\tilde{v}$, then

$$\bar{\partial}g = (\bar{\partial}\phi^2)v = \bar{\partial}(\log\phi)(g+\bar{g}) = \tilde{\alpha}g \quad \text{in } B_2\,,$$

where

$$\tilde{\alpha} = \begin{cases} (\bar{\partial}\log\phi)\left(1+\frac{\bar{g}}{g}\right) & \text{if } g \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Again, let us write g as $g = he^w$, where $\bar{\partial}w = \tilde{\alpha}$. We can proceed as before provided that we verify a similar estimate for $\tilde{\alpha}$ and we control g by u. For the former, it can be proved that $\|\nabla\log\phi\|_{B_{7/5}} \leq C\sqrt{M}$, which implies the same estimate for $\tilde{\alpha}$. For the latter it is used that $g = \phi u + i\tilde{v}$ and that $\|\tilde{v}\|$ can be also bounded by $\|u\|$.

Combining previous estimates with assumptions on u we see that $e^{-C\sqrt{M}} \leq \|u\|^{\theta}_{B_r}$, and hence $\|u\|_{B_r} \geq r^{C\sqrt{M}}$.

Proof of Theorem 2

In the case that $W = (W_1, W_2)$ does not vanish, the proof follows the same strategy with essentially the following changes:

• ϕ now satisfies $L^*\phi = 0$ and $e^{-2(\sqrt{M}+K)} \leq \phi(z) \leq e^{2(\sqrt{M}+K)}$ for $z \in B_2$.

• If $Lu = 0$, then $v = \frac{u}{\phi}$ verifies $\nabla(\phi^2(\nabla v - Wv)) = 0$ in B_2. We construct g in the same way but now

$$\bar{\partial}g = \left(\bar{\partial}\log\phi + \frac{1}{4}(W_1 + iW_2)\right)(g+\bar{g}) = \tilde{\gamma}g\,.$$

In this case, it can be proved similarly that $\|\bar{\partial}\log\phi\|_{B_{7/5}} \leq C(\sqrt{M}+K)$ and same bound holds for $\tilde{\gamma}$.

• If $L^*u = 0$, then we have $\Delta v + (2\nabla\log\phi + W)\cdot\nabla v = 0$ in B_2, which can be rewritten as $\bar{\partial}(\partial v) = \tilde{W}(\partial v)$. Now $\partial v = e^w h$ with $w \leq C(\sqrt{M}+K)$. Here the tricky (but not difficult) point is to bound $\|\nabla v\|$ from above and below by $\|u\|$.

10.3 Landis' Conjecture in an Exterior Domain

In this section we state Landi's conjecture in an exterior domain in a quantitative form. Although Theorem 2 remains true, the scaling argument fails.

We again reduce our problem to a Beltrami system, in this case inhomogeneous, what requires to use a Carleman estimate for $\bar{\partial}$.

Theorem 3. *Assume V is measurable, real valued and $V \geq 0$ a.e. in B_1^c and satisfies $\|V\|_{B_1^c} \leq 1$. Let u be a real solution of*

$$\Delta u - Vu = 0 \text{ in } B_1^c$$

such that

$$\|u\|_{B_1^c} \leq 1\,, \qquad \inf_{|z_0|=\frac{5}{2}} \int_{B_1(z_0)} |u|^2 \geq C_0\,.$$

Then

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq Ce^{-C' R(\log R)^2} \tag{5}$$

for $R \gg 1$, C depending on C_0 and C' an absolute constant.

Proof of Theorem 3

For $R \gg 1$, let us consider $z_0' = (R + \frac{5}{2})e_1$ and $u_R(z) = u(z_0' + ARz)$. This satisfies the equation $\nabla u_R - V_R u_R = 0$ in $B^c_{\frac{1}{AR}}(z_1)$, where $z_1 = \frac{z_0'}{AR}$, $V_R(z) = (AR)^2 V(z_0' + ARz)$. We take A large enough so $B_{\frac{1}{AR}(z_1)} \subset B_{7/5}$.

Let us write $u_R = \phi v$ as in the previous section. But now the equation $\nabla \cdot (\phi^2 \nabla v) = 0$ is just satisfied in the not simply-connected region $B_2 \backslash B_{\frac{1}{AR}}(z_1)$. Therefore, the stream function may not exist and we need to add a suitable cutoff function χ which vanishes on a neighborhood of $B_{\frac{1}{AR}}(z_1)$:

$$\tilde{v}(x,y) = \int_a^x -[\chi\phi^2\partial_y v](s,y)ds + \int_0^y [\chi\phi^2\partial_x v](a,s)ds\,,$$

where a is taken so that $\chi(a,y) = 1$. Notice that whereas $\partial_x \tilde{v}$ does not differ from the case on B_2 (up to χ), $\partial_y \tilde{v}$ has an extra term coming from ∂_y of the first integral of $\tilde{v}$. Therefore, if $g = \chi\phi^2 v + i\tilde{v}$, $\bar{\partial} g = \tilde{\alpha} g + F$, where $\tilde{\alpha}$ is given by (4) and F depends on $\nabla\chi$.

Similarly, we write $g = e^{-w}h$, with $\bar{\partial} w = -\tilde{\alpha}$ in $B_{7/5}$ and suitable estimate. But h is not longer an holomorphic function: $\bar{\partial} h = e^w F = H_1 + H_2$ in $B_{7/5}$ with

$$H_1 = e^w(\bar{\partial}\chi)\phi u_R\,, \qquad H_2 = \frac{e^w}{2}\int_a^x [\partial_y\chi\phi^2\partial_y v + \partial_x\chi\phi^2\partial_x v](s,y)ds\,.$$

Hadamard's three-circle theorem cannot be applied to h. Instead we use the following Carleman estimate [1, Proposition 2.1]: for any $f \in C_0^\infty(B_{7/5}\backslash\{0\})$,

$$\int |\bar{\partial} f|^2 e^{\varphi_\tau} \geq \frac{1}{4} \int (\Delta\varphi_\tau)|f|^2 e^{\varphi_\tau} = \int |f|^2 e^{\varphi_\tau}\,, \tag{6}$$

where $\varphi_\tau(z) = -\tau \log|z| + |z|^2$, which is decreasing in $|z|$ for $\tau > 8$. We apply it to $f = \zeta h$, where ζ is supported on $\hat{\mathcal{Z}} = \{\frac{1}{4AR} \leq |z| \leq \frac{6}{5}\}$ and $\zeta \equiv 1$ in $\mathcal{Z} = \{\frac{1}{2AR} \leq |z| \leq 1\}$. Taking A and R large enough then (6) becomes

$$\begin{aligned}\int_{B_{\frac{1}{AR}}(-\frac{e_1}{A})} |h|^2 e^{\varphi_\tau} &\leq \int_{\mathcal{Z}} |h|^2 e^{\varphi_\tau} \leq 2\int_{\hat{\mathcal{Z}}} \big(|\bar{\partial}\zeta h|^2 + |\zeta\bar{\partial} h|^2\big) e^{\varphi_\tau} \\ &\leq C(AR)^2 \int_{\mathcal{X}} |h|^2 e^{\varphi_\tau} + C\int_{\mathcal{Y}} |h|^2 e^{\varphi_\tau} + \int_{\hat{\mathcal{Z}}} (|H_1|^2 + |H_2|^2) e^{\varphi_\tau}\,,\end{aligned}$$

with $\mathcal{X} = \big\{\frac{1}{4AR} \leq |z| \leq \frac{1}{2AR}\big\}$ and $\mathcal{Y} = \big\{1 \leq |z| \leq \frac{6}{5}\big\}$.

Each term can be bounded properly through a careful analysis of the supports and the behaviour of their functions. Going back to the original function u and using assumptions on it, it can be seen that taking τ suitably and R large enough, the three last terms on the r.h.s. can be absorbed by the l.h.s. whereas $\int_{\mathcal{X}} |h|^2 e^{\varphi_\tau}$ involves $\int_{B_1(z_0)} |u|^2$, with $z_0 = Re_1$, which finally leads to

$$\int_{B_1(z_0)} |u|^2 \geq CC_0 e^{-CAR(\log(AR))^2}\,,$$

and hence (5) holds.

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11 L^2 Carleman estimates - Talk 2: Backwards Uniqueness for the Heat Equation in an Exterior Domain

A summary written by Max Hallgren

Abstract

Two L^2 Carleman estimates are proved for the backwards heat operator in $(\mathbb{R}^n \setminus B_R) \times [0,T]$. These estimates are then used to prove backwards uniqueness for functions satisfying general growth conditions and solving a heat-type equation.

11.1 Introduction

The focus of the paper is on solutions of the backwards heat equation on the exterior domain $Q_{R,T} := (\mathbb{R}^n \setminus \bar{B}_R) \times [0,T]$. The main goal is the following theorem.

Theorem 1. *Suppose $u \in C^\infty(Q_{R,T})$ satisfies $u(\cdot,0) = 0$ in $\mathbb{R}^n \setminus \bar{B}_R$ and*

$$|(\partial_t + \Delta)u| \leq C(|u| + |\nabla u|), \qquad |u(x,t)| \leq Ce^{C|x|^2}$$

in $Q_{R,T}$. Then $u = 0$ in $Q_{R,T}$.

Remark 2. *A similar problem was previous solved independently by C.C. Poon and X.Y. Chen, where $Q_{R,T}$ is replaced by $\mathbb{R}^n \times [0,T]$, without using Carleman estimates. That result is a corollary of the main theorem here by letting $R \to 0$.*

Remark 3. *Note that this theorem is false, even with stronger assumptions, if $\partial_t + \Delta$ is replaced by $\partial_t - \Delta$, as the heat kernel $p(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}$ is a counterexample.*

11.2 Carleman-Type Inequalities for the Backwards Heat Equation

The first goal is to prove a L^2 Carleman inequality for the backwards heat operator. In particular, we want to bound weighted L^2 norms of $u \in C_c^\infty(\mathbb{R}^n \times [0,1))$ and $|\nabla u|$ in terms of the weighted L^2 norm of $(\partial_t + \Delta)u$.

Theorem 4. *(First Carleman Estimate)There exists $\alpha_0 = \alpha_0(R, n) < \infty$ such that for all $\alpha \geq \alpha_0$ and all $u \in C_c^\infty(Q_{R,T})$ satisfying $u(\cdot, 0) = 0$, we have*

$$\begin{aligned}||e^{\alpha(T-t)(|x|-R)+|x|^2}u||_{L^2(Q_{R,T})} + ||e^{\alpha(T-t)(|x|-R)+|x|^2}\nabla u||_{L^2(Q_{R,T})} \\ \leq ||e^{\alpha(T-t)(|x|-R)+|x|^2}(\partial_t u + \Delta u)||_{L^2(Q_{R,T})} + ||e^{|x|^2}\nabla u(\cdot, T)||_{L^2(\mathbb{R}^n \setminus \bar{B}_R)}.\end{aligned}$$

For $G \in C^\infty(Q_{R,T})$, define $F := (\partial_t G - \Delta G)/G$. The $L^2(Gdxdt)$-self-adjoint part of $\partial_t + \Delta$ is

$$S = \Delta + \nabla \log G \cdot \nabla - \frac{1}{2}F,$$

and the $L^2(Gdxdt)$-skew-adjoint part of $\partial_t + \Delta$ is

$$A = \partial_t - \nabla \log G \cdot \nabla + \frac{1}{2}F.$$

We may compute the principal symbol of the commutator: by [2],

$$\sigma_{[S,A]}(x, t, \xi, \tau) = \{\sigma_S, \sigma_A\}(x, t, \xi, \tau) = -\nabla^2 \log G(\xi, \xi),$$

which suggests that (up to lower order terms) if G is log-convex, it should be possible to prove a priori estimates for $[S, A]$. In fact, for carefully chosen G, an elementary integration by parts argument shows that $u \mapsto \langle Su, Au\rangle_{L^2(Gdxdt)}$ has strong positivity properties:

$$\begin{aligned}\langle Su, Au\rangle_{L^2(Gdx)}(t) = \frac{1}{2}\int_{\mathbb{R}^n \setminus B_R} u^2(\partial_t F + \Delta F)Gdx - \int_{\mathbb{R}^n \setminus \bar{B}_R} |\nabla u|^2 Gdx \quad (1) \\ + 2\int_{\mathbb{R}^n \setminus B_R} \nabla^2 \log G(\nabla u, \nabla u)Gdx - \int_{\mathbb{R}^n \setminus \bar{B}_R} u^2 FGdx.\end{aligned}$$

Since $||(\partial_t+\Delta)u||^2_{L^2(Gdxdt)} = ||Au||^2_{L^2(Gdxdt)} + ||Su||^2_{L^2(Gdxdt)} + 2\langle Au, Su\rangle_{L^2(Gdxdt)}$, the desired Carleman inequality will follow (by integrating (1) from $t = 0$ to $t = T$) from finding G such that $F \leq 0$, $\partial_t F + \Delta F \geq 1$, and $\nabla^2 \log G \geq I$. The functions $G(x, t) := e^{2\alpha(T-t)(|x|-R)+2|x|^2}$ satisfy these properties for large $\alpha > 0$. The parameter α will be important later for dealing with our lack of information about u near $(\partial B_R) \times [0, T]$, making use of the important fact that, for fixed $t_1 < t_2$, $G(x, t_1)/G(x, t_2) \to \infty$ as $\alpha \to \infty$.

For the second Carleman estimate, we need to define the auxiliary functions $\sigma(t) := te^{-\frac{t}{3}}$ and $\sigma_a(t) := \sigma(t + a)$.

Theorem 5. *(Second Carleman Estimate) There exists $N = N(n) < \infty$ such that, for any $\alpha \geq 0$, $a \in (0,1)$, $y \in \mathbb{R}^n$, and $u \in C_c^\infty(\mathbb{R}^n \times [0,1))$ satisfying $u(\cdot,0) \equiv 0$, we have*

$$||\sigma_a^{\alpha-1/2} e^{-\frac{|x-y|^2}{8(t+a)}} u||_{L^2(\mathbb{R}^n\times(0,1)} + ||\sigma_a^{\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} \nabla u||_{L^2(\mathbb{R}^n\times(0,1)} \leq N||\sigma_a^{\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} (\partial_t u + \Delta u)||_{L^2(\mathbb{R}^n\times(0,1)}.$$

Note that in this case, the Gaussian weight $x \mapsto e^{-\frac{|x-y|^2}{8(t+a)}}$ is in fact log-concave rather than log-convex, so the σ^α term is essential for the second Carleman estimate. Set $G_a(x,t) := (4\pi(t+a))^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t+a)}}$, and apply (1) with G replaced by $\sigma_a^{-\alpha} G_a$. Assuming that $u(\cdot,0) \equiv 0$ and $u \in C_c^\infty(\mathbb{R}^n \times [0,1)$, multiplying (1) by $\sigma/\dot\sigma$, and integrating from $t = 0$ to $t = 1$ leads to

$$\int_{\mathbb{R}^n\times[0,1]} \frac{\sigma_a^{1-\alpha}}{\dot\sigma_a}(Su)(Au)G_a dxdt \tag{2}$$
$$= \int_{\mathbb{R}^n\times(0,1)} \frac{\sigma_a^{1-\alpha}}{\dot\sigma_a}\left(\left(\log(\frac{\sigma_a}{\dot\sigma_a})\right)' I + 2\nabla^2 \log G_a\right)(\nabla u, \nabla u)G_a dxdt.$$

We used here that G_a is an exact solution of the heat equation. Because (this is where the e^{-3t} term in σ is important)

$$\left(\log(\frac{\sigma_a}{\dot\sigma_a})\right)' I + 2\nabla^2 \log G_a = \frac{1}{3}\frac{1/3}{1-(t+a)/3)} I \geq \frac{1}{3} I$$

and $\frac{1}{3e} \leq \dot\sigma_a(t) \leq 1$ for $t \in [0,1]$, we conclude that

$$||\sigma_a^{-\alpha} G_a^{\frac{1}{2}} \nabla u||_{L^2(\mathbb{R}^n\times[0,1])} \leq N'||\sigma_a^{-\alpha} G_a^{\frac{1}{2}} (\partial_t u + \Delta u)||_{L^2(\mathbb{R}^n\times[0,1])}.$$

where we replaced α with $1 + 2\alpha$. Integrating $(\partial_t + \Delta)u^2 = 2u(\partial_t + \Delta u) + 2|\nabla u|^2$ against the weight $\sigma_a^{-2\alpha} G_a$, and integrating by parts and using Cauchy-Schwartz gives the claim. Note that here the nice formula for the heat operator applied to the weights rescues us, when a Poincare inequality in the unbounded domain would not be helpful.

11.3 Proving Quadratic Exponential Decay

We first recall two a priori estimates for solutions of second order parabolic equations.

Lemma 6. *(Gradient Estimate for the Nonhomogeneous Heat Equation) Set* $P_r := B_r \times [-r^2, 0)$ *for* $r > 0$*. There exists* $C^* = C^*(n) < \infty$ *such that, for any* $u \in C^\infty(P_r)$*, we have*

$$\sup_{(x,t)\in P_r} d_{x,t}|\nabla u(x,t)| \leq C^* \sup_{(x,t)\in P_r} (|u(x,t)| + d_{x,t}^2|(\partial_t + \Delta)u(x,t)|),$$

where $d_{x,t} := \min\{d(x, \partial B_r), |t|\}$.

Proof. Use a parabolic barrier function and the maximum principle. $\square$

Lemma 7. *(Parabolic Mean Value Inequality) There exists* $C = C(n) < \infty$ *such that, for any* $s > 0$ *and* $u \in C^\infty(B_{\sqrt{s}}(y) \times [s, 2s])$ *satisfying* $|(\partial_t + \Delta)u| \leq |u| + |\nabla u|$*, we have*

$$|u(y,s)|^2 \leq \frac{C}{s^{n+2}} \int_s^{2s} \int_{B_{\sqrt{s}}(y)} u^2 dxdt.$$

Proof. Use parabolic Moser iteration on u_+, u_-. $\square$

Lemma 8. *There exists* $\epsilon = \epsilon(n) > 0$ *and* $M = M(n) < \infty$ *such that the following holds. Suppose* $u \in C^\infty(Q_{R,1})$ *satisfies*

$$|\partial_t u + \Delta u| \leq \epsilon(|u| + |\nabla u|), \qquad |u(x,t)| \leq e^{\epsilon|x|^2},$$

and $u(\cdot, 0) = 0$ *in* $\mathbb{R}^n \setminus \bar{B}_R$ *for some* $R \geq 1$*. Then*

$$|u(y,s)|+|\nabla u(y,s)| \leq Me^{-\frac{|y|^2}{Ms}}(1+||u||_{L^\infty((B_{4R}\setminus B_R)\times(0,1))}) \qquad \textit{for } (y,s) \in Q_{6R,M^{-1}}.$$

Proof. Fix $y \in \mathbb{R}^n \setminus 6R$. The strategy is to first obtain an $L^2(dxdt)$ estimate on a forwards parabolic cylinder around y, by applying the second Carleman inequality with the Gaussian weight centered at y. This gives a good L^2 bound for u near y since the Gaussian weight is bounded below near y, while $\sigma^{-\alpha}(t)$ is large for t small. The claim then follows immediately from the above parabolic regularity theorems.

To get a right hand side of the Carleman inequality we can estimate effectively, we need to cutoff u appropriately, so that we only have to estimate $G_a\sigma_a^{-2\alpha}u^2$ where the weight $\sigma_a^{-2\alpha}G_a$ is relatively small. Define $u_r(x,t) = u(x,t)\varphi(t)\psi_r(x)$, where $\phi \in C^\infty(\mathbb{R})$ and $\psi_r \in C_c^\infty(\mathbb{R}^n)$ satisfy $\phi = 1$ on $(-\infty, 1/2]$, $\phi = 0$ on $[3/4, \infty)$, $\psi_r = 1$ on $B_{2r} \setminus B_{3R}$, $\psi_r = 0$ outside $B_{3r} \setminus B_{2R}$. Then

$$|(\partial_t + \Delta)u_r| \le \epsilon(|u_r| + |\nabla u_r|) + |\varphi' u| + \varphi(|u|(|\Delta\psi_r| + |\nabla\psi_r|) + 2|\nabla\psi_r| \cdot |\nabla u|),$$

so applying the first Carleman estimate gives

$$\begin{aligned}&||\sigma_a^{-\alpha-\frac{1}{2}}G_a u_r||_{L^2(\mathbb{R}^n\times[0,1])} + ||\sigma_a^{-\alpha}G_a\nabla u_r||_{L^2(\mathbb{R}^n\times[0,1])} \\ &\qquad \le C(n)(||\sigma_a^{-\alpha}G_a u||_{L^2((\mathbb{R}^n\setminus B_R)\times[\frac{1}{2},\frac{3}{4}]} + ||\sigma_a^{-\alpha}G_a(|u| + |\nabla u|)||_{L^2((A_1\cup A_2)\times[0,3/4]}),\end{aligned}$$

where $A_1 := B_{3R} \setminus B_{2R}$ and $A_2 := B_{3r} \setminus B_{2r}$. Now apply the gradient estimate for u to get (at scale 1, and assuming $\epsilon < \frac{1}{2}C^*$) to get $|\nabla u(x,t)| \le C(n)e^{\epsilon|x|^2}$, so the integral over A_2 vanishes as we let $r \to \infty$, for ϵ small. Also, we know $y \notin A_1$, so the right hand side stays bounded as $a \to 0$, and the left hand side converges by the monotone convergence theorem. Applying the gradient estimate in A_1 gives $M(n) < \infty$ such that

$$||\sigma^{-\alpha}G^{\frac{1}{2}}(|u|+|\nabla u|)||_{L^2(A_1\times[0,3/4])} \le M^\alpha \left(\sup_{t>0} t^{-\alpha}e^{-\frac{|y|^2}{16t}}\right)||u||_{L^\infty((B_{4R}\setminus B_R)\times[0,1]}),$$

and completing the square gives

$$||\sigma^{-\alpha}G^{\frac{1}{2}}u||_{L^2((\mathbb{R}^n\setminus B_R)\times[\frac{1}{2},\frac{3}{4}]} \le M^\alpha e^{|y|^2}.$$

By Stirling's formula,

$$\sup_{t>0} t^{-k}e^{-\frac{|y|^2}{16t}} = |y|^{-2k}(16k)^k e^{-k} \le |y|^{-2k}M^k k!,$$

so we can take $\alpha = k$, multiply by $|y|^{2k}(2M)^{-k}/k!$, and sum to get

$$||e^{\frac{|y|^2}{4Mt}}G^{\frac{1}{2}}u||_{L^2(Q_{3R,(8M)^{-1}})} \le C(n)(1 + ||u||_{L^\infty((B_{4R}\setminus B_R)\times[0,1]}).$$

$\square$

11.4 Completing the Proof of Theorem 1

Lemma 9. *With the same hypotheses as the previous lemma, we have $u = 0$ in $Q_{R,\epsilon}$.*

Proof. We now apply the second Carleman inequality to $u_{a,r} = u\psi_{a,r}$, where $\psi_{a,r} \in C_c^\infty(B_{2r} \setminus B_{(1+a)R})$, $|\nabla \psi a, r| \leq C(n)a^{-1}$, and $\psi_{a,r} = 1$ on $B_r \setminus B_{(1+2a)R}$. Take $T = 4\epsilon$ to get

$$\begin{aligned}
e^{10\alpha\epsilon aR}||u||_{L^2((B_r\setminus B_{(1+10a)R})\times[0,\epsilon]} &\leq C(n)e^{8\alpha\epsilon r+4r^2}|||u| + |\nabla u|||_{L^2((B_{2r}\setminus B_r)\times[0,4\epsilon])} \\
&\quad + C(a,n)e^{8\alpha\epsilon aR}|||u| + |\nabla u|||_{L^2((B_{(1+2a)R}\setminus B_{(1+a)R})\times[0,4\epsilon]} \\
&\quad + C(n)||e^{|x|^2}(|u| + |\nabla u|)||_{L^2(B_{\mathbb{R}^n\setminus B_R})}.
\end{aligned}$$

Note that we traded integrating over a larger region in return for a better exponent on the left hand side. Dealing with the gradient terms as before, we obtain

$$||u||_{L^2((B_r\setminus B_{(1+10a)R})\times[0,\epsilon])} \leq C \cdot (e^{\alpha r - r^2} + e^{-2\alpha\epsilon aR}).$$

Let $r \to \infty$, then $\alpha \to \infty$, then $a \to 0$. ☐

Proof of Theorem 1 Now we finish the proof of Theorem 1. By parabolic rescaling, we can assume the hypotheses of the previous lemmas (including $T \geq 1$), so $u = 0$ on $Q_{R,\epsilon}$. Repeat with ϵ as the new initial time, and keep repeating to get $u = 0$ on $Q_{R,a}$, where $T - a < 1$. Rescaling so that a is the initial time and T becomes 1, and applying the previous lemma gives $u = 0$ on $Q_{R,a+(T-a)\epsilon}$. Iterate, and see that we have $u = 0$ outside some region whose time interval is decreasing geometrically, hence $u = 0$ on all of $Q_{R,T}$. ☐

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12 L^P Carleman estimates and the Osculation Techniques - Talk 1: Carleman estimates and unique Continuation for second order elliptic equations with nonsmooth coefficients (SUCP, Talk 1)

A summary written by Zihui He

Abstract

This talk will show local Carlemen estimate in special case $P = \Delta$ with a radially symmetric exponential weight. Based on the Carleman estimates we can prove the strong unique continuation property for the corresponding second order elliptic operators.

12.1 Introduction

Consider the second order elliptic operator

$$P = \partial_i g^{ij}(x) \partial_j \tag{1}$$

in $\mathbb{R}^n$, the potential V and the vector fields W_1 and W_2. To these we associate the differential equation

$$Pu = Vu + W\nabla u + \nabla(W_2 u) \tag{2}$$

Definition 1. *Given a function $u \in L^2_{loc}$ and $x_0 \in \mathbb{R}^n$ we say that u vanishes of infinite order at x_0 if there exits $\mathcal{R}$ so that for each integer N we have*

$$\int_{B_r(x_0)} |u|^2 dx \leq c_N r^N, r < \mathcal{R} \tag{3}$$

Definition 2. *We say the problem (1) has the strong unique continuation property* (**SUCP**) *if for every H^1 function u satisfying (1) in a ball $B_R(x_0)$ the following is ture, if u vanishes of infinite order at x_0 the u=0 near x_0.*

We consider the following assumptions of metrics g, potentials V, W_1 and W_2, we consider metrics g uniformly bounded frrom above and below and satisfying

$$\| |x| \nabla g \|_{l^1_w(L^\infty)} < \epsilon, \epsilon small \tag{4}$$

this doesn not imply that g is close to rhe Euclidean metric. However, in our estimates later on we use a perturbation argument starting from estimates for the Euclidean metric. This requires a stronger form of (4), namely

$$\|g - I_n\|_{l^1_w(L^\infty)} + \||x|\nabla g\|_{l^1_w(L^\infty)} < \epsilon, \epsilon small \tag{5}$$

The reduction of (4) to (5) is carried out by using a suitable change of coordinates.
For potentials V, W_1 and W_2 we consider the following assumptions:

$$V \in l^\infty(L^{\frac{n}{2}}), \limsup_{r\to 0} \|V\|_L^{\frac{n}{2}}(\{r \leq |x| \leq 2r\}) \leq \epsilon, \epsilon small. \tag{6}$$

respectively,

$$\|W_1\|_{l^1_w(L^n)} + \|W_2\|_{l^1_w(L^n)} \leq \epsilon, \epsilon small. \tag{7}$$

A simple replacement of (4), (6) and (7) is,

$$|x||\nabla g| \in l^1(L^\infty), V \in c_0(L^{\frac{n}{2}}), W_1, W_2 \in l^1(L^n). \tag{8}$$

Now we state our main result.

Theorem 3. *Assume that (4), (6) and (7) hold. Then* (**SUCP**) *holds at 0 for H^1 solution u to (1).*

12.2 Carleman estimate

Recall the estimate of Jerison and Kenig [3],

$$\||x|^{-\tau}u\|_{L^p} \lesssim \||x|^{-\tau}\triangle u\|_{L^{p'}} \tag{9}$$

where p and p' are dual exponents satisfying the gap condition

$$\frac{1}{p'} - \frac{1}{p} = \frac{2}{n} \tag{10}$$

and for all u vanishing of infinite order at 0 and ∞, τ away from $\pm(\frac{n-2}{2} + \mathbb{N})$.

Theorem 4. *Assume that (5) holds. Then for each $\tau > 0$ there exists a convex function h satisfying $h' \in [\tau, \tau^2]$ so that*

$$\|e^{\varphi(x)}u\|_{l^{p'}(L^p)} \lesssim \|e^{\varphi(x)}P(x,\partial)u\|_{L^{p'}} \tag{11}$$

and for all u vanishing of infinite order at 0 and ∞.

Theorem 5. *Assume that (5) holds. Then for each $\tau > 0$, $W_1, W_2 \in l^1_w(L^n)$ and each function u vanishing of infinite order at o and ∞ there exists a function φ satisfying*

$$\tau \leq -r\partial_r\varphi \leq \tau^2, |\partial_\theta\varphi| \leq |r\partial_r\varphi| \tag{12}$$

so that

$$\|e^{\varphi(x)}u\|_{l^{p'}(X_\varphi)} + \frac{\|e^{\varphi(x)}W_1\nabla u\|_{L^{p'}}}{\|W_1\|_{l^1_w(L^n)}} + \frac{\|e^{\varphi(x)}W_2 u\|_{|\nabla\varphi|^{-1}L^{p'}}}{\|W_2\|_{l^1_w(L^n)}} \lesssim \|e^{\varphi(x)P(x,\partial)u}\|_{l^{p'}(X'_\varphi)}. \tag{13}$$

We can easily obtain the following corollary as a consequence of (12):

Corollary 6. *Assume that (4), (6) and (7) hold. Then for each $\tau > 0$ amd each function u vanishing of infinite order at 0 and ∞ which solves*

$$P(x,\partial)u - (Vu + W_1\nabla u + \nabla W_2 u) = f \tag{14}$$

there exists a function φ satisfying

$$\tau \leq -r\partial_r\varphi \leq \tau^2, |\partial_\theta\varphi| \leq |r\partial_r\varphi| \tag{15}$$

so that

$$\|e^{\varphi(x)}u\|_{L^p} \lesssim \|e^{\varphi(x)}f\|_{L^{p'}} \tag{16}$$

This implies the desired unique continuation result.

12.3 Polar coordinates and estimates for the flat case

We start from local estimate in special case $P = \Delta$ with a radially symmetric exponential weight. Introduce the polar coordinates

$$x = e^{-s}\theta, (s,\theta) \in \mathbb{R} \times \mathbb{S}^{n-1} := \mathcal{C} \tag{17}$$

We introduce the space $\widetilde{X}_{\tau,\epsilon}$of functions define onthe cylinder $\mathcal{C}$:

$$\widetilde{X}_{\tau,\epsilon} = \{v \in L^p \cap \tau^{-\frac{1}{2}}(1+\epsilon\tau)^{-\frac{1}{4}}L^2, \nabla v \in L^2 + \tau L^p \cap \tau^{\frac{1}{2}}(1+\epsilon\tau)^{-\frac{1}{4}}L^2\}$$

For the right hand side of the equation we use the dual space,

$$\widetilde{X}'_{\tau,\epsilon} = L^{p'} + \tau^{\frac{1}{2}}(1+\epsilon\tau)^{-\frac{1}{4}}L^2 + \nabla(L^2 \cap \tau^{-1}L^{p'}) + \tau^{-\frac{1}{2}}(1+\epsilon\tau)^{-\frac{1}{4}}L^2$$

We set,

$$\|u\|_{X_{\tau,\epsilon}} = \||x|^{\frac{n-2}{2}} u\|_{\widetilde{X}_{\tau,\epsilon}}$$

$$\|g\|_{X'_{\tau,\epsilon}} = \||x|^{\frac{n+2}{2}} g\|_{\widetilde{X}'_{\tau,\epsilon}}$$

Proposition 7. *Let $\tau \gg 1$. We consider a convex function h satisfying*

$$|h''| + dist(2h', \mathbb{Z}) \geq \frac{1}{4}$$

for which $|h'| \in [\tau, 2\tau]$. Then

$$\|e^{h(-in(|x|))} u\|_{X_{\tau,0}} \lesssim \|e^{h(-in(|x|))} \Delta u\|_{X'_{\tau,0}} \tag{18}$$

for all functions u vanishing of infinite order at 0 and ∞.

We consider in more detail the case when h is uniformly convex in some region:

Proposition 8. *Let $\tau^{-1} < \epsilon < 1$. We consider a convex function h satisfying*

$$|h'| \in [\tau, 2\tau], h'' \in [\epsilon\tau, \tau]$$

in some interval I. Then

$$\|e^{h(-in(|x|))} v\|_{X_{\tau,\epsilon}} \lesssim \|e^{h(-in(|x|))} \Delta v\|_{X'_{\tau,\epsilon}} \tag{19}$$

for all functions v supported in $\{x : |x| \in e^{-I}\}$.

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13 A nonlinear Plancherel theorem with applications to global well-posedness for the defocusing Davey-Stewartson equation and to the inverse boundary value problem of Calderón - Talk 2: Profile decomposition and Section 4

A summary written by Xian Liao

Abstract

In this section we study the scattering transform associated to the defocusing Davey-Stewartson equations, making use of the profile decomposition technics and the estimates for the fractional integrals and for the pseudo-differential operators with non-smooth symbols.

13.1 Introduction

We study the Cauchy problem for the integrable defocusing Davey-Stewartson equations for the unknown $q : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{C}$

$$\begin{cases} i\partial_t q + 2(\partial_{\bar{z}}^2 + \partial_z^2)q + q(g + \bar{g}) = 0, \\ \partial_{\bar{z}} g + \partial_z(|q|^2) = 0, \\ q(0, z) = q_0(z), \end{cases} \tag{1}$$

by means of the inverse scattering method. More precisely, we first solve the following two linear equations for the unknowns $m_\pm : \mathbb{C} \mapsto \mathbb{C}$

$$\partial_{\bar{z}} m_\pm = \pm e_{-k} q \overline{m_\pm} \text{ with } m_\pm(z) \to 1 \text{ as } |z| \to \infty, \quad e_k(z) := e^{i(zk + \overline{zk})}, \tag{2}$$

where $q(z)$ and $k \in \mathbb{C}$ can be viewed as the given potential function and the parameter respectively, such that we define the scattering transform of $q(z)$ as

$$\mathbf{s}(k) := \mathcal{S}q(k) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} e_k(z) \overline{q(z)} \big(m_+(z, k) + m_-(z, k)\big) dz, \tag{3}$$

where $z = x_1 + ix_2$ and $dz = dx_1 dx_2$. We then proceed as follows to solve the Cauchy problem (1):

$$\begin{cases} \mathbf{s}_0(k) = \mathcal{S}q_0(k), \\ \mathbf{s}(t, k) = e^{2i(k^2 + \bar{k}^2)t} \mathbf{s}_0(k), \\ q(t, z) = \mathcal{S}^{-1}(\mathbf{s}(t, k)). \end{cases} \tag{4}$$

We take the L^2-framework and aim to show the global well-posedness of Davey-Stewartson equations (1) in $L^2(\mathbb{R}^2)$ by means of the above procedure (4). To this end, we first solve the linear equations (2) and more generally we consider the following d-bar problem:

$$L_q u := \bar{\partial} u + q\bar{u} = f, \quad q \in L^2, \tag{5}$$

and we have

Theorem 1. *For each $f \in \dot{H}^{-\frac{1}{2}}$, there exists a unique solution $u \in \dot{H}^{\frac{1}{2}}$ of the inhomogeneous problem* (5) *with*

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2})\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{6}$$

We then prove the Plancherel theorem for the nonlinear scattering transform in $L^2(\mathbb{R}^2)$ as follows:

Theorem 2. *The nonlinear scattering transform $\mathcal{S} : q \mapsto \mathbf{s}$ is a C^1 diffeomorphism from L^2 to L^2, satisfying*

- *The Plancherel Identity $\|\mathcal{S}q\|_{L^2} = \|q\|_{L^2}$;*
- *The pointwise bound: $|\mathcal{S}q(k)| \leq C(\|q\|_{L^2})M\hat{q}(k)$, a.e. k, where M denotes the Hardy-Littlewood maximal function;*
- *Locally uniform bi-Lipschitz continuity:*

$$\begin{aligned}&\frac{1}{C}\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2} \leq \|q_1 - q_2\|_{L^2} \leq C\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2},\\ &C = C(\|q_1\|_{L^2})C(\|q_2\|_{L^2});\end{aligned}$$

- *Bound on the derivative: $\|D\mathcal{S}q\|_{L^2 \mapsto L^2} \leq C(\|q\|_{L^2})$;*
- *Inversion Theorem: $\mathcal{S}^{-1} = \mathcal{S}$.*

As a consequence of Theorem 2 we have the bound on $q(z)$ in terms of the Fourier transform of its scattering transform $\mathbf{s}(k) = \mathcal{S}q(k)$:

$$|q(z)| = |\mathcal{S}^{-1}\mathbf{s}(z)| = |\mathcal{S}\mathbf{s}(z)| \leq C(\|\mathbf{s}\|_{L^2})M\hat{\mathbf{s}}(z) = C(\|q\|_{L^2})M\left(\widehat{e^{2it(k^2+\bar{k}^2)}\mathcal{S}q_0}\right)(z).$$

The above bound allows us to transfer Strichartz estimates on the linearization of the Davey-Stewartson equations (1) to bounds on the nonlinear flow, which implies the global well-posedness and scattering results.

13.2 The d-bar problem

In this subsection we solve the d-bar problem (5) such that the bound (6) holds true.

13.2.1 Solvability of (5)

Recall the following solvability results for (5) (see the Lemmas at the beginning of Section 3 [1]):

Lemma 3. *Let $q \in L^2$. Then the operator $L_q : \dot{H}^{\frac{1}{2}} \mapsto \dot{H}^{-\frac{1}{2}}$ via $L_q u = \bar{\partial} u + q\bar{u}$ is invertible and there exists a constant $C = C(q)$ such that*

$$\|L_q^{-1} f\|_{\dot{H}^{\frac{1}{2}}} \leq C(q) \|f\|_{\dot{H}^{-\frac{1}{2}}}.$$

Furthermore, for any $q_0 \in L^2$ there exists $\epsilon > 0$ (depending only on $C(q_0)$) such that the map $q \mapsto L_q^{-1}$ is analytic on the ball $B_\epsilon(q_0) = \{q \in L^2 \mid \|q - q_0\|_{\dot{B}_\infty^{-\frac{1}{3},3}} \leq \epsilon\}$ and

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\|_{\dot{H}^{-\frac{1}{2}} \mapsto \dot{H}^{\frac{1}{2}}} \lesssim (C(q_0))^2 \|q_1 - q_2\|_{\dot{B}_\infty^{-\frac{1}{3},3}}, \quad \forall q_1, q_2 \in B_\epsilon(q_0),$$

where $\|f\|_{\dot{B}_\infty^{-\frac{1}{3},3}} = \sup_{k \in \mathbb{Z}} (2^{-\frac{1}{3}k} \|P_k f\|_{L^3})$ with P_k denoting the Littlewood-Paley projectors is the homogeneous Besov norm and $L^2(\mathbb{R}^2) \subset \dot{B}_\infty^{-\frac{1}{3},3}(\mathbb{R}^2)$.

13.2.2 A contradiction argument

We will use a contradiction argument to show $C(q) = C(\|q\|_{L^2})$, which together with Lemma 3 above implies Theorem 1.

Let
$$C(R) = \sup\{C(q) \mid \|q\|_{L^2} \leq R\} \quad C : \mathbb{R}^+ \mapsto [0, \infty],$$
then the function C is nondecreasing and continuous. We aim to show that

$$C(R) < +\infty \text{ for all } R > 0.$$

We argue by contraction. Let $R_0 \in (0, \infty)$ be the minimal constant such that $C(R_0) = \infty$. Then there exists a bounded sequence $\{q_n\} \subset L^2$ such that

$$R_0 > \|q_n\|_{L^2} \to R_0 \text{ and } \|L_{q_n}^{-1}\|_{\dot{H}^{-\frac{1}{2}} \mapsto \dot{H}^{\frac{1}{2}}} \to \infty. \tag{qn}$$

Taking into account of the symmetry of the function $C(q)$: $C(q) = C(S(\lambda, y)q)$, $S(\lambda, y)q = \lambda q(\lambda(\cdot - y))$, if there exist sequences (λ_n, y_n) such that

$$S(\lambda_n, y_n)q_n \to q^1 \text{ in } \dot{B}_\infty^{-\frac{1}{3},3}, \quad q^1 \in L^2, \tag{1p}$$

then by the estimates in Lemma 3,

$$\|L^{-1}_{S(\lambda_n,y_n)q_n}\|_{\dot{H}^{-\frac{1}{2}} \mapsto \dot{H}^{\frac{1}{2}}} \to \|L^{-1}_{q^1}\|_{\dot{H}^{-\frac{1}{2}} \mapsto \dot{H}^{\frac{1}{2}}} < +\infty,$$

which is a contradiction to (qn). We are going to make this argument more precise by use of profile decomposition technics below.

13.2.3 Profile decomposition

Proposition 4. *Let $\{q_n\}$ be a bounded sequence in L^2. Then up to the extraction of subsequence, for any $l \in \mathbb{N}$, the sequence can be decomposed as $q_n = \sum_{k=1}^l S(\lambda_n^k, y_n^k)q^k + q_n^l$, where*

$$\|q_n\|_{L^2}^2 = \sum_{k=1}^l \|q^k\|_{L^2}^2 + \|q_n^l\|_{L^2}^2 + o_n(1), \quad \lim_{l\to\infty} \limsup_{n\to\infty} \|q_n^l\|_{\dot{B}_\infty^{-\frac{1}{3},3}} = 0,$$

and the scale and core sequences (λ_n^k, y_n^k) satisfy that whenever $j \neq k$ then

$$\lim_{n\to\infty} \Big(\frac{\lambda_n^k}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^k} \Big) = \infty \quad \text{or} \quad \lambda_n^j = \lambda_n^k, \ \lim_{n\to\infty} |y_n^j - y_n^k| \lambda_n^j = \infty.$$

Let $\{q_n\}$ be the bounded sequence in (qn) and we do the above profile decomposition to it. Then as q_n^l plays a perturbative role, we will simply fix l and assume $q_n = \sum_{k=1}^l S(\lambda_n^k, y_n^k)q^k$. If $l = 1$, then (1p) holds and the contradiction argument works. We hence consider the case $l \geq 2$ such that

$$\sup_k \|q^k\|_{L^2} = R < R_0. \tag{7}$$

We choose a slowly increasing sequence $\{\mu_n\}$ and decompose f as

$$f = \sum f_n^k + f_n^{out}, \quad f_n^k = T^{\mu_n}(\lambda_n^k, y_n^k)f = \chi(\mu_n^{-2}\lambda_n^k(\cdot - y_n^k))P_{[\lambda_n^k/\mu_n, \lambda_n^k\mu_n]}(f),$$

such that f_n^k primarily interact only with $S(\lambda_n^k, y_n^k)q^k$ and f_n^{out} does not interact with either of them, and correspondingly we take the approximate solution for $L_{q_n}u = f$ as

$$u_n^{app} = \sum u_n^k + u_n^{out}, \quad u_n^k = L^{-1}_{S(\lambda_n^k,y_n^k)q^k} f_n^k, \quad u_n^{out} = L_0^{-1} f_n^{out},$$

such that $L_{q_n} u_n^{app} = f + \sum_{j \neq k} S(\lambda_n^k, y_n^k) q^k \overline{u_n^j}$. Then by virtue of (7),

$$\|u_n^{app}\|_{\dot{H}^{\frac{1}{2}}} \lesssim C(R)\|f\|_{\dot{H}^{-\frac{1}{2}}}, \quad \|L_{q_n} u_n^{app} - f\|_{\dot{H}^{-\frac{1}{2}}} = o_n(1)\|f\|_{\dot{H}^{-\frac{1}{2}}},$$

and hence $\limsup_{n\to\infty} C(q_n) \lesssim C(R)$ which is a contradiction to (qn).

13.3 The Plancherel theorem

Recall the following pointwise estimates for fractional integrals (see Corollary 2.2 [1] with $\alpha = 1$, $n = 2$):

$$\begin{aligned} &|\bar{\partial}^{-1}(e_{-k}q)(x)| \lesssim \big(M\hat{q}(k)\big)^{\frac{1}{2}}\big(Mq(x)\big)^{\frac{1}{2}}, \text{ if } q \in L^2(\mathbb{C}), \\ &\text{and hence } \|\bar{\partial}^{-1}(e_{-k}q)(x)\|_{L^4_x} \lesssim \|q\|_{L^2}^{\frac{1}{2}}\big(M\hat{q}(k)\big)^{\frac{1}{2}}. \end{aligned} \tag{8}$$

Recall the following pointwise estimates for the pseudo-differential operators with non-smooth symbols (see Theorem 2.3 in [1] with $\alpha = 1$, $n = 2$):

$$|a(x,D)f(x)| \lesssim \big(Mf(x)\big)^{\frac{1}{2}}\|\partial_\xi a(x,\cdot)\|_{L_\xi^{\frac{4}{3}}}\|f\|_{L^2}^{\frac{1}{2}}, \text{ if } \partial_\xi a \in L^4_x L_\xi^{\frac{4}{3}}. \tag{9}$$

Then we have the following estimates for $m_\pm - 1, m^1 - 1, m^2$ which satisfy $\bar{\partial} m_\pm = \pm e_{-k} q \overline{m_\pm}$, $\bar{\partial} m^1 = q m^2$, $(\partial + ik)m^2 = \bar{q} m^1$, $m_\pm = m^1 \pm e_{-k}\overline{m^2}$:

$$\|(m_\pm - 1)(\cdot,k)\|_{L^4} + \|\bar{\partial} m^1(\cdot,k)\|_{L^{\frac{4}{3}}} + \|m^2(\cdot,k)\|_{L^4} \leq C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}},$$

and hence $i\mathbf{s}(k) = \frac{1}{\pi}\int e_k\bar{q} + \frac{1}{\pi}\int e_k\bar{q}(m^1 - 1)$ with the first and second terms reading resp. as the Fourier transform of $\bar{q}$ and $(m^1-1)(D,k)\hat{\bar{q}}(k)$ satisfies

$$|\mathbf{s}(k)| \leq C(\|q\|_{L^2}) M\hat{q}(k).$$

The elegant difference formula

$$\mathbf{s}_1 - \mathbf{s}_2 = T_{q_1,q_2}(q_1 - q_2) \text{ with } \|T_{q_1,q_2}\|_{L^2 \mapsto L^2} \leq C(\|q_1\|_{L^2}) C(\|q_2\|_{L^2}),$$

together with the classical Plancherel identity and $\mathcal{S}^2 = I$ in the Schwartz function setting, implies Theorem 2.

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14 Boundary recovery: Determining conductivity by boundary measurements

A summary written by Yi-Hsuan Lin

Abstract

Based on the fundamental work from A. P. Calderón [1], one can determine the conductivity of an object from appropriate boundary measurements. In addition, one can prove the unique determination for the conductivity with its all derivatives at the boundary.

14.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^∞-smooth boundary $\partial\Omega$, for $n \geq 2$. Let $\gamma(x) \in L^\infty(\Omega)$ be a positive conductivity with $\gamma(x) \geq \lambda > 0$ for some universal constant $\lambda > 0$. Let L_γ be a second order elliptic operator

$$L_\gamma = \nabla \cdot (\gamma \nabla),$$

which acts on functions $u \in H^1(\Omega)$. It is well known that given any boundary Dirichlet datum $\phi \in H^{1/2}(\partial\Omega)$, one has the unique solution $u \in H^1(\Omega)$ to

$$\begin{cases} L_\gamma u = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Hence, one can define the energy operator by

$$Q_\gamma(\phi) := \int_\Omega \gamma |\nabla u|^2 dx,$$

where $u \in H^1(\Omega)$ is the unique solution of (1).

Calderón asked the question: Is the map $\Phi : \gamma \to Q_\gamma$ injective? Calderón demonstrated that the map Φ is analytic as a conductivity $\gamma \in L^\infty(\Omega)$ and its linearized map $d\Phi|_{\gamma=1}$ is injective.

- The goal of Kohn-Vogelius work is: The energy function Q_γ determines γ with its all derivative at the boundary, provided that γ is C^∞-smooth near the boundary.

In addition, knowing $Q_\gamma(\phi)$ for each $\phi \in H^{1/2}(\partial\Omega)$ is equivalent to knowing the *Dirichlet-to-Neumann map* $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$

$$\Lambda_\gamma(\phi) = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

where $u \in H^1(\Omega)$ is the unique solution of (1). Notice that the integration by parts formula yields that

$$\int_{\partial\Omega} \phi_2 \Lambda_\gamma(\phi_1) dS = \int_\Omega \gamma \nabla u_2 \cdot \nabla u_1 dx,$$

where u_j is the solutions of $L_\gamma u_j = 0$ in Ω with $u_j = \phi_j$ on $\partial\Omega$ for $j = 1, 2$.

14.2 Boundary determination

Theorem 1. *For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^∞-smooth boundary $\partial\Omega$. Let $\gamma_i \in L^\infty(\Omega)$ $(i = 1, 2)$ be positive conductivities. Given $x_0 \in \partial\Omega$, let B be a neighbourhood of x_0 relative to $\overline{\Omega}$. Suppose that*

$$\gamma_i \in C^\infty(B), \text{ for } i = 1, 2,$$

and

$$Q_{\gamma_1}(\phi) = Q_{\gamma_2}(\phi) \text{ for } \phi \in H^{1/2}(\partial\Omega) \text{ with } \operatorname{supp}\phi \subset B \cap \partial\Omega.$$

Then one has

$$D^k\gamma_1(x_0) = D^k\gamma_2(x_0) \text{ for any } k = (k_1, k_2, \cdots, k_n),$$

where D^k stands for the derivative $(\frac{\partial}{\partial x_1})^{k_1} \cdots (\frac{\partial}{\partial x_n})^{k_n}$.

It is easy to see the following result.

Corollary 2. *Φ is injective on the set of real-analytic functions, which are bounded away from zero.*

Remark 3. *For the C^∞-smooth conductivities, Sylvester-Uhlmann [3] has proved the global uniqueness result by using the* complex geometrical optics solutions.

14.3 Lemmas

In order to prove Theorem 1, we need following lemmas.

Lemma 4. *Let $M \in \mathbb{N}$ and $z \in \partial\Omega$. Then there exists a sequence $\{\phi_N\}_{N\in\mathbb{N}} \subset C^\infty(\partial\Omega)$ such that*

$$\|\phi_N\|_{H^{1/2+t}(\partial\Omega)} \leq C_t N^t, \text{ for } t \geq -M,$$
$$\|\phi_N\|_{H^{1/2}(\partial\Omega)} = 1,$$

and

$$\text{supp}(\phi_N) \to \{z\} \text{ as } N \to \infty.$$

Lemma 5. *Let $D \subset \Omega$ with $\rho(x) :=dist(z, D) > 0$. Suppose that $\gamma \in C^\infty$ near an neighbourhood $\mathcal{U}$ of $z \in \partial\Omega$, then*

(a) We have

$$\|u_N\|_{H^1(D)} \leq CN^{-M}, \text{ for all } N \geq 1,$$

where u_N is the solution of $L_\gamma u_N = 0$ in Ω with $u_N = \phi_N$ on $\partial\Omega$ and $C > 0$ is a constant independent of N.

(b) Given $l \geq 0$ and $\epsilon > 0$, there is a constant $C_{l,\epsilon} > 0$ such that

$$\int_{\mathcal{U}} \rho^l |\nabla u_N|^2 dx \geq C_{l,\epsilon} N^{-(n+\epsilon)l}$$

for $N \in \mathbb{N}$ large enough.

Combining above two lemmas, we can prove Theorem 1.

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15 Recovery: Reconstructions from boundary measurements - Talk 1: Sections 1,2,3

A summary written by Itamar Oliveira

Abstract

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a $C^{1,1}$ boundary. Consider the operator $L_\gamma u = \nabla \cdot (\gamma \nabla u)$, where γ is a real-valued function in $C^{1,1}(\overline{\Omega})$ with a positive lower bound. Define the quadratic form Q_γ on $H^{\frac{1}{2}}(\partial\Omega)$ by $Q_\gamma(f) = \int_\Omega \gamma(x)|\nabla u(x)|^2 \mathrm{d}x$, where $u \in H^1(\Omega)$ is the unique solution to $L_\gamma u = 0$ in Ω, $u_{|\partial\Omega} = f$. A. P. Calderón posed the problem of deciding whether γ is uniquely determined by Q_γ and, if so, if one can calculate it explicitly. This talk will discuss some reductions done by A. Nachman in [1], where he provides a positive answer to the latter problem.

15.1 Introduction

Consider the problem

$$L_\gamma u = \nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega, \quad u_{|\partial\Omega} = f,$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a $C^{1,1}$ boundary. Let Q_γ be the quadratic form

$$Q_\gamma(f) = \int_\Omega \gamma(x)|\nabla u(x)|^2 \mathrm{d}x$$

where $u \in H^1(\Omega)$ is the unique solution to the problem above. Integrating by parts in the definition of Q_γ we get the alternate expression

$$Q_\gamma(f) = \int_{\partial\Omega} \overline{f}\gamma \frac{\partial u}{\partial \nu} \mathrm{d}\sigma = \int_{\partial\Omega} \overline{f}(\Lambda_\gamma f) \mathrm{d}\sigma \tag{1}$$

where ν is the outward normal on $\partial\Omega$, $\mathrm{d}\sigma$ is the usual surface measure and the **Dirichlet to Neumann** operator Λ_γ is given by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}. \tag{2}$$

A. P. Calderón posed the following problem:

- Decide whether γ is uniquely determined by Q_γ.
- If so, calculate γ in terms of Q_γ.

The first part was answered affirmatively for $\partial\Omega \in C^\infty$ and γ piecewise analytic by R. Kohn and M. Vogelius (recovery on the boundary, [2] and [3]), and for $\gamma \in C^\infty(\overline{\Omega})$ by J. Sylvester and G. Uhlmann (in [4], [5] and [6]). Here we will deal with the second problem, i.e. determining γ in terms of Q_γ (equivalently, in terms of Λ_γ).

By a change of variables,

$$\nabla \cdot (\gamma \nabla u) = 0 \iff -\Delta w + qw = 0, \text{ where } q = \gamma^{-\frac{1}{2}} \Delta \gamma^{\frac{1}{2}}.$$

For the Schrödinger operator $-\Delta + q$ define

$$\Lambda_q(f) := \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega}$$

where w is the solution to $-\Delta w + qw = 0$ in Ω, $w\big|_{\partial\Omega} = f$. We also have

$$\Lambda_q(f) = \gamma^{-\frac{1}{2}} \Lambda_\gamma(\gamma^{-\frac{1}{2}} f) + \frac{1}{2}\gamma^{-1}\frac{\partial \gamma}{\partial \nu} f \tag{3}$$

Thus, to recover Λ_q, it is enough to first find γ and $\frac{\partial\gamma}{\partial\nu}$ on $\partial\Omega$ given Λ_γ. For the first of a sequence of two talks, we will assume that we have recovered this data on $\partial\Omega$ and focus on the problem of recovering q given Λ_q above (sections 2 and 3 of [1]). In the second lecture we will learn how to obtain γ once we have q (Section 5 of [1]).

15.2 Recovering q from Λ_q

An intermediate step to find q is to recover the *scattering amplitude* t given by

$$t(\xi, \zeta) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} q(x) \psi(x, \zeta) \mathrm{d}x,$$

where $\psi(\cdot, \zeta)$ are solutions of

$$-\Delta\psi + q\psi = 0 \quad \text{in } \mathbb{R}^n. \tag{4}$$

Suppose for a moment that solutions to the PDE above exist. By applying Green's formula to $\psi(x,\zeta)$ and $e^{-ix\cdot(\xi+\zeta)}$ we get

$$t(\xi,\zeta) = \int_\Omega e^{-ix\cdot(\xi+\zeta)}\Delta\psi \mathrm{d}x = \int_{\partial\Omega} e^{-i\cdot(\xi+\zeta)}\left[\frac{\partial\psi}{\partial\nu} + i(\xi+\zeta)\cdot\nu\right]\mathrm{d}\sigma$$

when $(\xi+\zeta)^2 = 0$. To recover t, we are then led to study the behavior of ψ on $\partial\Omega$. Solutions of (4) satisfy the following integral equation:

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\mathbb{R}^n} G_\zeta(x-y)q(y)\psi(y,\zeta)\mathrm{d}y \tag{5}$$

where

$$G_\xi(x) = \frac{1}{(2\pi)^n}e^{ix\cdot\xi}\int \frac{e^{ix\cdot\xi}}{\xi^2+2\zeta\cdot\xi}\mathrm{d}\xi.$$

The problem now becomes the solvability of (5).

A key step in the proof is showing that solutions to (5) are closely related to the ones of the following problem:

(a) $\Delta\psi = 0 \quad \text{in } \Omega'$,

(b) $\psi \in H^2(\Omega'_\rho) \quad \text{for any } \rho > \rho_0$,

(c) $\psi(x,\zeta) - e^{ix\cdot\zeta}$ satisfies a technical condition to be explained in the talk.

(d) $\frac{\partial\psi}{\partial\nu_+} = \Lambda_q\psi \quad (\text{on } \partial\Omega)$.

where $\Omega' := \mathbb{R}^n\backslash\Omega$ and $\Omega'_\rho := \{x \notin \overline{\Omega}, |x| < \rho\}$.

By getting a solution to the problem above in terms of layer potentials, we will be able to find a solution to the PDE (5), hence we have will have t. Finally, Theorem 3.4 of [1] gives an explicit inverse formula to compute $\hat{q}$ in terms of t, which is enough to conclude this step of the proof.

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16 Stability - Talk 1: Exponential instability in an inverse problem for the Schrödinger equation

A summary written by Lisa Onkes

Abstract

We show that the problem of determining the potential from the Dirichlet to Neumann map of the Schrödinger operator is severely ill-posed. This is connected to the problem of electrical impedance tomography.

16.1 Introduction

Let $d \geq 2$ and $B := B(0,1) \subset \mathbb{R}^d$ the unit ball. Consider the boundary value problem

$$\begin{cases} u|_{\partial B} = f \\ (-\Delta + q)u = 0 \text{ in } B \end{cases}, \tag{1}$$

where the potential q is bounded, and 0 is not a Dirichlet eigenvalue of $-\Delta + q$. If $f \in H^{1/2}(\partial B)$ then (1) has a unique solution $u \in H^1(B)$.
We define the Dirichlet to Neumann operator $\Lambda_q f = \frac{\partial u}{\partial \nu}\big|_{\partial B}$, where $\partial \nu$ describes the normal derivative to the boundary. The question now is, what can one find out about q, when only Λ_q is known?

16.1.1 Connection to the problem of electrical impedance tomography

The problem of electrical impedance tomography (EIT) is the problem of determining the isotropic electrical conductivity γ of an object from measurements at its boundary. EIT has several applications in medicine and most prominently is used for monitoring the lungs, since the lungs and surrounding tissue differ highly in conductivity.
The mathematical description of this problem reads as follows. We want to retrieve γ from the voltage to current map $\tilde{\Lambda}_\gamma$ defined by $\tilde{\Lambda}_\gamma f = \gamma \frac{\partial v}{\partial \nu}\big|_{\partial B}$,

with

$$\begin{cases} v|_{\partial B} = f \\ \operatorname{div} \gamma \nabla v = 0 \text{ in } B. \end{cases} \tag{2}$$

One can reduce problem (2) to (1). However this requires to know the restriction of γ and its normal derivative to the boundary.

16.2 The main stability result

The results in [1],[2] and the methods of [4] show that for any $d \geq 3$ and $m > 0$, there exists an $\alpha > 0$, such that for every $M > 0$ there is $C(M) > 0$, so that $\|q_1\|_{C^m}, \|q_2\|_{C^m} \leq M$ implies

$$\|q_1 - q_2\|_{L^\infty} \leq C(M)(\log(1 + \|\Lambda_{q_1} - \Lambda_{q_2}\|^{-1}_{H^{1/2} \to H^{-1/2}}))^{-\alpha}. \tag{3}$$

We will show, that the result (3) is optimal in the sense, that there exist counter-examples in the case $\alpha > m(2d-1)/d$. Our instability result reads as follows.

Theorem 1. *In dimension $d \geq 2$ for any $m > 0$ and any $s \geq 0$, there is a constant $\beta > 0$, such that for any $\varepsilon \in (0,1)$ and $q_0 \in L^\infty$ with $\|q_0\|_\infty \leq 1$ and* supp $q_0 \subset B(0,1/2)$, *there are real-valued potentials $q_1, q_2 \in C^m$, also supported in $B(0,1/2)$ such that*

$$\begin{cases} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{-s} \to H^s} \leq \exp\left(-\varepsilon^{-\frac{d}{(2d-1)m}}\right) \\ \|q_1 - q_2\|_\infty = \varepsilon \\ \|q_1 - q_0\|_{C^m}, \|q_2 - q_0\|_{C^m} \leq \beta \\ \|q_1 - q_0\|_\infty, \|q_2 - q_0\|_\infty \leq \varepsilon \end{cases} . \tag{4}$$

In the following we will give a sketch of the proof of Theorem 1.

16.3 The basic estimate

Let $\{f_{jp} : j \geq 0, 1 \leq p \leq p_j\}$ be an orthonormal basis in $L^2(S^{d-1})$, with f_{jp} a spherical harmonic of degree j and p_j the dimension of the space of spherical harmonics of degree j. We make the following estimate.

Lemma 2. *Let $r_0 \in (0,1)$. Suppose that q is bounded,* $\operatorname{supp} q \subset B(0, r_0)$ *and 0 is not an eigenvalue of $-\Delta + q$. Denote $\Gamma(q) := \Lambda_q - \Lambda_0$. Then there is a constant $\rho = \rho(r_0, d)$, such that for any $0 \leq j, 1 \leq p \leq p_j$ and $0 \leq k, 1 \leq q \leq p_k$, we have*

$$|\langle \Gamma(q) f_{jp}, f_{kq} \rangle| \leq \rho r_0^{\max(j,k)} \|q\|_\infty \|(-\Delta + q)^{-1}\|_{L^2}. \tag{5}$$

Sketch of proof. Consider the problem (1) with the potential q and the boundary value $f = f_{jp}$ and denote the solution by u. The solution for the zero potential and again the boundary value $f = f_{jp}$ is $u_0(r, w) = r^j f_{jp}(w)$. Then

- $u - u_0 = -(-\Delta + q)^{-1} q u_0$ and
- $\Gamma(q) f_{jp} = \left.\frac{\partial(u-u_0)}{\partial \nu}\right|_{\partial B(0,1)}$.

Now estimate $\frac{\partial(u-u_0)}{\partial \nu}|_{\partial B(0,1)}$ in terms of $u - u_0$ to achieve the desired estimate. □

In the following we fix $r_0 = 1/2$ and denote $\rho(1/2, d)$ by ρ.

16.4 A fat metric space and a thin metric space

Definition 3. *Let (X, d) be a metric space and $\varepsilon > 0$. We say that a set $Y \subset X$ is an ε-net for $X_1 \subset X$ if for any $x \in X_1$ there is $y \in Y$ such that $d(x, y) \leq \varepsilon$. A set $Z \subset X$ is called ε-discrete if for any distinct $z_1, z_2 \in Z$, we have $d(z_1, z_2) \geq \varepsilon$.*

Lemma 4. *Let $d \geq 2$ and $m > 0$. For $\varepsilon, \beta > 0$, consider the metric space*

$$X_{m\varepsilon\beta} := \{f \in C_0^m(B(0, 1/2)) : \|f\|_\infty \leq \varepsilon, \|f\|_{C^m} \leq \beta\}, \tag{6}$$

with the metric induced by L^∞. Then there is a $\mu > 0$ such that for any $\beta > 0$ and $\varepsilon \in (0, \mu\beta)$ there is an ε-discrete set $Z \subset X_{m\varepsilon\beta}$ with at least $\exp(2^{-d-1}(\mu\beta/\varepsilon)^{d/m})$ *elements.*

Sketch of proof. The elements of Z are constructed by adding up bump-functions. □

Given an operator $A : H^{-s}(S^{d-1}) \to H^s(S^{d-1})$, we denote its matrix elements in a basis of spherical harmonics (f_{jp}) by $a_{jpkq} := \langle Af_{jp}, f_{kq}\rangle$. One can estimate

$$\|A\|_{H^{-s}\to H^s} \leq 4 \sup_{j,p,k,q} (1+\max(j,k))^{2s+d-1/2}|a_{jpkq}|. \tag{7}$$

Therefore we introduce the Banach space

$$X_s := \left\{ (a_{jpkq}) \;\middle|\; \|(a_{jpkq})\|_{X^s} := \sup_{j,p,k,q} (1+\max(j,k))^{2s+d}|a_{jpkq}| < \infty \right\}.$$

Let us denote by B^∞ the unit ball of $L^\infty(B(0,1/2))$.

Lemma 5. *Γ maps B^∞ into X_s for any s. There is a constant $0 < \eta = \eta(s,d)$, such that for every $\delta \in (0, e^{-1})$, there is a δ-net Y for $\Gamma(B^\infty)$ in X_s, with at most $\exp(\eta(-\log\delta)^{2d-1})$ elements.*

Sketch of proof. The embedding is proven by the estimate (5).
In order to define the δ-net Y let $l_{\delta s}$ be the smallest integer such that $(1+l)^{2s+d}\rho 2^{-l} \leq \delta$ for any $l \geq l_{\delta s}$. Then

$$Y := \{(a_{jpkq}) \mid a_{jpkq} \in \left((1+l_{\delta s})^{-2s-d}\delta\mathbb{Z}\right) \bigcap [-\rho,\rho] \text{ for } \max(j,k) \leq l_{\delta s}, \\ a_{jpkq} = 0 \text{ otherwise}\}$$

fulfills the requirements. This, again, is proven by estimate (5). □

Now we are in the position to prove the main Theorem 1.

Proof sketch of Theorem 1. Take $q_0 \in L^\infty(B(0,1/2))$, $\|q_0\|_\infty \leq 1$ and $\varepsilon \in (0,1)$. Set $\delta = \frac{1}{8}\exp(-\varepsilon^{\frac{-d}{(2d-1)m}})$ and show that the set $q_0 + X_{m\varepsilon\beta}$ has an ε-discrete set $q_0 + Z$ and $\Gamma(q_0 + X_{m\varepsilon\beta})$ has a δ-net Y in X_s with $|Z| > |Y|$. This implies that there are two points in $q_0 + Z$ with images under Γ in the same X_s-δ-ball centered at a point of Y. These are the looked-for potentials q_1 and q_2. The sets Z and Y exist by Lemma 4 and Lemma 5. □

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17 Counterexamples - Talk 3: A Remark on Gradients of Harmonic Functions in Dimension ≥ 3.

A summary written by João Pedro G. Ramos

Abstract

We show that there are $C^{1+\epsilon}$ harmonic functions on $\mathbb{R}^d_+$ ($d \geq 3$) for which ∇f vanishes on a boundary set of positive measure, generalizing to higher dimensions an earlier result of Wolff [3].

17.1 Main Result

The main motivation of this work is to generalize the previous work of Wolff [3] about boundary values of gradients of harmonic functions. In fact, we shall prove the following:

Theorem 1. *If $d \geq 3$ there is a harmonic function $f : \mathbb{R}^d_+ \to \mathbb{R}$ which is C^1 up to the boundary and such that f and ∇f vanish on a common boundary set of positive measure.*

17.2 Outline of the methods

In order to prove this result, we shall begin with an arbitrary smooth function u_0 on $\mathbb{R}^{d-1}$ that vanishes on an open subset of the boundary. The procedure basically consists then on an application of a correction theorem multiple times to decrease the normal derivative of a specific function on a large set where it vanishes. Of course, we shall identify a function in $\mathbb{R}^{d-1}$ to its harmonic extension to $\mathbb{R}^d_+$ canonically.

In order to do this "correction theorem" procedure, we must select well the functions we are adding at each step of the process. We must pcik them with a small compact support, as well as make our normal derivastive diminish at the same time. The main idea to construct them is due to A. B. Aleksandrov – P. Kargaev [1]:

Lemma 2. *Let $p > 0$ be small enough. Then for all sufficiently small ε there is $F_\varepsilon : \mathbb{R}^{d-1} \to \mathbb{R}$ with supp$F_\varepsilon \subset D_{\mathbb{R}^{d-1}}(0, \varepsilon^{1/2})$, and, if $\widehat{F_\varepsilon}$ denotes its harmonic extension to $\mathbb{R}^d_+$,*

$$\int_{\mathbb{R}^{d-1}} \left(\left| 1 + \frac{d\widehat{F_\varepsilon}}{dn} \right|^p - 1 \right) dx < -\eta,$$

with $\eta > 0$ independent of $\varepsilon > 0$. Moreover, one can also assume $|\nabla \widehat{F_\varepsilon}| \lesssim \min(\varepsilon^{-d}, |x|^{-d})$ on $\mathbb{R}^{d-1}$.

We need, however, a sharper, 'perturbed' version of the last Lemma:

Lemma 3. *If N is large enough, there is a constant $\beta = \beta(N) > 0$ such that, if $Q \subset \mathbb{R}^{d-1}$ is a cube, a_Q its center and $I : Q \to \mathbb{R}$ is a function such that $N^{d-1}|I(a_Q)|^{-1} \sup_{x \in Q} |I(x) - I(a_Q)|$ is sufficiently small (independently of N, ε), then*

$$\left(\int_Q \left| I(x) + \frac{d\widehat{F_\varepsilon}}{dn}(N\ell(Q)^{-1}(x - a_Q)) \right|^p dx \right)^{1/p} < e^{-2\beta} |I(a_Q)| |Q|^{1/p}.$$

As promised, we shall iterativelly construct our sequence of functions. Begin, as stated, with a smooth function u_0 which vanishes on the open unit cube $Q(1)$. For shortness, we shall sometimes identify a harmonic function on the upper half space to its boundary value.

We suppose further that, at stage n, we are provided with a number $\delta_n > 0$ such that $\delta_n^{-1} \in \mathbb{Z}$, a subcollection $\mathcal{G}_n$ of cubes of side δ_n contained in $Q(1)$, and a smooth function u_n such that

$$\left(\int_{V_n} \left| \frac{du_n}{dn} \right|^p \right)^{1/p} \leq A e^{-\beta n},$$

where $V_n = \cup_{Q \in \mathcal{G}_n} Q$, β is as above, and A is a large constant not depending on n. For instance, we can begin with $\delta_0 = 1$, $\mathcal{G}_0 = \{Q(1)\}$.

In order to pass to the next stage, we choose δ_{n+1} small such that δ_n / δ_{n+1} in an integer. Moreover, we define the cubes of the collection $\mathcal{G}_{n+1}$ to be the children of cubes of the family $\mathcal{G}_n$ such that

$$\left(\frac{1}{|Q|}\int_Q |du_n/dn|^p\right)^{1/p} < K_{n+1}e^{-\beta n},$$

where we will specify what "small" and K_{n+1} mean later. If a_Q is the center of the cube Q, as above, then we define the function

$$u_{n+1}(x) = u_n(x) + \sum_{Q\in\mathcal{G}_{n+1}} \frac{du_n}{dn}(a_Q)F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x-a_Q))\frac{\delta_{n+1}}{N}.$$

Next, we prove an estimate that allows us to have good control on the norms of gradients of u_{n+1} :

Lemma 4.

$$\sum_{Q\in\mathcal{G}_{n+1}:|x-a_Q|>\rho} |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x-a_Q))| \leq CN^{-d}\delta_{n+1}\rho^{-1},$$

for all $\rho > c\delta_{n+1}$ and $x \in \mathbb{R}^{d-1}$.

Corollary 5. *If δ_{n+1} is small enoguh, then, on $\mathbb{R}^{d-1}$,*

$$|\nabla u_{n+1} - \nabla u_n| \leq CK_{n+1}e^{-\beta n}\varepsilon_{n+1}^{-d}.$$

With these, we can then verify, although we shall omit the details here, that u_{n+1} satisfies the same assumption as u_n:

Lemma 6. *If we pick A to be large enough, then*

$$\left(\int_{V_{n+1}} |du_{n+1}/dn|^p\right)^{1/p} < Ae^{-\beta(n+1)}.$$

We remark only that, for the proof of this lemma, one must choose δ_{n+1} small enough, but in a harmless way to the rest of our argument.

Finally, in order to conclude the proof, we must pick K_n, ε_n suitably: they must satisfy

1. $\sum \varepsilon_{n+1}^{-d}K_{n+1}e^{-\beta n} < \infty$,

2. $\sum[K_{n+1}^{-p} + \varepsilon_{n+1}^{(d-1)/2}]$ is sufficiently small.

We can pick, for instance, $\varepsilon_n = C^{-1}n^{-2}$ and $K_n = Cn^{2/p}$ for a big constant $C > 1$, and those properties are immediately verified.

By the first condition above, we can ensure that the functions form a Cauchy sequence in $C^1(\mathbb{R}_+^{d-1})$. Let u be the limit function. By the support of the functions F_{ε_n} bein contained in $D_{\mathbb{R}^{d-1}}(0, \varepsilon_n^{1/2})$ and by the fact that $u_{n+1} \neq u_n$ in (at most) $\delta_{n+1}^{-(d-1)}$ discs of radius $\delta_{n+1}\varepsilon_{n+1}^{1/2}$, we conclude that

$$|\{x \in Q(1) : \ u(x) \neq 0\}| \leq C \sum \varepsilon_{n+1}^{(d-1)/2}.$$

Also, by the recursive definition of the functions u_n, one can estimate the measure of $|V_{n+1} \backslash V_n| \leq A^p K_{n+1}^{-p}$. This plainly implies that

$$|\{x \in Q(1) : \ d\widehat{u}/dn(x) \neq 0\}| \leq A^p \sum K_{n+1}^{-p}.$$

As we chose K_n, ε_n satisfying condition (2) above, we get that

$$|Q(1) \cap \{x : (d\widehat{u}/dn)(x) = 0\} \cap \{x : u(x) = 0\}| > 0,$$

as desired, which concludes the result. We notice that this proof does not yield immediately the stated regularity, but a more careful analysis of the way we defined our sequence of functions gives us, in a fashion similar to [3], that the limit u is in some $C^{1+\epsilon}$ class.

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18 The 2D Calderón Problem - Talk 1: The Calderón problem with L^∞ conductivity

A summary written by Olli Saari

Abstract

Calderón's inverse problem can be solved for conductivities with no smoothness in the plane using quasiconformal methods. The higher dimensional analogue is not known. This is summary of the exposition of the solution to planar Calderón problem in [1]. The result was originally due to Astala and Päivärinta [2].

18.1 Introduction

Let $\Omega \subset \mathbb{C}$ be the unit disc. For $\sigma > 0$ with $\log \sigma \in L^\infty(\mathbb{C})$, the Dirichlet problem

$$\begin{aligned} \nabla \cdot \sigma \nabla u &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= \phi \in W^{1/2,2} \end{aligned} \tag{1}$$

always admits a unique solution $u \in W^{1,2}(\Omega)$. Here $W^{1/2,2}$ is the class of boundary traces of $W^{1,2}(\Omega)$ functions. We define the normal derivative of a solution u through

$$\langle \sigma \frac{\partial u}{\partial \nu}, \psi \rangle = \int_\Omega \sigma \nabla u \cdot \nabla \psi, \quad \psi \in W^{1,2}(\Omega),$$

and call σ the conductivity, ϕ the Dirichlet data (voltage) and $\sigma \dfrac{\partial u}{\partial \nu}$ the Neumann data (current). The Dirichlet-to-Neumann map of σ is the operator

$$\Lambda_\sigma(\phi) = \sigma \frac{\partial u}{\partial \nu}. \tag{2}$$

The inverse problem of Calderón consists in showing that given two conductivities σ and $\tilde{\sigma}$ as above, the coincidence of Dirichlet-to-Neumann maps $\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$ implies $\sigma = \tilde{\sigma}$ almost everywhere in the sense of the Lebesgue measure.

When additional smoothness assumptions are imposed on σ, one may expand the differential operator and reduce the problem to a Schrödinger type equation

$$\Delta v - qv = 0, \quad q = \sigma^{-1/2}\Delta\sigma^{1/2}.$$

Such a reduction is not possible under the mere assumption of boundedness, and one has to choose another path. By exploiting quasiconformal methods, Astala and Päivärinta solved Calderón's problem in the plane.

Theorem 1 (Astala and Päivärinta [2])**.** *Let $\sigma, \tilde{\sigma} > 0$ be bounded from above and from below. If $\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$ as defined in* (2)*, then $\sigma = \tilde{\sigma}$ almost everywhere.*

18.2 Outline of the proof

18.2.1 Beltrami equation

We call solutions to (1) σ-harmonic. Every σ-harmonic u has a ($1/\sigma$-harmonic) conjugate v such that $f = u + iv$ satisfies the $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar{z}} f = \mu(z)\overline{f_z}, \quad \frac{1-\sigma}{1+\sigma}. \tag{3}$$

Such an f is quasiregular. The proof aims at showing that the mappings f corresponding to the conductivities coincide. The uniqueness of μ and σ will then follow.

18.2.2 Complex geometric optics

There is a family of solutions to the Beltrami equation such that $f_\mu(z,\xi) = e^{i\xi z}M_\mu(z,\xi)$ and $M_\mu(z,\xi) - 1 = \mathcal{O}(z^{-1})$ as $z \to \infty$. They are called complex geometric optics solutions (CGO) and for any given ξ the corresponding CGO is unique. CGO for (3) give rise to corresponding CGO for (1).

18.2.3 Hilbert transform

The Hilbert transform (special to this problem) is defined as $\mathcal{H}_\sigma : u|_{\partial\Omega} \mapsto v|_{\partial\Omega} - \int_{\partial\Omega} v$ when v is the conjugate of u. Through integration by parts one sees

$$\partial_T \mathcal{H}(u) = \Lambda_\sigma(u)$$

so that Λ_σ determines the Hilbert transform uniquely. This information leads to unique determination of CGOs outside the unit disc and the main part of the problem is to extend this inside the disc.

18.2.4 Nonlinear Fourier transfrom

The next step is to prove that CGOs depend smoothly on the parameter ξ. Once this is known, one may differentiate a CGO for (1) with respect to ξ parameter to obtain

$$\partial_{\bar{\xi}} u_\sigma = -i\tau_\sigma(\xi) e^{-i\bar{\xi}\bar{z}} \left(1 + \mathcal{O}(|z|^{-1})\right).$$

Here the coefficient τ is independent of z so that $e^{-i\bar{\xi}\bar{z}}\left(1 + \mathcal{O}(|z|^{-1})\right)$ part is again a solution to σ-harmonic equation. It obeys the asymptotics of a CGO so that it must be u_σ itself. Hence

$$\partial_{\bar{\xi}} u_\sigma(z,\xi) = -i\tau_\sigma(\xi)\overline{u_\sigma(z,\xi)}. \tag{4}$$

The coefficient τ is called the nonlinear Fourier transform coming from σ and it is determined by Λ_σ. In particular u_σ and $u_{\tilde{\sigma}}$ solve the same differential equation in ξ.

18.2.5 Subexponential growth

The final passage of the proof consists in showing that the CGO's $u_\sigma(z,\xi)$ differ asymptoticaly from $e^{i\xi z}$ by a factor growing less than exponentially in ξ. Hence its logarithm is continuous and close to a multiple of the identity for z large, which implies surjectivity as a mapping $z \mapsto \log u_\sigma(z,\xi)$. The final claim $\log u_\sigma(z,\xi) = \log u_{\tilde{\sigma}}(z,\xi)$ follows from (4) and the argument principle.

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19 The Calderón problem with partial data - Talk 2

A summary written by Gennady Uraltsev

Abstract

In this paper the authors show an L^∞ potential on a smooth domain in dimension $n \geq 3$ is uniquely determined by the boundary values of the solutions to the Sch??dinger equation. They show that this uniqueness holds for partial knowledge of the boundary data. In particular if one considers solutions whose Dirichlet boundary values $u_{\restriction \partial\Omega}$ are supported on some open subsets $\Gamma_D \subset \partial\Omega$ of the boundary and the knowledge of the Neumann boundary values $\partial_\nu u_{\restriction \partial\Omega}$ is restricted to an appropriate neighborhood $\Gamma_N \supset \partial\Omega \setminus \Gamma_D$ of the complement of Γ_D. Sufficient properties of such neighborhoods (Γ_D, Γ_N) are given.

This result relies on studying the injectivity of a transform of the potential associated to interactions with products of so-called complex geometrical optics solutions constructed using Carleman estimates with limiting Carleman weights.

19.1 Main results

Let $n \geq 3$ and let $\Omega \subset\subset \mathbb{R}^n$ be an open connected domain with compact closure and C^∞ boundary $\partial\Omega$. Let $\nu(x)$ for $x \in \partial\Omega$ be the external normal vector to Ω. Let $\Gamma_D, \Gamma_N \subset \partial\Omega$ be two open subsets of the boundary. The partial Cauchy data for a potential $q \in L^\infty(\Omega)$ is the set

$$C^{\Gamma_D,\Gamma_N}_{\Delta,q} = \left\{ (u_{\restriction \partial\Omega}, \partial_\nu u_{\restriction \Gamma_N})\colon u \in H_\Delta(\Omega), \begin{array}{l} (-\Delta + q)u = 0 \text{ on } \Omega \\ \operatorname{spt}(u_{\restriction \partial\Omega}) \subset \Gamma_D \end{array} \right\} \tag{1}$$

where

$$H_\Delta(\Omega) = \{u \in D'(\Omega)\colon u \in L^2(\Omega),\, \Delta u \in L^2(\Omega)\} \tag{2}$$

so that

$$C^{\Gamma_D,\Gamma_N}_{\Delta,q} \subset H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega). \tag{3}$$

For a given choice of Γ_D and Γ_N one would like to understand if $q_1, q_2 \in L^\infty(\Omega)$ and

$$C^{\Gamma_D,\Gamma_N}_{\Delta,q_1} = C^{\Gamma_D,\Gamma_N}_{\Delta,q_2}$$

implies that $q_1 = q_2$.

(†) We assume that the potential $q \in L^\infty(\Omega)$ is such that 0 is not an eigenvalue of $-\Delta+q$ seen as a compact perturbation of $-\Delta$ on $H^2(\Omega)\cap H^1_0(\Omega)$.

Let $x_0 \in \mathbb{R}^n \setminus \overline{\mathrm{ch}(\Omega)}$ where $\mathrm{ch}(\Omega)$ is the convex hull of Ω. Then the front, back, and tangential points of $\partial\Omega$ with respect to x_0 are given by

$$\begin{aligned} F(x_0) &= \{x \in \partial\Omega \colon (x - x_0) \cdot \nu(x) < 0\} \\ B(x_0) &= \{x \in \partial\Omega \colon (x - x_0) \cdot \nu(x) > 0\} \\ T(x_0) &= \{x \in \partial\Omega \colon (x - x_0) \cdot \nu(x) = 0\}. \end{aligned}$$

The following result holds

Theorem 1 (Uniqueness with partial data)**.** *If $\Gamma_D \supset F(x_0) \cup T(x_0)$ and $\Gamma_N \supset B(x_0) \cup T(x_0)$ are open subsets of $\partial\Omega$ and $q_1, q_2 \in L^\infty(\Omega)$ are two potentials† if*

$$C_{q_1}^{\Gamma_D,\Gamma_N} = C_{q_2}^{\Gamma_D,\Gamma_N} \tag{4}$$

then $q_1 = q_2$ where $C_q^{\Gamma_D,\Gamma_N} := C_{\Delta,q}^{\Gamma_D,\Gamma_N} \cap H^{1/2}(\Omega) \times H^{-1/2}(\Gamma_N)$

Notice that if Ω is convex then Γ_D can be made arbitrarily small. Then however Γ_N consists of most of the boundary.

The assumption on the potential† imply [2] that $C_q^{\Gamma_D,\Gamma_N}$ is the graph of the restriction to Γ_N of Dirichlet to Neumann map for $-\Delta + q$ given by

$$\begin{aligned} &\mathcal{DN}_q \colon H^{1/2}(\Omega) \to H^{-1/2}(\Omega) \\ &\mathcal{DN}_q(f) := \partial_\nu u_{\restriction\partial\Omega} \end{aligned} \qquad \text{where} \qquad \begin{cases} (-\Delta + q)u = 0 & u \in H^1(\Omega) \\ u_{\restriction\partial\Omega} = f. \end{cases} \tag{5}$$

Green's formula implies that $\mathcal{DN}_q^* = \mathcal{DN}_{\bar q}$ so the role of $F(x_0)$ and $B(x_0)$ can be interchanged in the above statement. As a matter of fact, first the authors obtain the following preliminary result

Theorem 2 (Full Dirichlet data case)**.** *If $\Gamma_N \supset F(x_0) \cup T_{(x_0)}$ is an open subset of $\partial\Omega$ and for $q_1, q_2 \in L^\infty(\Omega)$ two potentials as above it holds that*

$$\mathcal{DN}_{q_1}(f)_{\restriction\Gamma_N} - \mathcal{DN}_{q_2}(f)_{\restriction\Gamma_N} = 0 \quad \forall f \in H^{1/2}(\Omega) \tag{6}$$

then $q_1 = q_2$.

We will concentrate now on this statement. For simplicity let us suppose that $\Omega \subset\subset \mathbb{R}^n_+ := \{x \in \mathbb{R}^n, x_n > 0\}$ and that $\Gamma_N \supset F(0)$.

19.2 Complex geometrical optics solutions

As seen in *part 1*, for any potential† $q \in L^\infty(\Omega)$ we are able to construct the following family of functions, the so called "complex geometrical optics" solutions,

$$v_{\pm,y,h}(x) = e^{\pm\frac{1}{h}(\phi_y(x)\pm i\psi_y(x))}(a_y(x) + r_{\pm,y,h}(x)) \tag{7}$$

where $\phi_y(x) := \ln|x-y|$ are instances of smooth on $\overline{\Omega}$ limiting Carleman weights, $0 < h \ll 1$ is a small semi-classical parameter, and $y \in \mathbb{R}^n_- := \{y \in \mathbb{R}^n : y_n < 0\}$.

Solutions (7) satisfy

- $(-\Delta + q)v_{\pm,y,h} = 0$,
- $\nabla\phi_y \perp \nabla\psi_{\pm,y}$ and $|\nabla\phi_y| = |\nabla\psi_{\pm,y}|$,
- the function $a_y \in C^\infty(\overline{\Omega})$ is non-vanishing,
- $\|r_{\pm,y,h}\|_{H^1(\Omega)} \lesssim h$ for $h \to 0$ locally uniformly in y

Only the correction terms $r_{\pm,y,h}$ depend on the potential† q and on the semi-classical parameter h. Furthermore the choice of a_y and ψ_y is not unique but one has an analytic family of real analytic candidates for which the last estimate above is locally uniform.

19.2.1 Integral testing

The following crucial integral testing condition holds for the above solutions

Proposition 3. *For any two solutions $v_{1,y,h} = v_{-,y,h}$ associated to q_1 and $v_{2,y,h} = v_{+,y,h}$ associated to q_2 it holds that*

$$\int_\Omega (q_1 - q_2)v_{1,y,h}v_{2,y,h} = O(h) \tag{8}$$

as $h \to 0$ locally uniformly for all $y \in \mathbb{R}^n_-$ such that $F(y) \subset \Gamma_N$.

This result uses that $C^{\Omega,\Gamma_N}_{q_1} = C^{\Omega,\Gamma_N}_{q_2}$ to construct a solution $\tilde{v}_{1,y,h} \in H^1(\Omega)$ satisfying

$$(-\Delta + q_1)\tilde{v}_{1,y,h} = 0 \qquad \begin{cases} \tilde{v}_{1,y,h} = v_{2,y,h} & \text{on } \Gamma_D = \partial\Omega \\ \partial_\nu\tilde{v}_1 = \partial_\nu v_2 & \text{on } \Gamma_N. \end{cases}$$

As a matter of fact a straightforward integration by parts yields that

$$\int_{\Omega}(q_1 - q_2)v_1 v_2 = -\int_{\partial\Omega\setminus\Gamma_N} v_{1,y,h}\nabla(v_{2,y,h} - \tilde{v}_{1,y,h}) \cdot d\nu(x)$$

and the the right-hand side follows can be bound using Carleman estimates due to the fact that $\nabla\phi_y \cdot \nu(x)$ is positive and bounded away from 0 on $\partial\Omega \setminus \Gamma_N$.

The bounds on the remainders $r_{1,y,h}$ and $r_{2,y,h}$ allow us to then conclude that

$$\int_{\Omega}(q_1(x) - q_2(x))e^{-i\frac{\psi_{1,y}(x)-\psi_{2,y}(x)}{h}} a_{1,y}(x)a_{2,y}(x)dx = O(h). \tag{9}$$

where we have a crucial amount of freedom when choosing $\psi_{1,y}, \psi_{2,y}$ and corresponding $a_{1,y}, a_{2,y}$

19.3 Injectivity of the complex geometrical optic transform

Concluding the proof now relies on showing that the family of complex geometrical optics solutions available is rich enough to be able to conclude from (9) that $q_1 - q_2$ vanishes. The construction of complex geometrical optic solutions shows that $\psi_y(x)$ may be chosen to be

$$\Psi_{(y,\omega)}(x) := d_{S^{n-1}}\left(\frac{x-y}{|x-y|};\omega\right)$$

where $d_{S^{n-1}}$ is the geodesic distance on S^{n-1} and $\omega \in S^{n-1}$ is in the complement of the bi-cone generated by

$$\mathcal{B}_y(\Omega) = \{\lambda(x-y) \in \mathbb{R}^n \colon \lambda \in \mathbb{R},\, x \in \overline{\Omega}\}.$$

If $A_{1,y,\omega}(x)$ and $A_{2,y,\omega}(x)$ are any two functions associated to $\phi_y(x)$ and $\Psi_{y,\omega}(x)$ in the construction of the complex geometrical optic solution then it follows from (9) that

$$\int_{\Omega}(q_1(x) - q_2(x))e^{-i\lambda F_{y,\omega,d\omega}(x)} A_{1,y,\omega}(x)A_{2,y,\omega}(x)dx = 0.$$

where $F_{y,\omega,d\omega}(x) = D_\omega\Psi_{y,\omega}(x)[d\omega]$ where $d\omega \in \mathbb{R}^{n-1}$ is any vector with $d\omega \perp \omega$ and $\lambda \in \mathbb{R}$ since $F_{y,\omega,d\omega}$ is linear in $d\omega$.

Recall that $A_{1,y,\omega}(x)$ and $A_{2,y,\omega}(x)$ are non-vanishing and may be chosen to depend analytically on (y,ω). Analytic micro-local analysis ([3], [4]) techniques that depend on appropriate non-degeneracy properties of F that stem from the above geometric construction allow one to conclude that the transform

$$q \mapsto \int_{\Omega} q(x) e^{-i\lambda F_{y,\omega,d\omega}(x)} A_{1,y,\omega}(x) A_{2,y,\omega}(x) dx = 0. \tag{10}$$

is injective on any set of parameters of the form

$$\{(\lambda, y, \omega, d\omega)\colon (y,\omega) \in U(y_0,\omega_0),\ \lambda \in \mathbb{R}, d\omega \in T_\omega S^{n-1}\}$$

where $U(y_0,\omega_0) \subset \mathbb{R}^n_- \times S^{n-1}$ is any open neighborhood of (y_0,ω_0) with $\omega_0 \notin \mathcal{B}_{y_0}(\Omega)$.

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20 Counterexamples - Talk 1: Sharp counterexamples in unique continuation for second order elliptic equations

A summary written by Julian Weigt

Abstract

For $n \geq 3$ and $p < \frac{n}{2}$ there is a function $u \in C^\infty(\mathbb{R}^n)$ with nonempty compact support and $\frac{\Delta u}{u}\mathbb{1}_{u\neq 0} \in L^p(\mathbb{R}^n)$. For $n = 2$ it also holds for $p = 1$.

20.1 Main result

The existence of a nontrivial compactly supported function u with $\frac{\Delta u}{u}\mathbb{1}_{u\neq 0} \in L^p$ can be used to provide a counterexample to the *unique continuation property.* A PDE

$$\Delta u = Vu \qquad \text{in } \mathbb{R}^n. \tag{1}$$

with some potential V has said unique continuation property if every solution that is 0 on a nonempty open set is 0 everywhere. This means that for any solution u and any nonempty open set U, u is the only solution that equals u on U. So, asking for which p every PDE (1) with $V \in L^p$ has the unique continuation property is equivalent to asking for which p there is a nontrivial u which is 0 on a nonempty open set and $\frac{\Delta u}{u}\mathbb{1}_{u\neq 0} \in L^p$.

Theorem 1. *For each $n \geq 2$, $p < \frac{n}{2}$ there exists a $u \in C^\infty(\mathbb{R}^n)$ with nonempty compact support and*

$$\frac{\Delta u}{u}\mathbb{1}_{u\neq 0} \in L^p(\mathbb{R}^n).$$

Also for $n = 2$, $p = 1$ such a function exists.

The proof goes roughly as follows: Start with a smooth bump supported on $B(0, 2)$. Then redistribute the L^1-mass of Δu onto balls arranged in a fine grid, where the radius of the balls is so tiny that that $\operatorname{supp}(\Delta u)$ becomes small. Since the kernel $K(x, y)$ of the Laplace equation tends to infinity for $y \to x$, this means that also $\left\|\frac{\Delta u}{u}\mathbb{1}_{u\neq 0}\right\|_1$ becomes small. The rate of divergence of K determines the range of p in Theorem 1.

In [1] they prove a number of statements like Theorem 1. However the functions that satisfy the respective statements are all constructed according to the same scheme described above. For simplicity we also prove Theorem 1 only for $n \geq 3$ here.

20.2 Inductive Construction

Take an increasing sequence of radii $(r_k)_k$ that tends to 2, and a sequence of thicknesses $(a_k)_k$ with $a_0 = r_0$ and such that with

$$A_k := \{x \mid r_k - a_k \leq |x| \leq r_k\}$$

we have

$$\operatorname{dist}(A_k, A_{k+1}) = a_k,$$

which means $r_{k+1} = r_k + a_k + a_{k+1}$ and $2 - r_k \geq a_k$. Then take a sequence of smooth functions $(\chi_k)_k$ that are 1 for $|x| \leq r_k - a_k + \frac{\delta}{2\sqrt{n}}a_k$ and 0 for $|x| \geq r_k - \frac{\delta}{2\sqrt{n}}a_k$ with δ to be chosen later, and which for each α satisfy

$$\|\partial^\alpha \chi_k\|_\infty \lesssim a_k^{-|\alpha|}. \tag{2}$$

Throughout the presentation the implicit constant in $\lesssim$ and such may depend only on n. We will produce u as the limit of the sequence $(u_k)_k$ with $u_0 = \chi_1$. Given u_k set $g_k := \Delta u_k$. We will come up with some f_k and define v_k by

$$\begin{aligned} \Delta v_k &= \begin{cases} g_k & \text{outside } A_k \\ f_k & \text{in } A_k \end{cases}, \\ v_k &= 0 \qquad \text{on } \partial B(0, r_{k+1}), \end{aligned}$$

and then set $u_{k+1} := \chi_{k+1} v_k$.

In order to define f_k define the grid

$$X_k := A_k \cap \delta a_k \mathbb{Z}^n.$$

Note that $|X_k| \sim_n \delta^{-n} a_k^{1-n}$. Then take positive smooth functions φ, ψ where φ is supported on $B(0,1)$ and has integral 1, and ψ is supported on $[-\frac{3}{4}, \frac{3}{4}]^n$ and satisfies for all x that

$$\sum_{m\in\mathbb{Z}^n} \psi(x - z) = 1.$$

Let $(\varphi_k^z)_{z\in X_k}, (\psi_k^z)_{z\in X_k}$ be versions of φ, ψ distributed along X_k according to

$$\varphi_k^z(x) := \varepsilon_k^{-n}\varphi\big(\frac{x-z}{\varepsilon_k}\big),$$
$$\psi_k^z(x) := \psi\big(\frac{x-z}{\delta a_k}\big),$$

for ε_k to be chosen later. Then on A_k we have

$$\sum_{z\in X_k}\psi_k^z = 1.$$

Now define the values

$$f_k^z := \begin{cases}\int \psi_k^z g_k & |\int \psi_k^z g_k| \geq \delta^{n+1}a_k^{n-1}2^{-2k}\\ \delta^{n+1}a_k^{n-1}2^{-2k} & \text{otherwise}\end{cases}$$

and set

$$f_k := \sum_{z\in X_k} f_k^z\varphi_k^z.$$

The following proposition provides the basic estimates for the sequence $(v_k)_k$. We will not prove it here, but it follows from computations involving the fundamental solution for the Laplacian on $B(0,r)$ for $n\geq 3$,

$$K_r(x,y)\sim \frac{1}{|x-y|^{n-2}} - \frac{|x|^{n-2}}{r^{n-2}|\bar{x}-y|^{n-2}},$$

where $\bar{x} = r^2\frac{x}{x^2}$.

Proposition 2. *There are constants $\delta, C>0$ such that for all k the following holds: Assume that v_{k-1} is harmonic in A_k and satisfies for $|\alpha|\leq 1$ that*

$$|\partial^\alpha v_{k-1}| \leq 2^{-2k}a_k^{1-|\alpha|} \qquad \text{in } A_k. \tag{3}$$

Then

$$|g_k| \leq C2^{-2k}a_k^{-1} \qquad \text{in } A_k,$$
$$|f_k^z| \leq C2^{-2k}\delta^n a_k^{n-1} \qquad \text{for all } z\in X_k, \tag{4}$$

and whenever $\varepsilon_k\leq \delta a_k$ and $|\alpha|\leq 1$ we have

$$\partial^\alpha(v_k - v_{k-1})| \leq 2^{-2(k+1)}a_k^{1-|\alpha|} \qquad \text{in } B(0, r_k - 2a_k), \tag{5}$$
$$|\partial^\alpha v_k| \leq 2^{-2(k+1)}a_{k+1}^{1-|\alpha|} \qquad \text{in } A_{k+1}. \tag{6}$$

20.3 Convergence

Proposition 3. *Assume that $u_k, v_k, X_k, \varepsilon_k, f_k^z$ are chosen as in Proposition 2. Then there is a function u supported in $B(0,2)$ with $\Delta u \in L^1$ and*

- *$\Delta u_k \to \Delta u$ in L^1*
- *$u_k \to u$ uniformly on sets with positive distance to $\partial B(0,2)$*

Proof.

$$\Delta u_{k+1} - \Delta u_k = \begin{cases} f_k - g_k & \text{in } A_k \\ g_{k+1} & \text{in } A_{k+1} \\ 0 & \text{else} \end{cases}.$$

Therefore by Proposition 2

$$\|\Delta u_{k+1} - \Delta u_k\|_1 \lesssim 2^{-2k}$$

so that $(\Delta u_k)_k$ converges in L^1 to

$$f := \sum_k \sum_{z \in X_k} f_k^z \varphi_k^z. \tag{7}$$

Define u by $\Delta u = f$, $u = 0$ on $\partial B(0,2)$. Because $\Delta u_{k+1} - \Delta u_k$ is supported outside of $B(0, r_k - a_k)$ and $r_k - a_k \to 2$, it is easy to see that u_k converges uniformly to u on $B(0,\rho)$ for all $\rho < 2$. □

20.4 Bounds on $\Delta u/u$

Proposition 4. *Let $u_k, v_k, X_k, \varepsilon_k, f_k^z$ be chosen as in Proposition 2. If $n \geq 3$ then on $B(z, \varepsilon_k)$ with $z \in X_k$ we have*

$$\left|\frac{\Delta u}{u}\right| \lesssim \varepsilon_k^{-2} \tag{8}$$

Proof. Recall that Δu equals (7) and that φ_k^z is supported on $B(z, \varepsilon_k)$. We have

$$|\Delta u| \lesssim |f_k^z| \frac{1}{\varepsilon_k^n}.$$

The dominant contribution to $u(x)$ is

$$\int K_2(x,y) f_k^z \varphi_k^z(y) \mathrm{d}y,$$

all other contributions are negligible. For $z \in X_k$, $x, y \in B(z, \varepsilon_k)$ we have

$$\begin{aligned} K_2(x,y) &\gtrsim \frac{1}{|x-y|^{n-2}} - \frac{|x|^{n-2}}{2^{n-2}(2-|y|)^{n-2}} \\ &\gtrsim \frac{1}{\varepsilon_k^{n-2}} - \frac{|x|^{n-2}}{(a_k^z)^{n-2}} \\ &\gtrsim \frac{1}{\varepsilon_k^{n-2}}, \end{aligned}$$

and thus

$$|u(x)| \gtrsim |f_k^z| \frac{1}{\varepsilon_k^{n-2}}$$

so that (8) follows. $\square$

20.5 Conclusion

Proof of Theorem 1. Recall that $|X_k| \sim a_k^{1-n}$. Thus by Proposition 4

$$\|\frac{\Delta u}{u} \mathbb{1}_{u \neq 0}\|_p^p \lesssim \sum_k a_k^{1-n} \varepsilon_k^{n-2p}. \tag{9}$$

Since we assumed $p < \frac{n}{2}$ we can make (9) finite by taking $(a_k)_k$ polynomially decreasing and $(\varepsilon_k)_k$ polynomially decreasing fast enough.

In order to prove the smoothness of u we need to show that f is smooth on $\mathbb{R}^n$. Since by construction f is smooth on $B(0,2)$ it suffices to show that for each α $\partial^\alpha f(x)$ goes to 0 for $|x| \to 2$ faster than linearly. Using (2) and (4) we obtain

$$|\partial^\alpha f(x)| \lesssim 2^{-2k} \delta^n a_k^{n-1-m} \varepsilon_k^{-n} \tag{10}$$

for $x \in A_k$. And since $(a_k)_k$ and $(\varepsilon_k)_k$ decrease polynomially in k, (10) decreases exponentially in k. And since $2 - r_k \geq a_k$, this means that (10) also decreases at least exponentially in $(2 - |x|)^{-1}$. $\square$

References

[1] Koch, H. and Tataru, D., *Sharp counterexamples in unique continuation for second order elliptic equations.* Journal für die Reine und Angewandte Mathematik (2002), 133–146

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21 L^P Carleman estimates and the Osculation Techniques - Talk 3: Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients (SUCP, Talk 2)

A summary written by Immanuel Zachhuber

Abstract

This is the second talk about [1], where we present sections 5-7.

21.1 Introduction

We are interested in proving the strong unique continuation property(SUCP) for solutions to PDEs of the form

$$\Delta u = Vu + W_1 \nabla u + \nabla(W_2 u) \tag{1}$$

for some suitable potentials V, W_1, W_2. In fact the main result is

Theorem 1 (Thm 1.1)**.** *Assume $V \in c_0(L^{\frac{n}{2}})$ and $W_1, W_2 \in l^1_w(L^n)$ hold. Then SUCP holds at 0 for H^1-solutions to* (1)*.*

Here we have used the notation $l^p(L^q)$ to mean L^q on dyadic annuli of size 2^j and l^q summation w.r.t. the j-parameter.
We briefly recall some notations from the previous sections, $h : \mathbb{R} \to \mathbb{R}$ is a convex function which, in addition, satisfies

$$|h''| + \operatorname{dist}(2h', \mathbb{Z}) \geq \frac{1}{4}. \tag{2}$$

For a function φ, which will be a perturbation of h, we define the space X_φ with norm

$$\|v\|_{X_\varphi} := \|v\|_{L^p} + \|\nabla v\|_{L^2 + |\nabla\varphi| L^p}$$

and its dual

$$X'_\varphi = L^{p'} + \nabla(L^2 \cap |\nabla\varphi|^{-1} L^{p'})$$

Moreover we introduce the following spaces for $\tau > 0$ and $0 < \varepsilon < 1$ on the "cylinder" $\mathbb{R} \times \mathbb{S}^{n-1}$

$$\tilde{X}_{\tau,\varepsilon} := \{v \in L^p \cap \tau^{-1/2}(1+\varepsilon\tau)^{-1/4}L^2, \nabla v \in (L^2 + \tau L^p) \cap \tau^{1/2}(1+\varepsilon\tau)^{-1/4}L^2\}$$

and by undoing the transformation $x = e^s\theta$, we set

$$\|u\|_{X_{\tau,\varepsilon}} := \||x|^{\frac{n-2}{2}} u\|_{\tilde{X}_{\tau,\varepsilon}} \tag{3}$$

for functions on $\mathbb{R}^n$. We further define the dual spaces

$$\tilde{X}'_{\tau,\varepsilon} = L^{p'} + \tau^{1/2}(1+\varepsilon\tau)^{1/4}L^2 + \nabla(L^2 \cap \tau^{-1}L^{p'}) + \tau^{-1/2}(1+\varepsilon\tau)^{1/4}\nabla L^2$$

and correspondingly

$$\|g\|_{X'_{\tau,\varepsilon}} := \||x|^{\frac{n+2}{2}} g\|_{\tilde{X}'_{\tau,\varepsilon}} \tag{4}$$

Note that in the case $|\nabla\varphi| \approx \tau$ these spaces are a refinement of the X_φ in the sense

$$X_{\tau,\varepsilon} \subset X_\varphi \text{ and } X'_\varphi \subset X'_{\tau,\varepsilon}.$$

With this notation in hand, we can state the main result of the paper which implies Theorem 1 by a standard argument.

Theorem 2. *[Thm. 3.2]For any $\tau > 0$, $W_1, W_2 \in l^1_w(L^n)$ and each function u vanishing of infinite order at 0 and ∞, there exists a function φ satisfying*

$$\tau \le -r\partial_r\varphi \le \tau^2, \ |\partial_\theta\varphi| \le |r\partial_r\varphi|$$

so that

$$\|e^\varphi u\|_{l^{p'}(X_\varphi)} + \frac{\|e^\varphi W_1 \nabla u\|_{L^{p'}}}{\|W_1\|_{l^1_w(L^n)}} + \frac{\|e^\varphi W_2 u\|_{|\nabla\varphi|^{-1}L^{p'}}}{\|W_2\|_{l^1_w(L^n)}} \lesssim \|e^\varphi \Delta u\|_{l^{p'}(X'_\varphi)}$$

21.2 A perturbation argument

In this section we extend the results of section 4 to suitable small perturbations of the function h which is allowed to have a radial dependence. We set

$$\varphi(x) = h(-\ln|x|) + k(-\ln|x|, \theta).$$

The first result is analogous to Proposition 4.1, which did not include the k.

Proposition 3 (Prop. 5.1)**.** *Let $\tau \gg 1$. Consider a convex function h satisfying (2) and $|h'| \in [\tau, 2\tau]$. Assume that k is s.t.*

$$|k| + |x||\nabla k| \ll 1.$$

Then

$$\|e^{\varphi} u\|_{X_{\tau,0}} \lesssim \|e^{\varphi} \Delta u\|_{X'_{\tau,0}}$$

for all functions u vanishing at infinite order at 0 and ∞.

We need a further refinement of this, the next result is similar to Prop. 4.2.

Proposition 4 (Prop.5.2)**.** *Let $\tau \geq 1$ and $0 < \varepsilon < 1$ s.t. $\varepsilon\tau > 1$. Consider h s.t.*

$$|h'| \in [\tau, 2\tau], \;\; h'' \in [\varepsilon\tau, \tau], \;\; |h'''| \leq \tau$$

in some interval I of length $|I| \lesssim 1$. Assume that

$$k + |x|^2 |\nabla^2 k| << \varepsilon\tau.$$

Then

$$\|e^{\varphi} u\|_{X_{\tau,\varepsilon}} \lesssim \|e^{\varphi} \Delta u\|_{X'_{\tau,\varepsilon}}$$

for all functions u supported in $\{x : |x| \in e^{-I}\}$.

21.3 The construction of h and global estimates

This part details the construction of the function h and the proof of a sharper result than Theorem 3.1.

Lemma 5 (Lem. 6.1)**.** *Let $(a_j)_{j \in \mathbb{Z}}$ be a non-negative sequence s.t.*

$$\|a\|_{l^1_w} \leq \frac{1}{4}$$

which is slowly varying, i.e.

$$\frac{1}{2} \leq \frac{a_{j+1}}{a_j} \leq 2 \;\forall j \in \mathbb{Z}.$$

Then for any τ large there exists a function $h : \mathbb{R} \to \mathbb{R}$ such that

1. *$\tau \leq \partial_s h(s) \leq \tau^2$.*

2. $h'(s)a_j \le h''(s) \le 2h'(s)a_j$ *if* $j \le s \le j+1$ *and* $a_j \ge C\tau^{-1}$.

3. $\mathrm{dist}(2h'(s), \mathbb{Z}) \ge \frac{1}{4}$ *if* $j \le s \le j+1$ *and* $a_j \le C\tau^{-1}$.

4. $|h'''(s)| \le 4h'(s)a_j$ *if* $j \le s \le j+1$.

This gives us a specific function h which will be perturbed by a small function k as above. The next theorem is the main global estimate and is a sharper result than Thm. 3.1 in [1].

Theorem 6 (Thm. 6.2)**.** *Let* $(a_j)_{j\in\mathbb{Z}}$ *and* h *as above and let* k *satisfy*

$$|x||\nabla k| + |x|^2|\nabla^2 k| \ll a_j h'(j) \quad \textit{for } s \in [j, j+1].$$

Then the estimate

$$\|e^{\varphi}u\|_{l^{p'}X_{h',a}} \lesssim \|e^{\varphi}\Delta u\|_{l^{p'}X_{h',a}}$$

holds for all functions u *vanishing at infinite order at* 0 *and* ∞.

Here the spaces $X_{h',a}$ are defined as $X_{h'(j),a_j}$ on dyadic annuli $2^{j-1} \le |x| \le 2^j$ and then the l^q norm is taken with respect to the parameter j.

21.4 Wolff's lemma and the gradient term

In order to also treat the gradient terms and prove the bounds in Theorem 2 we need some additional ideas, since the Carleman-type estimates from the previous sections are not enough. The strategy is to use the fact that we still have some liberty in choosing the function φ. Roughly speaking, we will construct a φ for which $\|e^{\varphi}W_1\nabla u\|^{p'}$ and $\|e^{\varphi}W_2 u\|^{p'}$ are concentrated on a "small" set so that they can be controlled by the $\|e^{\varphi}u\|_{l^{p'}X_{\varphi}}$ term.
We state the following result due to Wolff [2], which is needed to find the exact form of the perturbation k s.t. $\varphi(x) = h(|x|) + k(|x|, \theta)$ does the job and h is chosen as in Lemma 5 with a suitable sequence (a_j).

Lemma 7 (Wolf,[2])**.** *Let* μ *be a positive, compactly supported measure in* $\mathbb{R}^n$. *Define* μ_k *by* $\mathrm{d}\mu_k(x) := e^{k\cdot x}\mathrm{d}\mu(x)$. *Suppose* B *is a convex set in* $\mathbb{R}^n$. *Then there exists a sequence* $\{k_i\} \subset B$ *and, for each* i, *a convex set* E_{k_i} *with*

$$\mu_{k_i}(\mathbb{R}^d \backslash E_{k_i}) \le \frac{1}{2}\|\mu_{k_i}\|$$

such that the E_{k_i} are pairwise disjoint and

$$\sum_i |E_{k_i}|^{-1} \geq C|B|$$

for $C > 0$ depending only on the dimension and where $|\cdot|$ denotes the Lebesgue measure.

We then apply this to the measure

$$\mathrm{d}\mu = \chi_{F_j} \left| |\tilde{\nabla}\varphi| e^{\varphi(s,\vartheta)} \tilde{W} v \right|^{p'} \mathrm{d}s\mathrm{d}\theta$$

where we have changed variables via $x = e^s\theta$ and F_j is some suitably chosen compact set.

References

[1] Herbert Koch and Daniel Tataru, *Carleman estimates and unique continuation for second???order elliptic equations with nonsmooth coefficients.* Communications on Pure and Applied Mathematics(2001): 339-360.

[2] Thomas H. Wolff, *A property of measures in $\mathbb{R}^N$ and an application to unique continuation.* Geometric & Functional Analysis GAFA 2.2 (1992): 225-284.

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22 CGO, Uniqueness and Limiting Carleman Weights - Talk 1: A global uniqueness theorem for an inverse boundary value problem

A summary written by Wiktoria Zaton

Abstract

In this talk we shall see that the knowledge of the Dirichlet integrals for any boundary data on a region $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, associated to the elliptic operator $L_\gamma = \nabla \cdot \gamma \nabla$ is enough to determine the smooth coefficient γ. The proof relies on the construction of complex geometrical optics solutions.

22.1 Introduction

We work on a bounded domain Ω with smooth boundary and for a smooth coefficient $0 < \varepsilon < \gamma \in C^\infty(\overline{\Omega})$ consider the elliptic operator $L_\gamma = \nabla \cdot \gamma \nabla$. Let Q_γ denote the associated Dirichlet integral

$$Q_\gamma \colon H^{1/2}(\mathbb{R}^n) \ni f \mapsto \int_\Omega \gamma |\nabla u|^2 = \int_{\partial\Omega} f \gamma \frac{\partial u}{\partial \nu},$$

where u is the unique solution to

$$\begin{aligned} L_\gamma u &= 0 \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega. \end{aligned}$$

The question is whether the knowledge of the values of Q_γ for all boundary data f enables us to determine the coefficient γ. For later use let us define the Dirichlet to Neumann map $\Lambda_\gamma \colon f \mapsto \gamma \partial_\nu u|_{\partial\Omega}$. The following result we saw last time gives an affirmative answer in case of real analytic coefficients.

Theorem 1 (R. Kohn, M. Vogelius)**.** *Let γ_0, $\gamma_1 \in C^\infty(\overline{\Omega})$ and $x^* \in \partial\Omega$. Suppose there exists a neighbourhood B of x^* in $\overline{\Omega}$ such that $Q_{\gamma_0}(f) = Q_{\gamma_1}(f)$ holds for all $f \in H^{1/2}(\partial\Omega)$ supported in B. Then for any multiindex $\alpha \in \mathbb{N}^n$*

$$\partial^\alpha \gamma_0(x^*) = \partial^\alpha \gamma_1(x^*).$$

In this talk we solve the problem for $n \geq 3$ using the technique of complex geometrical optics solutions, inspired by Calderón's calculations (see the survey [3]). For this we construct special solutions of $L_\gamma u = 0$ of the form

$$e^{x\cdot\xi}\gamma^{-1/2}(1+\psi(x,\xi)),$$

where $\xi \in \mathbb{C}^n$ is a parameter with $\xi\cdot\xi = 0$ and $\psi(\cdot,\xi) \to 0$ for $|\xi| \to \infty$. Those functions are highly oscillatory for $|\xi|$ large. We observe that if $n \geq 3$ than for any $k \in \mathbb{R}^n$ there are $\xi_1,\ \xi_2$ with $\xi_j \cdot \xi_j = 0$ such that $e^{ix\cdot k} = e^{ix\cdot\xi_1}e^{ix\cdot\xi_2}$ arises in the product of the two corresponding special solutions. We can enforce $|\xi_j|$ to be arbitrarily large and so hope to be able to gain information about the Fourier transform of some function involving γ and thus about the coefficient itself.

22.2 The main result

We postpone the construction of special solutions and explain first how they are used in the main proof.

Theorem 2. *Let $n \geq 3$. Suppose $\gamma_0,\ \gamma_1 \in C^\infty(\overline{\Omega})$ satisfy $\gamma_0,\ \gamma_1 > 0$ in $\overline{\Omega}$ and $Q_{\gamma_0}(f) = Q_{\gamma_1}(f)$ holds for all $f \in H^{1/2}(\partial\Omega)$. Then $\gamma_0 = \gamma_1$.*

For γ_0 and γ_1 as above let us define $\gamma = (1-t)\gamma_0 + t\gamma_1$ for $t \in [0,1]$.

Proposition 3. *Let $s > n/2$ and $\xi \in \mathbb{C}^n$ satisfy $\xi \cdot \xi = 0$. There exist constants C_1 and C_2 such that if $|\xi| \geq C_1\|q\|_{H^s(\Omega)}$ for $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$ then there exists a solution $u(x,\xi,t)$ of $L_{\gamma(t)}u(\cdot,\xi,t) = 0$ of the form*

$$u(x,\xi,t) = e^{x\cdot\xi}\gamma^{-1/2}(1+\psi(x,\xi,t)),$$

for some $\psi(\cdot,\xi,t) \in H^s(\Omega)$, $t \in [0,1]$, with $\|\psi(\cdot,\xi,t)\|_{H^s(\Omega)} \leq \frac{C_2}{\xi}\|q(t)\|_{H^s(\Omega)}$. We have $u(x,\xi,0) = u(x,\xi,1)$ on $\partial\Omega$ in the trace sense.

Theorem 2. As explained before given $k \in \mathbb{R}^n$ and $r > 0$ we can choose

$$\xi_1 = i\left(\frac{k}{2} + r\eta\right) + \zeta \ \text{ and }\ \xi_2 = i\left(\frac{k}{2} - r\eta\right) - \zeta,$$

where $\eta,\ \zeta \in \mathbb{R}^n$ satisfy $\langle k,\eta\rangle = \langle k,\zeta\rangle = \langle \eta,\zeta\rangle = 0$, $|\eta| = 1$ and $|\zeta|^2 = \frac{|k|^2}{4} + r^2$, such that $\xi_j\cdot\xi_j = 0$ and $|\xi_j| > r$ for $j = 1,2$. The bilinear forms

Q_{γ_1} and Q_{γ_2} coincide, thus, putting $w(t) := u(\cdot, \xi_1, t)$ and $v(t) := u(\cdot, \xi_2, t)$

$$\begin{aligned}
0 &= Q_{\gamma_0}(w(0)|_{\partial\Omega}, v(0)|_{\partial\Omega}) - Q_{\gamma_1}(w(0)|_{\partial\Omega}, v(0)|_{\partial\Omega}) \\
&= Q_{\gamma_0}(w(0)|_{\partial\Omega}, v(0)|_{\partial\Omega}) - Q_{\gamma_1}(w(1)|_{\partial\Omega}, v(1)|_{\partial\Omega}) \\
&= \int_0^1 (Q_{\gamma_t}(w(t)|_{\partial\Omega}, v(t)|_{\partial\Omega}))' \, dt \\
&= \int_0^1 \left[\int_\Omega \gamma'(t) \nabla w(t) \cdot \nabla v(t) + \int_{\partial\Omega} \gamma(t) \left(w'(t) \partial_\nu v(t) + v'(t) \partial_\nu w(t) \right) \right] dt,
\end{aligned}$$

where we have used that the traces of $v(0)$ and $v(1)$, corresp. $w(0)$ and $w(1)$, coincide, and that the map $\gamma \to Q_\gamma$ is analytic (Calderón) with the Frechét derivative at γ_0 in the direction h given by $dQ_{\gamma_0} h(v, w) = \int_\Omega h \nabla w \cdot \nabla v$. It can be shown ([2] lemma 2.8) using integration by parts that the second double integral in the sum vanishes and thus $\int_0^1 \int_\Omega \gamma'(t) \nabla w(t) \cdot \nabla v(t) dt = 0$. Now, all derivatives of $\gamma' = \gamma_1 - \gamma_0 \in C^\infty(\overline{\Omega})$ vanish on $\partial\Omega$ by theorem 1, so we may extend it trivially by 0 to $\mathbb{R}^n$. We use the product rule, integrate by parts a few times using $L_\gamma v = 0,\ L_\gamma w = 0$ and arive at

$$\begin{aligned}
0 &= \int_0^1 \int_{\mathbb{R}^n} L_\gamma \left(\frac{\gamma'}{\gamma} \right) vw \\
&= \int_0^1 dt \int_{\mathbb{R}^n} dx \; L_\gamma \left(\frac{\gamma'}{\gamma} \right) \frac{1}{\gamma} e^{ix \cdot k} (1 + \psi(x, \xi_1, t))(1 + \psi(x, \xi_2, t)).
\end{aligned}$$

We now use the Hölder inequality together with the Sobolev norm estimate for ψ and let $r \to \infty$. We are left with

$$0 = \int_{\mathbb{R}^n} dx \int_0^1 dt \; L_\gamma \left(\frac{\gamma'}{\gamma} \right) \frac{1}{\gamma} e^{ix \cdot k}.$$

Since $k \in \mathbb{R}^n$ was arbitrary, we conclude

$$0 = \int_0^1 L_\gamma \left(\frac{\gamma'}{\gamma} \right) \frac{1}{\gamma} \, dt.$$

We can calculate that the right hand side equals

$$\Delta(\log \gamma_1 - \log \gamma_0) + \frac{1}{2} \nabla(\log \gamma_1 + \log \gamma_0) \cdot \nabla(\log \gamma_1 - \log \gamma_0).$$

Combining this with the fact $\log \gamma_1 = \log \gamma_0$ on $\partial\Omega$ we conclude by the weak maximum principle $\log \gamma_1 = \log \gamma_0$ in Ω. $\square$

22.2.1 Existence of the special solutions

Proof. 3 Since all derivatives of γ_0 and γ_1 coincide on $\partial\Omega$, we may extend $\gamma(t) = (1-t)\gamma_0 + t\gamma_1$ smoothly to $\mathbb{R}^n$ and can assume that outside of a ball around Ω, $\gamma = 1$. We plug a special solution into $\nabla \cdot \gamma \nabla u = 0$ and see that we need to solve

$$\Delta\psi + 2\xi \cdot \nabla\psi - q\psi = q, \quad \text{where} \quad q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}. \tag{1}$$

This is done by Fourier transform methods and weighted Sobolev spaces. For $m \in \mathbb{N}$ let $H^m_\delta = H^m(\mathbb{R}^n, (1+|x|^2)^\delta dx)$. We first find solutions to

$$Lw = \Delta w + 2\xi \cdot \nabla w = f \quad \text{in} \quad \mathbb{R}^n, \tag{2}$$

where $f \in L^2_{\delta+1}$ for some $-1 < \delta < 0$. By change of coordinates we can assume that $2\xi = s(e_1 + ie_2)$ and thus on the Fourier side (2) reads

$$l(\eta)\hat{w} = (-|\eta|^2 - s\eta_2 + is\eta_1)\hat{w} = \hat{f}. \tag{3}$$

Now $\mathscr{M} = \{\eta \in \mathbb{R}^n |\ l(\eta) = 0\}$ is a codimension-2 sphere and we can introduce a finite covering (V_j) of $\mathbb{R}^n$ (with a subordinate partition of unity (ρ_j)) and new local (dual-)coordinates on the V_j's so that (3) locally becomes $\hat{w}_j = (\beta_j + i\beta_1)^{-1}\rho_j \hat{f}$. This Fourier multiplier is a bounded map from $L^2_{\delta+1}$ to L^2_δ, which is seen in a direct calculation and we put $\hat{w} = \sum_j \hat{w}_j$. We have

$$\|w\|_{L^2_\delta} \leq \frac{C_\delta}{|\xi|}\|f\|_{L^2_{\delta+1}} \text{ and } \|w\|_{H^m_\delta} \leq \frac{C_\delta}{|\xi|}\|f\|_{H^m_{\delta+1}}$$

by differentiation. This is a unique solution in L^2_δ ([2] corollary 3.5) and so we have proven the existence of the solution operator $S\colon L^2_{\delta+1} \to L^2_\delta$. We turn our attention back to (1) and apply the solution operator to obtain

$$(I + S \circ M_{(-q)})\psi = Sq, \text{ where } M_{(-q)}w := -qw. \tag{4}$$

We need to invert the operator $I + S \circ M_{(-q)}$ on L^2_δ. There exists a constant C_1 depending on Ω, n, s such that if $s > \frac{n}{2}$ and $|\xi| \geq C_1\|q\|_{H^s}$ the following estimate is true

$$\|S \circ M_{(-q)}\psi\|_{L^2_\delta} \leq \frac{C_\delta}{|\xi|}\|q\psi\|_{L^2_{\delta+1}} \leq \frac{C_\delta}{|\xi|}\|(1+|x|^2)^{1/2}q\|_{L^\infty}\|\psi\|_{L^2_\delta} < \frac{1}{2}\|\psi\|_{L^2_\delta},$$

since by Sobolev embeddings and the compact support of q we can estimate $\|(1+|x|^2)^{1/2}q\|_{L^\infty} \leq C_\Omega \|q\|_{L^\infty} \leq C_{\Omega,n,s}\|q\|_{H^s} \leq C|\xi|$ (the constant C varies). Thus we may invert the operator by Neumann series. Similarly we obtain an estimate of the H^s_δ-norm of ψ.

To complete the proof we need to show that for ξ as above

$$u(x,\xi,0) = u(x,\xi,1) \quad \text{on} \quad \mathbb{R}^n \setminus \Omega.$$

To this end let $z \in H^1(\Omega)$ be the unique solution of

$$\begin{aligned} L_{\gamma_1} z &= 0 && \text{in} \quad \Omega \\ z &= u(x,\xi,0) && \text{on} \quad \partial\Omega. \end{aligned}$$

Recall that we extended γ to $\mathbb{R}^n$ independently of t. In particular $u(x,\xi,0)$ solves $\nabla \cdot \gamma_1 \nabla u = 0$ in $\mathbb{R}^n \setminus \Omega$. We claim

$$w := \begin{cases} z & \text{in } \Omega \\ u(x,\xi,0) & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

solves $L_{\gamma_1} w = 0$ in $\mathbb{R}^n$. Since $\gamma^{1/2} e^{-x\cdot\xi} w - 1$ satisfies the required Sobolev estimates, we get $w = u(x,\xi,1)$ on $\mathbb{R}^n$, thus $u(x,\xi,0) = u(x,\xi,1)$ on $\mathbb{R}^n \setminus \Omega$. For our claim we only need to show $\partial_\nu z = \partial_\nu(u(x,\xi,0))$ on $\partial\Omega$. Since the Dirichlet forms Q_{γ_0} and Q_{γ_1} coincide, so do the Dirichlet to Neumann maps Λ_{γ_i}, $i = 0,\ 1$. Thus, using $\Lambda_{\gamma_0}(u(x,\xi,0)|_{\partial\Omega}) = \gamma_0 \partial_\nu u(x,\xi,0)|_{\partial\Omega}$

$$\gamma_0 \partial_\nu z|_{\partial\Omega} = \gamma_1 \partial_\nu z|_{\partial\Omega} = \Lambda_{\gamma_1}(u(x,\xi,0)|_{\partial\Omega}) = \gamma_0 \partial_\nu u(x,\xi,0)|_{\partial\Omega}. \qquad \square$$

22.2.2 Remarks abd susbequent results

We refer to the survey [3] where further details and references can be found.

In a different work the authors proved theorem 1 for merely Lipschitz continuous conductivities. Using this fact we can replace the smoothness condition in the main theorem by γ being C^2. However, this is not the weakest possible assumption under which the result holds.

The main theorem is a consequence of a more general result. We have

$$\gamma^{-\frac{1}{2}} L_\gamma(\gamma^{-\frac{1}{2}}) = \Delta - q \ \text{ with } \quad q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$$

and the knowledge of the Cauchy data for the conductivity equation is enough to determine the Cauchy data for the Schrödinger equation. It can be shown that in this case the boundary measurements are enough to determine any merely bounded potential q. Surprisingly it is sufficient to do the measurements on slightly more than half of the boundary only. The following theorem is obtained as a corollary of this result.

Theorem 4 ([1])**.** *For $\xi \in S^{n-1}$ and a bounded domain with C^2 boundary define $\partial\Omega_-(\xi) = \{x \in \partial\Omega | \langle \nu(x), \xi \rangle < 0\}$. Assume the potentials $\gamma_i \in C^2(\overline{\Omega})$, $i = 1, 2$ are strictly positive and that for some $\xi \in S^{n-1}$*

$$\Lambda_{\gamma_1}(f)|_{\partial\Omega_-(\xi)} = \Lambda_{\gamma_2}|_{\partial\Omega_-(\xi)}(f)$$

holds for all $f \in H^{-1/2}(\partial\Omega)$. Then $\gamma_1 = \gamma_2$.

The assumptions above can be relaxed even further, as we will see in the next talk.

Finally we wish to remark that in two dimensions the main theorem holds even for L^∞ conductivities. This case requires some new methods and a different construction of complex geometrical optics solutions.

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23 A nonlinear Plancherel theorem with applications to global well-posedness for the defocusing Davey-Stewartson equation and to the inverse boundary value problem of Calderón - Talk 3: section 6

A summary written by Pavel Zorin-Kranich

Abstract

We indicate how the Dirichlet-to-Neumann map for the Calderón problem in dimension 2 with conductivity σ can be used to recover the scattering transform of $q = -\frac{1}{2}\partial \log \sigma$. By invertibility of the scattering transform for L^2 data this implies that σ can be recovered from the Dirichlet-to-Neumann map assuming $\log \sigma \in \dot{H}^1$.

Let $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ be a simply connected domain with $C^{1,1}$ boundary. Denote the outside normal unit vector field by ν and the tangential vector field by τ. Consider the Dirichlet problem

$$\nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = g. \tag{1}$$

Here we interpret u as electrostatic potential, σ as conductivity, ∇u as electric field, and $\sigma \nabla u$ as current density. Then the equation describes conservation of charge and the boundary condition is an externally applied voltage. We consider conductivities $\sigma > 0$ that satisfy

$$\log \sigma \in \dot{H}^1(\Omega), \quad \sigma|_{\partial\Omega} \equiv 1. \tag{2}$$

Theorem 1 ([1, Theorem 1.7])**.** *Assume* (2)*. Then for every real-valued* $g \in H^1(\partial\Omega)$ *there exists a unique solution to* (1) *with* $\sigma^{1/2}\nabla u \in H^{1/2}(\Omega)$*. Also,* $\nu \cdot \nabla u \in L^2(\partial\Omega)$*.*

The existence of the solution does not follow from elliptic theory because σ is not assumed to be bounded above or below. 1/2 derivative of ∇u is necessary to define the trace on the boundary in L^2.

The solutions of (1) are in one-to-one correspondence to solutions of

$$\bar{\partial} v - q\bar{v} = 0 \text{ in } \Omega, \quad \Im(\nu v|_{\partial\Omega}) = g_0 \tag{3}$$

with data

$$q = -\frac{1}{2}\partial \log \sigma \in L^2(\Omega), \quad g_0 = -\frac{1}{2}\frac{\partial g}{\partial \tau} \in L^2(\partial\Omega). \tag{4}$$

Here $\partial = (\partial_x - i\partial_y)/2$ and $\bar{\partial} = (\partial_x + i\partial_y)/2$. The correspondence is given by

$$v = \sigma^{1/2}\partial u. \tag{5}$$

One can verify that (1) for u implies (3) for v. Conversely, suppose that (3) holds for v. In order to find a real-valued function u such that (5) holds it suffices to check that the vector field $\sigma^{-1/2}v$ is rotation-free, that is, that

$$\Im(\bar{\partial}(\sigma^{-1/2}v)) = \partial_y \Re(\sigma^{-1/2}v) + \partial_x \Im(\sigma^{-1/2}v)$$

vanishes. This can be in turn checked using (3).

Theorem 2 ([1, Theorem 1.9])**.** *Suppose $q \in L^2(\Omega)$ is as in* (4) *with σ as in (2) and $g_0 \in L^2(\partial\Omega)$ is real-valued with integral* 0*. Then (3) has a unique solution v with*

$$\|v\|_{H^{1/2}(\Omega)} + \|v\|_{L^2(\partial\Omega)} \le C(q)\|g_0\|_{L^2(\partial\Omega)}.$$

In particular there is a bounded operator

$$H_q(g_0) := \Re(\nu u|_{\partial\Omega}) \tag{6}$$

acting on $L^2(\partial\Omega)$. We do not claim that its norm is controlled by $\|g_0\|_{L^2}$.

First we solve (3) in the case $q = 0$, that is,

$$\bar{\partial}v = 0 \text{ in } \Omega, \quad \Im(\nu v|_{\partial\Omega}) = g + c \text{ on } \partial\Omega.$$

To do this write $v = \partial u$ and solve a boundary value problem for $\Delta u = 0$. The solution operator $g_0 \mapsto (v, c)$ is bounded from $L^2(\partial\Omega)$ to $H^{1/2}(\Omega) \times \mathbb{C}$. Denote the first component of the solution operator by $Bg_0 := v$.

Next consider the inhomogeneous problem

$$\bar{\partial}v = f_0 \text{ in } \Omega, \quad \Im(\nu v|_{\partial\Omega}) = c \text{ on } \partial\Omega$$

with $f_0 \in L^{4/3}(\Omega)$. This can be solved by first solving $\Delta u = f_0$ on $\mathbb{R}^2$, then $v_0 = \partial u \in W^{1,4/3}(\mathbb{R}^2)$, and presumably this space embeds into $H^{1/2}(\Omega) \cap L^2(\partial\Omega)$. The possibly non-trivial boundary trace can be eliminated by setting

$v = v_0 - B(v_0|_{\partial\Omega})$. The solution operator $f_0 \mapsto (v, c)$ is bounded from $L^{4/3}(\Omega)$ to $H^{1/2}(\Omega) \times \mathbb{C}$. Denote its first component by $Tf_0 = v$.

The equation (3) can now be written as

$$v = T(q\bar{v}) + B(g_0), \quad v \in H^{1/2}(\Omega) \cap L^2(\partial\Omega)$$

(a priori this equation has an additional constant term on the boundary that cancels out eventually). The right-hand side makes sense because $\|q\bar{v}\|_{L^{4/3}} \lesssim \|q\bar{v}\|_{\dot{H}^{-1/2}} \lesssim \|q\|_{L^2}\|v\|_{\dot{H}^{1/2}}$. If q is smooth, then $T(q\bar{v})$ has one more derivative than v, so $v \mapsto T(q\bar{v})$ is a compact operator. Approximating $q \in L^2(\Omega)$ by smooth functions we see that $v \mapsto T(q\bar{v})$ is a norm limit of compact operators, hence compact.

By Fredholm's alternative existence (and also uniqueness) of solutions will follow from absence of non-zero solutions of

$$v = T(q\bar{v}). \tag{7}$$

If $\|q\|_{L^2}$ is sufficiently small, then the operator on the right-hand side of (7) is strictly contractive on $H^{1/2}(\Omega)$, and this implies $v = 0$. A general function $q \in L^2(\Omega)$ can be split as $q = q_{sm} + q_{er}$ into a smooth part and an error part with small L^2 norm. Then starting with a solution of (7) we can construct a solution of the same equation with q replaced by q_{er} as for the equation on the full space. That solution has to be 0, so that $v = 0$. This finishes the proof of Theorem 2.

The Dirichlet-to-Neumann map for the problem (1) can be related to the operator (6). Reconstructing σ from the Dirichlet-to-Neumann map is the same as reconstructing q from (6). The next result tells that q is uniquely determined by H_q.

Theorem 3 ([1, Theorem 1.10])**.** *The map $q \mapsto H_q$ from the subset of $L^2(\Omega)$ given by (4) to the space of bounded operators on $L^2(\partial\Omega)$ is injective.*

Proof. We will show that the scattering transform $\mathcal{S}q$ is uniquely determined by H_q, where we identify q with its extension by 0 outside Ω. Recall

$$\mathcal{S}q(k) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}(m_+(z,k) + m_-(z,k))\mathrm{d}z,$$

where $m_\pm(\cdot, k)$ are the Jost solutions of the equation $\bar{\partial}m_\pm = \pm e_{-k}q\overline{m_\pm}$ and $e_k(z) = e^{i(zk+\overline{zk})}$ (as in the 2nd talk on this article). Let $k \in \mathbb{R}^2$ be such that

$M\hat{q}(k) < \infty$. Substituting the equation determining $m_\pm$ and using Stokes's theorem we obtain

$$\mathcal{S}q(k) = \frac{1}{2\pi i}\int_\Omega \overline{\bar\partial m_+(z,k) - \bar\partial m_-(z,k)}\mathrm{d}z = \frac{1}{4\pi i}\int_{\partial\Omega} \bar\nu\overline{(m_+(z,k) - m_-(z,k))}\mathrm{d}z.$$

Hence it suffices to show that $m_\pm(\cdot,k)|_{\partial\Omega}$ is uniquely determined by H_q. In [1] this is done for m_+.

Let $\psi_\pm(z) := e^{izk}m_\pm(z,k)$. Then $\psi_\pm$ satisfy

$$\bar\partial\psi_\pm = e^{izk}\bar\partial m_\pm = \pm e^{izk}e_{-k}q\overline{m_\pm} = \pm e^{i(zk-zk-\overline{zk})}q\overline{m_\pm} = \pm q\overline{m_\pm e^{izk}} = \pm q\overline{\psi_\pm}.$$

In particular ψ_+ satisfies (3) in Ω, and it follows that

$$\Re(\nu\psi_\pm|_{\partial\Omega}) = H_{\pm q}(\Im(\nu\psi_+|_{\partial\Omega})).$$

We also have

$$\bar\partial\psi_\pm = 0 \text{ in } \mathbb{C}\setminus\overline{\Omega}, \quad \psi_\pm(z)e^{-izk} - 1 \in L^4(\mathbb{C}\setminus\overline{\Omega}) \cap W^{1,4/3}_{loc}.$$

The last two displays only involve q via $H_{\pm q}$. We claim that the last two displays uniquely characterize ψ_+ in $\mathbb{C}\setminus\Omega$. Let h_1, h_2 be functions with

$$\Re(\nu h_j|_{\partial\Omega}) = H_q(\Im(\nu h_j|_{\partial\Omega})), \quad \bar\partial h_j = 0 \text{ in } \mathbb{C}\setminus\overline{\Omega}, \quad h_j(z)e^{-izk}-1 \in L^4(\mathbb{C}\setminus\overline{\Omega})\cap W^{1,4/3}_{loc}$$

for $j = 1, 2$. Then their difference $h = h_1 - h_2$ satisfies

$$\Re(\nu h|_{\partial\Omega}) = H_q(\Im(\nu h|_{\partial\Omega})), \quad \bar\partial h = 0 \text{ in } \mathbb{C}\setminus\overline{\Omega}, \quad h(z)e^{-izk} \in L^4(\mathbb{C}\setminus\overline{\Omega})\cap W^{1,4/3}_{loc}.$$

We claim that $h = 0$. Indeed by Theorem 2 we can solve (3) with $g_0 = \Im(\nu h|_{\partial\Omega})$. Then by definition of H_q we obtain $\Re(\nu v|_{\partial\Omega}) = \Re(\nu h|_{\partial\Omega})$. Extend h to be the solution v inside Ω. Then

$$\bar\partial h = q\bar h \text{ on } \mathbb{C},$$

so $m = e^{-izk}h \in L^4(\mathbb{C})$ satisfies $\bar\partial m = e^{-izk}q\bar h = e_{-k}q\bar m$. The solution of this problem is unique, so $m = 0$, hence $h = 0$. Hence ψ_+ is uniquely determined by H_q. $\square$

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